The Inefficiency of Diversification in Economies With Endogenous Liquidation Costs*

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Abstract

We consider a two-asset economy in which consumers are subject to liquidity shocks. Consumers (or banks in the case of delegation) may liquidate these assets in order to finance shortfalls of liquidity. The costs of liquidation for a consumer depend on the liquidity positions of other consumers since those are potential purchasers of assets. We show that diversification of consumers' portfolios is inefficient in this economy. In the absence of situations of insolvency it is even efficient to have their portfolios completely undiversified. Efficiency further requires consumers to specialize in different portfolios such that the economy's *aggregate* portfolio is diversified. We also show that equilibrium portfolio allocations may not be efficient: Consumers have a tendency to invest more in the market portfolio than is optimal.

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1 Introduction

The principle of diversification is an essential paradigm of economics. It is based on the simple observation that by investing in a combination of assets rather than a single asset, return volatility can be reduced. It has been most prominently formulated in the form of portfolio theory (Markovitz (1957)), which stipulates that investors should only hold shares of the market portfolio. It has also been applied to a wide range of other situations, such as investment decisions at firms, lending activities at banks, or to the industrial policies of countries.

An assumption shared by many models of diversification is that assets are held until they mature. Portfolio diversification (that is, diversification of asset payoffs at maturity) is then equivalent to diversification of the payoffs an investor is receiving. This assumption does not hold in many contexts. Assets may have to be liquidated before they mature, for example, due to liquidity shocks or insolvency. The payoffs in such liquidations may not be equal to the assets' final value. More importantly, in general equilibrium the payoffs may depend on the portfolio allocations of other investors in the economy since those are potential purchasers of assets.

The purpose of this paper is to show that diversification may no longer be desirable when assets have to be liquidated before maturity. The basic idea is the following. If all portfolios in an economy are diversified, portfolios values are perfectly correlated. When liquidation decisions are contingent on portfolio values (such as is the case with insolvency) portfolios may then have to be liquidated jointly in some states. In such states there will then also be only few potential buyers of assets. Asset prices will hence be low, resulting in misallocations of assets and inefficient risk sharing. The costs that arise from this may then make it optimal to hold only specialized portfolios in order to reduce the occurrence of such joint liquidations.

We study an economy where consumers are subject to idiosyncratic liquidity shocks (as in Diamond and Dybvig (1983)). There are two banks which insure consumers against these shocks and invest on their behalf in risky assets. The downside of insurance is that at low bank values consumers find it optimal to run on a bank, forcing the bank to liquidate its portfolio. In the absence of any interbank markets in this economy the standard result of the efficiency of full diversification at each bank obtains in this economy. However, when banks can trade assets among each other, this is no longer the case. The possibility of asset sales reduces the costs of a bank experiencing an individual run because a bank can now sell its portfolio to the other bank rather than having to prematurely liquidate assets. If the bank is solvent it may even avoid such a run because it may be able to raise the required liquidity by selling assets to the other bank. For these two reasons it is hence relatively more costly for banks to encounter low realizations of portfolio values jointly, rather than individually. Banks then benefit from specializing into different assets because this reduces the likelihood of such realizations. Since it can be shown that close to full diversification any losses from imperfect diversification are only of second order importance, full diversification is hence always inefficient.

When banks in this economy only fail due to illiquidity, the optimal degree of diversification is even zero. This is because due to the ability of the interbank market to smooth out idiosyncratic liquidity problems, inefficiencies only arise when there are aggregate shortages of liquidity. There are then no longer benefits from bank diversification because banks can completely specialize in (different) assets without increasing the likelihood of aggregate shortages. However, when there are also insolvency problems the optimal degree of diversification may be larger than zero (but still incomplete) because diversification then also benefits banks by reducing their likelihood of insolvency.

There are two important ingredients for these results to obtain in an otherwise standard economy. The first one is that the costs of simultaneous failures (or joint shortages of resources) at institutions are larger than of individual failures.[1] The second ingredient

[1] Besides for the two reasons present in our model this may be, for example, because a joint failure creates externalities outside the banking sector (as in Kahn and Santos (2008)) or causes inefficient liquidations to outsiders (Shleifer and Vishny (1992) and Acharya, Shin and Yorulmazer (2008)). It may also be because

is that an institutional failure induces a fixed cost (or, more generally, that the cost of shortages of resources at an institution are concave). In our economy this cost arises because when a bank's portfolio value falls below a certain threshold, it becomes optimal for all consumers to run on the bank, thus forcing the bank to liquidate its entire portfolio. It is this discrete cost which makes diversity beneficial: conditional on having an aggregate shortage of resources, efficiency losses can be lowered if institutions are diverse because then one institution may survive and hence a joint failure can be avoided. If efficiency losses were proportional to individual shortfalls, the total costs of shortages would only depend on the size of the aggregate shortfall and diversity would obviously have no benefit. Concave costs arising from liquidations are arguably present in many situations. For example, we show in an extension that the inefficiency of diversification continues to hold when consumers invest themselves rather than delegating to banks. Concave costs in this extension are due to consumers suffering a loss when liquidation of their assets is no longer sufficient to satisfy their liquidity needs.[2]

Even though we have shown that it is not optimal in our economy to diversify individual portfolios, this does not imply that the economy as a whole should not be diversified. In fact we show that efficiency requires that individual portfolios should be specialized such that the economy's aggregate portfolio is fully diversified with respect to the set of investable projects.[3] We believe this to be an important result since it suggests that in order to evaluate whether portfolios are efficiently diversified, one has to consider whether there is a lack of diversification at the aggregate level and not necessarily at the individual level.[4]

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joint liquidations lower asset prices due to fire-sales (e.g., Allen and Gale (2004)) or market microstructure reasons (e.g., Grossman and Miller (1988) and Bernardo and Welch (2004)).

[2]We also show that concave costs may arise when partial discontinuations of portfolios distort effort choices (yet another example of concave costs are bankruptcy costs).

[3]An immediate application of this is that the home bias observed in investors' portfolios (French and Poterba (1991)) does not necessarily indicate any inefficiencies.

[4]While it is widely documented that individual portfolios are not diversified (see, e.g., Heaton and Lucas (2000)), we are not aware of studies that measure aggregate diversification.

We also show that the efficient outcome may not be obtained in equilibrium. The reason is that when an investor (or bank) invests more in the market portfolio (and hence becomes less specialized) it increases the likelihood that other investors in the economy will encounter shortages jointly. While this lowers the utility of the investor itself, it also lowers the utility of other investors. Investment in the market portfolio hence entails an externality and in equilibrium there may hence be more of it than is efficient. We note that this provides a rationale for regulators to discourage investment in the market portfolio.[5] While this result concerns the efficiency of allocating investments among risky assets, previous literature has mostly considered efficiency in economies with a risky and a liquid asset.[6] Allen and Gale (2004) have shown the generic inefficiency of such economies when markets are incomplete. Other papers have identified several mechanisms through which an inefficient mix of risky assets and liquidity may be obtained in equilibrium (e.g., Bhattacharya and Gale (1987), Holmström and Tirole (1998), Gromb and Vayanos (2002), Gorton and Huang (2004) and Acharya, Shin and Yorulmazer (2007)).

Our results connect to an extensive literature that has studied why investors (or institutions) may rationally not diversify. Two main explanations have been brought forward. First, there may be some direct cost associated with diversification, such as for example a transaction cost (e.g., Constantinides, (1986)). Second, investors may find it optimal not to diversify if they are heterogenous, for example because of different background risk (e.g., Heaton and Lucas (2000)). The explanation in the present paper is not based on either reason as investors can both diversify at no costs and are, moreover, homogenous. The inefficiency of diversification arises because it is inefficient to hold identical portfolios.

[5] It also provides support for recent proposals to make bank capital requirements conditional on their likelihood of contributing to a systemic crisis, see for example, Pederson and Roubini (2009).

[6] An exception is Acharya and Yorulmazer (2006, 2007) where banks can also choose between two risky assets (but cannot diversify among them). Acharya and Yorulmazer consider a setting where a regulator faces a time-inconsistency problem that makes it optimal ex-post to bail out jointly failing banks. It is shown that this creates an incentive for banks to (inefficiently) specialize in the same asset, resulting in them not investing enough in the market portfolio.

The rest of the paper is organized as follows. In Section 2 we introduce the basic economy and study optimal diversification in the presence of illiquidity runs. Section 3 extends the analysis to the case where consumers invest themselves, rather than delegating to banks. In Section 4 we study the case of insolvency runs. Section 4.1 extends the analysis to a general form of consumer preferences and to a liquidity choice. Section 5 concludes.

2 The Basic Economy

In this section we describe a simple economy in which full diversification is neither an equilibrium nor efficient. The model is in the spirit of Allen and Gale (1998) and the ensuing literature in that it considers consumers which are subject to idiosyncratic liquidity shocks. These shocks create a role for banks in insuring consumers. There is also aggregate uncertainty because asset returns are random. This uncertainty exposes banks to runs. There is an interbank asset market at which banks can insure against this uncertainty. We depart from Allen and Gale (1998) in that we consider two risky assets (instead of one) and allow banks to hold a combination of these assets. Additionally, we also depart from the assumption of a representative bank by considering two banks and their interactions.

There are two regions (1 and 2) and three dates ($0, 1$ and 2). In each region there is a continuum of (ex-ante identical) consumers of equal mass and a bank. Consumers can only invest in the bank of their region. There is a single good, which can be used for both consumption and production. Consumers each have one unit of endowment of the good at date 0, but none at the other dates. Consumers face uncertainty about their preferences. We capture this in the simplest possible form (in Section 4.1 we consider standard Diamond-Dybvig preferences). A consumer faces at date 1 with probability λ ($0 < \lambda \leq 1$) a consumption need of $\overline{d}$ (early consumer). If he cannot satisfy this need, he suffers a utility loss of $k > 0$ and can only consume at date 1. In all other cases his utility is simply linear in consumption at dates 1 and 2. A consumer's utility can hence be summarized as follows

$$U(c_1, c_2) = \begin{cases} \text{with probability } \lambda: & u(c_1, c_2) = \begin{cases} c_1 + c_2 \text{ if } c_1 \geq \overline{d} \\ c_1 - k \text{ if } c_1 < \overline{d} \end{cases} \\ \text{with probability } 1 - \lambda: & u(c_1, c_2) = c_1 + c_2 \end{cases} \quad (1)$$

Consumers learn about their type (that is, whether they are an early consumer) at the beginning of date 1 and this information is private.

The role of banks is to make investments on behalf of the consumers and to insure them against the preference shocks. Banks have two investments at their disposal: asset X and asset Y (these assets are real non-tradeable projects in infinite supply). Each asset transforms one unit of the good at date 0 into an uncertain return at date 1 (denoted with x and y for assets X and Y, respectively) and a return R at date 2 (which is certain for simplicity). The date 1 asset returns of each asset are independently drawn from a density function $f(.)$, where f is assumed to be continuous with full support on $[0, \infty)$. We denote the mean of the distribution with μ. We assume for the date 2 return R:

$$R > \overline{d}. \quad (2)$$

As we will show later, this assumption ensures that in equilibrium a bank is always solvent when the economy has sufficient resources to pay out early consumers at the intermediate 1. This implies that there will only be bank failures due to illiquidity (we consider the alternative case of insolvency problems in Section 4).

Consumers themselves cannot invest in the assets because the assets need input from a bank (e.g., monitoring) between dates 0 and 1 and between dates 1 and 2. Without this input, an asset becomes worthless (in Section 4 we relax this assumption by allowing for premature liquidation of the asset at date 1). Assets, however, can at the intermediate date be transferred to another bank and be continued there. There is also a storage technology (available to both consumers and banks) which shifts one unit of the good from one period to the next. Since we want to focus on the diversification problem of banks, we assume that investing in the storage technology at date 0 is never optimal. Note that this will always be the case for sufficiently high asset returns (in Section 4.1 we consider the case where investing in the storage technology may be optimal, allowing banks adjust their risk by varying the amount of (risky) assets in their portfolio).

Banks offer standard deposit contracts to consumers of their region in return for their endowments. Without loss of generality we assume that consumers invest their entire wealth at their bank. There is free entry in each region, hence banks offer contracts to consumers that maximize consumers' utility. These contracts can neither be contingent on a consumer's type nor on the state of the economy (that is, the intermediate returns x and y). The deposit contract of bank i ($i = 1, 2$) thus offers a fixed amount d_i to consumers who wish to withdraw at date 1. Consumers who withdraw at date 2 receive the residual value of the bank. When a bank is not able to satisfy early withdrawals with its liquid resources, the bank has to sell its assets to the other bank (if this is possible). If, after having done so, the bank still cannot meet early withdrawals in full, the bank is declared bankrupt and ceases to exist. The bank's liquid resources are then distributed among the withdrawing depositors on a pro-rata basis (any assets that could not be sold to the other bank are then worthless since bank input can no longer be provided for them).

Each bank chooses a deposit contract d_i and how to allocate its funds among the two assets. A bank will never chose $d_i < \overline{d}$ as it then does not provide any liquidity insurance. A bank will also not choose $d_i > \overline{d}$ because this does not provide any more insurance to consumers (since above $\overline{d}$ utility is linear) but will expose the bank to additional runs. We can hence set $d_i = \overline{d}$.[7] We denote with α_i ($\alpha_i \in [0, 1]$) the share of funds a bank invests in asset Y (the share invested in asset X is then $1 - \alpha_i$). This parameter hence summarizes a bank's portfolio choice, where $\alpha_i = \frac{1}{2}$ denotes the full diversification portfolio and $\alpha_i = 0$ and $\alpha_i = 1$ the fully polarized portfolios. We denote with v_i the resulting portfolio return at date 1 which is given by

[7] In Section 4.1 (where we consider a more general form of risk averse preferences) the optimal deposit contract will be less straightforward.

$$v_i = \alpha_i y + (1 - \alpha_i)x. \tag{3}$$

We further denote with $\widetilde{v}$ ($\widetilde{v} = \frac{v_1+v_2}{2}$) the average return in the economy at this date. The final date return on a portfolio is R, irrespective of the portfolio allocation.

We next derive the utility for the consumers at both banks given portfolio choices α_1 and α_2 and deposit contracts $d_1 = d_2 = \overline{d}$. Without loss of generality we assume that bank 1 has invested at least as much in asset Y as bank 2 ($\alpha_1 \geq \alpha_2$). In order to avoid a (complete) run at date 1 a bank has to be both liquid and solvent. Liquidity requires that the bank has sufficient resources at date 1 to pay out all early consumers $\overline{d}$. If the bank is not liquid, withdrawal by early consumers will cause the failure of the bank. Late consumers, anticipating that they then would be left with nothing at date 2, will then also run on the bank. Solvency requires that after having paid early consumers at date 1 there are still sufficient resources left to pay the remaining consumers at least $\overline{d}$ at date 2. Otherwise, those consumers would find it optimal to withdraw at date 1 as well and to store the proceeds for consumption at date 2.[8]

Consider first the case of $\widetilde{v} \geq \lambda\overline{d}$ at date 1, that is, there are sufficient (liquid) resources in the economy to pay out early consumers at both banks. Two situations can arise: either we have $v_i \geq \lambda d$ at each bank, or there is one bank with $v_i < \lambda d$. In the first situation each bank has sufficient resources to pay out early consumers without resorting to asset sales. There exists then an equilibrium without runs in which the (shadow) price of an asset at date 1 is $p = R$.[9] To see this, note first that from date 1 on there is effectively

[8] Note that we do not consider panic runs, which rules out contagion among banks. It has been shown that diversification may also cause contagion. For example, Goldstein and Pauzner (2004) show that diversification of investors' portfolios may induce contagion among countries through a wealth effect.

[9] We exclude panic runs by assuming that banks can select the equilibrium preferred by depositors. The main results also hold if panic runs are permitted (calculations available on request). Then, contagion among banks may occur because the failure of one bank reduces the value of the other bank in liquidation (since assets can no longer be sold) and may trigger another run (relatedly, Dasgupta (2004) has shown that full cross-insurance between banks may not be desirable in the presence of contagion).

only one asset in the economy since both asset X and asset Y return R at date 2. Each bank holds exactly one unit of this asset, regardless of its original portfolio composition. Furthermore, note that the resources available for a late consumer at bank i at date 2 (after all early consumers have withdrawn at date 1) are $\frac{R+v_i-\lambda\overline{d}}{1-\lambda}$ since the bank can store any excess resources at date 1 at a return of one. Since $v_i \geq \lambda\overline{d}$ we have $\frac{R+v_i-\lambda\overline{d}}{1-\lambda} \geq \frac{R}{1-\lambda}$, which is larger than $\overline{d}$ by condition (2). Hence, each bank remains solvent and no runs occur. Since the return on storage is one, the equilibrium asset price is $p = R$. The payoffs at a bank are then $\overline{d}$ for early consumers and $\frac{R+v_i-\lambda\overline{d}}{1-\lambda}$ for late consumers.

In the second situation, one bank has a liquidity deficit. However, since there is overall no deficit in the economy, this bank can finance the deficit by selling assets to the other bank. There exists then again an equilibrium without runs and $p = R$. To see this, presume that we have $p = R$. The deficit bank first uses its liquid resources (v_i) to pay out early consumers as much as possible. The remaining shortfall ($\lambda\overline{d} - v_i$) then has to be generated through asset sales. For this the bank needs to sell a share q of its assets to the surplus bank, where q is given by $\lambda\overline{d} - v_i = pq$. After having done so, the return available to its late consumers is

$$\frac{(1-q)R}{1-\lambda} = \frac{R + v_i - \lambda\overline{d}}{1-\lambda}. \tag{4}$$

This is the same expression as in the first situation. Hence, no run will occur. Moreover, since there is no liquidity shortage in the economy and the return on the storage technology is one, $p = R$ constitutes an equilibrium price. The utilities of the consumers at each bank are then the same as in the first situation: $\overline{d}$ and $\frac{R+v_i-\lambda\overline{d}}{1-\lambda}$ for early and late consumers, respectively.

We consider next the case of $\widetilde{v} < \lambda\overline{d}$ at the intermediate date, that is, there are not enough liquid resources to satisfy the early consumers at both banks. Again we have to distinguish between two situations: either one of the banks has a surplus of resources or no bank has a surplus. In the first situation the deficit bank can again sell assets to the surplus bank. However, since there is an aggregate shortage, this can never generate enough liquidity to satisfy early consumers. Since this would leave zero for late consumers after early consumers have withdrawn, it becomes optimal for late consumers to run. The bank will thus face withdrawals by all consumers, forcing it to sell all its assets. The liquidation will occur at fire-sale prices since the available liquidity at the surplus bank (denoted with i) is not sufficient to purchase all assets of the deficit bank at their date 2 value R (this can be verified from that we have for the surplus liquidity $v_i - \lambda\overline{d}$: $v_i - \lambda\overline{d} < v_i + v_j - \lambda\overline{d} < \lambda\overline{d} < R$, where the first inequality follows from $x, y \geq 0$, the second from $\frac{v_i + v_j}{2} < \lambda\overline{d}$, and the third from condition (2)). The equilibrium asset price p will then be such that when the surplus bank purchases all assets from the deficit bank, its excess liquidity is just sufficient for this. We hence have $p = v_i - \lambda\overline{d}$. The reason for this is that $p = v_i - \lambda\overline{d}$ is the only price at which the demand for the asset by the surplus bank is one and hence equals the (inelastic) supply by the deficit bank. There is hence cash-in-the-market pricing (e.g., Allen and Gale (1998), Gorton and Huang (2004) and Acharya and Yorulmazer (2006)) because asset prices are determined by the available liquidity in the economy. The utilities of the consumers at the banks are then as follows. Since the deficit bank faces a (complete) run, all its consumers get the same amount. Early consumers additionally suffer the costs k since they cannot satisfy their consumption needs. Utility is hence $v_j + p - k = v_j + v_i - \lambda\overline{d} - k$ for early consumers, and $v_j + p = v_j + v_i - \lambda\overline{d}$ for late consumers. At the surplus bank early consumers get $\overline{d}$ as before. Late consumers get $\frac{2R}{1-\lambda}$ since the bank now holds the assets from both banks. This pay-off is strictly larger than the pay-off for a late consumer in the absence of fire-sales since $v_i - \lambda\overline{d} < R$.

In the second situation there is a liquidity deficit ($v_i < \lambda\overline{d}$) at both banks. Each bank then faces a run and will cease to exist. Since there is then no other bank that can acquire assets and assets need bank input between date 1 and 2, bank assets become worthless.[10] The utility is then $v_i - k$ for early consumers and v_i for late consumers at each bank.

We denote in the following with $\widetilde{y}_i(x)$ the critical return on asset y that ensures that

[10] In Section 4 banks will be allowed to prematurely liquidate their assets in this case.

bank i has enough liquidity to serve its early consumers for a given realization of asset X. This function is implicitly defined by $v_i = \alpha_i \widetilde{y}_i(x) + (1-\alpha_i)x = \lambda\overline{d}$. Solving for $\widetilde{y}_i(x)$ gives

$$\widetilde{y}_i(x) = \frac{\lambda\overline{d}}{\alpha_i} - \frac{1-\alpha_i}{\alpha_i}x. \tag{5}$$

Equivalently, we define with $\widetilde{y}(x)$ the critical return that ensures that there is enough liquidity in the economy to serve early consumers. This function is defined by $v_1 + v_2 = \alpha_1\widetilde{y}(x) + (1-\alpha_1)x + \alpha_2\widetilde{y}(x) + (1-\alpha_2)x = 2\lambda\overline{d}$. Solving for $\widetilde{y}(x)$ gives

$$\widetilde{y}(x) = \frac{2\lambda\overline{d}}{\alpha_1+\alpha_2} - \frac{1-\alpha_1+1-\alpha_2}{\alpha_1+\alpha_2}x. \tag{6}$$

Furthermore, we denote with $\widetilde{x}_i(0)$ $(= \frac{\lambda\overline{d}}{1-\alpha_i})$ and $\widetilde{x}(0)$ $(= \frac{2\lambda\overline{d}}{1-\alpha_1+1-\alpha_2})$ the x at which $\widetilde{y}_i(x) = 0$ and $\widetilde{y}(x) = 0$ (that is, the largest x for which there can be bank-specific and aggregate liquidity shortages).

Using these definitions, we can summarize the expected utility for consumers at bank 1 (analogous for bank 2):

1. There is no aggregate shortage ($y \geq \widetilde{y}(x)$). The total expected utility for a consumer of bank 1 is $\lambda\overline{d} + (1-\lambda)(\frac{R+v_1-\lambda\overline{d}}{1-\lambda}) = R + v_1$.

2. There is an aggregate shortage ($y < \widetilde{y}(x)$). Three cases arise:

   (a) Bank 1 has a surplus ($y \geq \widetilde{y}_1(x)$). Bank 2 then fails and the expected utility for consumers at bank 1 is $\lambda\overline{d} + (1-\lambda)\frac{2R}{1-\lambda} = 2R + \lambda\overline{d}$. Note that this situation can only occur when $x < \lambda\overline{d}$ (since bank 2 is more invested in X).

   (b) Bank 1 has a deficit, but bank 2 has a surplus ($y \geq \widetilde{y}_2(x)$). The expected utility is then $\lambda(v_1 + v_2 - \lambda\overline{d} - k) + (1-\lambda)(v_1 + v_2 - \lambda\overline{d}) = v_1 + v_2 - \lambda\overline{d} - \lambda k$. This situation can only occur when $x > \lambda\overline{d}$.

   (c) Both banks have a deficit ($y < \widetilde{y}_1(x)$ and $y < \widetilde{y}_2(x)$). The expected utility is then $\lambda(v_1 - k) + (1-\lambda)v_1 = v_1 - \lambda k$.

From this we have that the total expected utility of a consumer at bank 1 is[11]

$$EU_1(\alpha_1,\alpha_2) = \int_0^{\widetilde{x}(0)} \int_{\widetilde{y}(x)}^{\infty} (R+v_1) f(y) f(x) dy dx + \int_{\widetilde{x}(0)}^{\infty} \int_0^{\infty} (R+v_1) f(y) f(x) dy dx$$

$$+ \int_0^{\lambda\overline{d}} \int_{\widetilde{y}_1(x)}^{\widetilde{y}(x)} (2R+\lambda\overline{d}) f(y) f(x) dy dx + \int_{\lambda\overline{d}}^{\widetilde{x}_2(0)} \int_{\widetilde{y}_2(x)}^{\widetilde{y}(x)} (v_1+v_2-\lambda\overline{d}-\lambda k) f(y) f(x) dy dx$$

$$+ \int_{\widetilde{x}_2(0)}^{\widetilde{x}(0)} \int_0^{\widetilde{y}(x)} (v_1+v_2-\lambda\overline{d}-\lambda k) f(y) f(x) dy dx$$

$$+ \int_0^{\lambda\overline{d}} \int_0^{\widetilde{y}_1(x)} (v_1-\lambda k) f(y) f(x) dy dx + \int_{\lambda\overline{d}}^{\widetilde{x}(0)} \int_0^{\widetilde{y}_2(x)} (v_1-\lambda k) f(y) f(x) dy dx. \quad (7)$$

This can be rearranged to

$$EU_1(\alpha_1,\alpha_2) = R + \mu + \int_0^{\lambda\overline{d}} \int_{\widetilde{y}_1(x)}^{\widetilde{y}(x)} (R-(v_1-\lambda\overline{d})) f(y) f(x) dy dx$$

$$- \int_{\lambda\overline{d}}^{\widetilde{x}_2(0)} \int_{\widetilde{y}_2(x)}^{\widetilde{y}(x)} (R-(v_2-\lambda\overline{d})+\lambda k) f(y) f(x) dy dx - \int_{\widetilde{x}_2(0)}^{\widetilde{x}(0)} \int_0^{\widetilde{y}(x)} (R-(v_2-\lambda\overline{d})+\lambda k) f(y) f(x) dy dx$$

$$- \int_0^{\lambda\overline{d}} \int_0^{\widetilde{y}_1(x)} (R+\lambda k) f(y) f(x) dy dx - \int_{\lambda\overline{d}}^{\widetilde{x}_2(0)} \int_0^{\widetilde{y}_2(x)} (R+\lambda k) f(y) f(x) dy dx. \quad (8)$$

The first two terms $(R+\mu)$ give the expected return on the bank's portfolio if held until maturity (which is independent of the portfolio choice α_1 since the assets have the same mean). Note that this would be the expected utility of the bank's consumers if there were never aggregate shortages. The other integrals then give the utility gains (or losses) associated with the various cases that arise in an aggregate shortage, relative to a situation without a shortage. The first integral term refers to case 2a and represents the gains from being a surplus bank when there is an aggregate liquidity shortage. These gains arise because assets can be acquired from the other bank at fire-sale prices. The second and the third integral term refer to case 2b and give the loss from having a deficit at a time when the other bank has a surplus that is not sufficient to serve early consumers. These losses arise from the fire-sale nature of the asset sales and the fact that early consumers cannot meet their consumption needs. Note that the loss incurred due to fire-sales depends on the liquidity surplus of the other bank $(v_2-\lambda\overline{d})$ due to cash-in-the-market pricing. The fourth

[11] More detailed derivations are found in an extra appendix at the end of the paper.

and the fifth integral term refer to case 2c and represent the losses when both banks have a shortage. These losses are at least as high as in case 2b since now the other bank can no longer purchase the assets, thus making them worthless.

The different cases are illustrated in Figure 1a. Northeast of $\widetilde{y}(x)$ (bold line) there is no aggregate shortage and consumers at bank 1 simply get the fundamental value of the bank's portfolio (case 1). In area A we have $\widetilde{y}_1(x) \leq y < \widetilde{y}(x)$, hence bank 1 survives while bank 2 fails (case 2a). Bank 1 hence benefits from fire-sales at bank 2. In area B we have $\widetilde{y}_2(x) \leq y < \widetilde{y}(x)$, hence bank 2 survives while bank 1 fails and sells its assets to bank 2 (case 2b). In area C both banks fail (case 2c). The figure suggests that the utility of one bank depends on the portfolio choice of the other bank. For example, in area F (where $\widetilde{y}(x) \leq y < \widetilde{y}_1(x)$) bank 1 only survives because bank 2 has a sufficient surplus. If this were not the case, bank 1 would experience a run in this area.

Definition 1 *An equilibrium in this economy is a pair of portfolio choices (α_1^*, α_2^*), such that α_1^* maximizes (8) taking as given α_2^*, and α_2^* maximizes the equivalent of (8) for bank* 2 *taking as given α_1^*.*

Proposition 1 *Full diversification at each bank is not an equilibrium.*

Proof. *We prove this statement by showing that when both banks are fully diversified ($\alpha_1 = \alpha_2 = \frac{1}{2}$), bank* 1 *can strictly improve the utility of its consumers by deviating from this allocation. To this end, consider the impact of an increase in α_1 on the utility of bank* 1. *From (8) we have*

$$\frac{\partial EU_1}{\partial \alpha_1} = -\int_0^{\lambda\bar{d}} \int_{\widetilde{y}_1(x)}^{\widetilde{y}(x)} (y-x) f(y) f(x) dy dx + \int_0^{\lambda\bar{d}} \frac{\partial \widetilde{y}(x)}{\partial \alpha_1} (R - (v_1(\widetilde{y}(x)) - \lambda\bar{d})) f(\widetilde{y}(x)) f(x) dx$$
$$- \int_{\lambda\bar{d}}^{\widetilde{x}(0)} \frac{\partial \widetilde{y}(x)}{\partial \alpha_1} (R - (v_2(\widetilde{y}(x)) - \lambda\bar{d}) + \lambda k) f(\widetilde{y}(x)) f(x) dx$$
$$- \int_0^{\lambda\bar{d}} \frac{\partial \widetilde{y}_1(x)}{\partial \alpha_1} (2R - (v_1(\widetilde{y}_1(x)) - \lambda\bar{d}) + \lambda k) f(\widetilde{y}_1(x)) f(x) dx. \quad (9)$$

We now evaluate this expression at $\alpha_1 = \alpha_2 = \frac{1}{2}$ (full diversification). We have from (5) and (6) that $\widetilde{y}_1(x) = \widetilde{y}(x)$, hence the first integral term vanishes. We also have that

$v_i(\widetilde{y}_1(x)) = v_i(\widetilde{y}(x)) = \lambda\overline{d}$, *hence we have* $v_i = \lambda\overline{d}$ *in the remaining integrals. Furthermore, we have that* $\frac{\partial \widetilde{y}}{\partial \alpha_1} f(\widetilde{y}(x))f(x) \mid_{x=b} = -\frac{\partial \widetilde{y}}{\partial \alpha_1} f(\widetilde{y}(x))f(x) \mid_{x=2\lambda\overline{d}-b}$ *since at* $\alpha_1 = \alpha_2 = \frac{1}{2}$ *we have* $\frac{\partial \widetilde{y}}{\partial \alpha_1} = 2(x - \lambda\overline{d})$ *and* $\widetilde{y}(x) = 2\lambda\overline{d} - x$*. This allows us to combine the second and the third integral term into a single integral:* $\int_0^{\lambda\overline{d}} \frac{\partial \widetilde{y}(x)}{\partial \alpha_1}(2R+\lambda k)f(\widetilde{y}(x))f(x)dydx$*. The derivative can hence be written as*

$$\frac{\partial EU_1}{\partial \alpha_1} \mid_{\alpha_1=\alpha_2=\frac{1}{2}} = \int_0^{\lambda\overline{d}} (\frac{\partial \widetilde{y}(x)}{\partial \alpha_1} - \frac{\partial \widetilde{y}_1(x)}{\partial \alpha_1})(2R + \lambda k)f(\widetilde{y}(x))f(x)dx. \quad (10)$$

Since $\frac{\partial \widetilde{y}(x)}{\partial \alpha_1} > \frac{\partial \widetilde{y}_1(x)}{\partial \alpha_1}$ *(this can be verified from (5) and (6)) and* $f > 0$ *under the integration bounds, we have that this expression is strictly positive. Hence, moving away from full diversification improves the utility of bank* 1*. It follows that full diversification is not an equilibrium.* ■

The intuition behind this result can be understood by considering Figure 1b. If both banks are fully diversified, the conditions for aggregate and bank-specific shortages coincide (bold-line). An increase in α_1 (that is investing more in Y and less in X) then causes a counterclockwise rotation of $\widetilde{y}(x)$ and $\widetilde{y}_1(x)$, creating the areas A, B, C, D. In area A (where $y > x$) bank 1 previously failed jointly with the other bank but there is now an aggregate surplus. The bank hence saves $R + \lambda k$ whenever a realization in this area occurs. In area D (where $y < x$) there was previously an aggregate surplus. Now there is a deficit and the bank experiences a run while the other bank survives. The bank hence loses $R - (v_2 - \lambda\overline{d}) + \lambda k$ (the difference between the utility in case 1 and 2b developed above). Close to full diversification, these two effects cancel out. The reason is, first, because we then have $v_2 = \lambda d$ and hence the loss in area D is $R + \lambda k$ and thus the same as the gain in area A. Second, close to full diversification also the probabilities associated with both areas are identical (intuitively this is because full diversification in the economy minimizes portfolio variance; hence a small deviation does not have a first-order effect on the variance and hence on the likelihood of shortages).

It remains to consider areas B and C. In area B the bank previously did not experience any gains or losses. Now it has a liquidity deficit, but this is not costly since there is an aggregate surplus in these situations and hence assets can be sold at $p = R$. Finally, in area C the bank previously failed but now survives. When it failed its loss was $R+\lambda k$ since the other bank failed at the same time. Now it survives and can even acquire assets from the other bank; the gain from this close to full diversification is $R - (v_1(\widetilde{y}_1(x)) - \lambda d) = R$. The total gain in this area is hence $2R + \lambda k$. Thus, whenever area C is associated with a positive probability mass (which is guaranteed because of $f > 0$), the deviation from full diversification is worthwhile for the bank.[12]

Summarizing, the intuition why a deviation from diversification is beneficial for the bank is as follows. When the bank moves away from full diversification, there will be return realizations for which the bank now faces a deficit, but also realizations where it no longer has a deficit. The new deficit cases are only partly costly because the other bank then has a surplus and there may hence be an aggregate surplus. By contrast, in all situations where the bank now survives it previously incurred large costs since it failed jointly with the other bank. In addition, it now may also be able to purchase assets from the other (failing) bank at discounted prices. Note that the benefits from a reduction in diversification are thus due to the bank becoming more different from the other bank.

We next show that a lack of diversification at banks is also (socially) efficient. In fact, it turns out that the efficient outcome in this economy is for both banks to be completely undiversified.

Proposition 2 *The efficient allocation requires both banks to fully specialize in different assets ($\alpha_1 = 1$ and $\alpha_2 = 0$).*

Proof. *We show that starting from an arbitrary symmetric allocation (symmetric in the sense that $\alpha_1 = 1 - \alpha_2$) lowering diversification at either bank increases welfare. Without loss of generality we focus on a reduction in diversification at bank* 1 *(the result for bank* 2

[12] The deviation benefits for a bank further increase when there are a large number of banks. The reason is that when moving away from diversification, bank 1 will then no longer affect the aggregate threshold $\widetilde{y}(x)$. Hence, it will no longer fail for realization in area D. Moreover, the bank will then in addition also benefit from fire-sales in area A because there is then an aggregate shortage in this area.

follows then from the symmetry of the problem). Recall that we have taken the convention that $\alpha_1 \geq \alpha_2$*, thus we have* $\alpha_1 \geq \frac{1}{2}$ *in any symmetric allocation and hence a reduction in diversification at bank* 1 *is achieved by increasing* α_1*. Welfare in the economy is given by the sum of the utilities of the consumers at both banks,* $EU_1 + EU_2$*. We hence have to show that* $\frac{\partial(EU_1+EU_2)}{\partial\alpha_1} > 0$*. From summing equation (8) and its equivalent for bank* 2 *we get*

$$\begin{aligned} EU_1 + EU_2 = 2(R+\mu) - \int_0^{\lambda\bar{d}} \int_{\widetilde{y}_1(x)}^{\widetilde{y}(x)} \lambda k f(y) f(x) dy dx \\ - \int_{\lambda\bar{d}}^{\widetilde{x}_2(0)} \int_{\widetilde{y}_2(x)}^{\widetilde{y}(x)} \lambda k f(y) f(x) dy dx - \int_{\widetilde{x}_2(0)}^{\widetilde{x}(0)} \int_0^{\widetilde{y}(x)} \lambda k f(y) f(x) dy dx \\ -2 \int_0^{\lambda\bar{d}} \int_0^{\widetilde{y}_1(x)} (R+\lambda k) f(y) f(x) dy dx - 2 \int_{\lambda\bar{d}}^{\widetilde{x}_2(0)} \int_0^{\widetilde{y}_2(x)} (R+\lambda k) f(y) f(x) dy dx. \end{aligned} \tag{11}$$

Differentiating with respect to α_1 *gives*

$$\frac{\partial(EU_1+EU_2)}{\partial\alpha_1} = -\int_0^{\widetilde{x}(0)} \frac{\partial\widetilde{y}(x)}{\partial\alpha_1} \lambda k f(\widetilde{y}(x)) f(x) dx - \int_0^{\lambda\bar{d}} \frac{\partial\widetilde{y}_1(x)}{\partial\alpha_1} (2R+\lambda k) f(\widetilde{y}_1(x)) f(x) dx. \tag{12}$$

Evaluating at $\alpha_1 = 1 - \alpha_2$ *gives*

$$\frac{\partial(EU_1+EU_2)}{\partial\alpha_1} |_{\alpha_1=1-\alpha_2} = -\int_0^{\lambda\bar{d}} \frac{\partial\widetilde{y}_1(x)}{\partial\alpha_1} (2R+\lambda k) f(\widetilde{y}_1(x)) f(x) dx, \tag{13}$$

which is larger than zero since $\frac{\partial\widetilde{y}_1(x)}{\partial\alpha_1} < 0$*. Thus, reducing diversification starting from any symmetric situation is always socially desirable. It follows that no diversification is the efficient outcome.* ■

The reason for this result is the following. The effect of fire-sales cancels out on the level of the economy since a loss to one bank is a gain to the other. Welfare is thus solely determined by the likelihood of single and joint bank failures. Single bank failures (areas A and B in Figure 1a) induce a cost of λk. Joint failures (area C) induce costs of $2R+2\lambda k$ in total (note that this is more than twice as large as the costs of a single failure). Suppose that both banks equally reduce their diversification starting from a symmetric allocation. This will not affect the condition for an aggregate shortage ($\widetilde{y}(x)$ is unchanged). However, it will reduce the area of joint failures (area C) by increasing the areas of single failures (areas A

and B) (there will be a counterclockwise rotation of $\widetilde{y}_1(x)$ and a clockwise rotation of $\widetilde{y}_2(x)$ in Figure 1a). Since joint failures are more costly than individual failures, this improves welfare.[13] Lowering the amount of diversification at banks is thus always beneficial.

Remark 1 *Note that there are two independent reasons for the inefficiency of diversification. First, less diversification is beneficial since joint failures incur more than twice the costs of individual failures (the term* $2R$ *in the integral in equation (13)). Second, less diversification also reduces the expected number of bank failures since in a joint failure twice as many banks fail (this is represented by the term* λk *in the integral in 13). Thus, to obtain the inefficiency of full diversification higher costs of joint failures are not necessary. Note also that at the efficient solution we have* $\alpha_1 + \alpha_2 = 1$*. Hence, the combined portfolio of the two banks is* $x + y$*, which is a diversified portfolio. Therefore, the optimal outcome requires diversification at the level of the economy and for banks to completely specialize in different assets.*

Remark 2 *To simplify the derivations, we have presumed that consumers can only invest in one bank. It turns out that it is not optimal for consumers to mix between two banks. To see this, suppose that in a symmetric allocation* ($\alpha_1 = 1 - \alpha_2$) *a consumer who previously fully invested in bank* 1 *now invests a share of his funds at bank* 2*. In the absence of any date* 1 *consumption requirements (and the potentially resulting utility loss) this will not affect his utility since due to symmetry the expected payouts at each bank are the same. It will, however, change the probability that he is not able to satisfy his consumption needs when he is an early consumer. Previously this probability was given by the probability of bank* 1 *failing. Now it is given by the likelihood that any of the two banks fails. This is because already in the case of one bank failing he will not be able to withdraw* $\overline{\mathrm{d}}$ *in total. Thus, he will never benefit from the switch. Unless there is full diversification, he will even*

[13] Joint failures may also induce costs outside the banking system because they cause a systemic crisis (for example, there may be network externalities on producers, see Kahn and Santos (2008)). When this is the case, the desirability of diversification may be further reduced (Shaffer (1994) and Wagner (2006)).

be strictly worse off because banks then have different portfolios and hence the probability of any of the two banks failing is strictly larger than the probability of bank 1 *failing.*

Remark 3 *It is easy to see that in an economy without an interbank market full diversification is efficient. In such an economy a bank* i *will fail if* $y < \widetilde{y}_i(x)$ *and incur costs of* $R + \lambda k$*. Both the likelihood of failures and the cost of failures are then independent of the portfolio of the other bank. Since diversification will reduce the likelihood of realizations for which* $y < \widetilde{y}_i(x)$*, full diversification is then efficient.*

Remark 4 *The efficiency of full specialization may no longer obtain when consumers are risk-averse beyond the consumption need* $\overline{d}$*. This is because in the absence of bank runs, higher diversification then directly benefits consumers by reducing payoff variance. However, it is not clear that such risk-aversion generally increases the desirability of diversification. The reason is that more diversification also increases the likelihood of low consumption states (arising when there are joint liquidations), which become more costly when consumers are risk-averse. Furthermore, risk-aversion can never restore the efficiency of full diversification, as will be shown in Section 4.1.*

Remark 5 *It is easy to see that the efficiency of full specialization continues to hold when there is a large (but even) number of banks. By contrast, Wagner (2009) analyzes a setting with many investors that all face exogenous liquidation constraints for their portfolios and where the costs of liquidation are assumed to be (exogenously) increasing in the number of investors liquidating. In this setting investors optimally avoid complete specialization (that is, half of the investors investing in one asset, and the other half in the other asset) since it would imply that they have to liquidate jointly with all investors specialized in their asset if the asset performs poorly. This effect does not arise here due to the endogeneity of liquidation (that is, bank runs): specialized banks may still survive a poor performance of their asset by obtaining liquidity from the banks invested in the other asset.*

We have shown that it is efficient for banks not to diversify at all. The question we address next is whether this outcome also forms an equilibrium. We first note that banks indeed have an incentive to specialize in different assets, as required for efficiency. To see this, suppose that bank 2 plays the same portfolio allocation as bank 1: $\alpha_2 = \alpha_1$. The conditions for shortages then all coincide and can be jointly represented by $\widetilde{y}_1(x)$ in Figure 1a. Suppose now that bank 2 deviates by playing $\alpha_2 = 1 - \alpha_1$ instead. This causes a rotation of the threshold functions for bank 2 and the economy to $\widetilde{y}_2(x)$ and $\widetilde{y}(x)$ in the figure. This changes the utility of bank 2 in areas A and B (in areas E and F there are no changes since we then have an aggregate surplus). In area A the bank now fails alone, while previously there was no aggregate shortage in the economy and hence it survived. The bank thus loses $R - (v_1 - \lambda d) + \lambda k$ when there are return realizations that fall in this area. In area B the bank previously failed jointly with the other bank, but now survives and can purchase assets at fire-sale prices. The gain is $2R - (v_2 - \lambda d) + \lambda k$, which is larger than the loss for a corresponding realization in area A. Since the areas A and B are identical due to symmetry, it follows that the bank profits from this deviation. The intuition behind why the deviation is desirable is straightforward: by moving to $1 - \alpha_1$ the bank becomes more different from the other bank (while retaining its degree of diversification), which is beneficial for the bank for the various reasons outlined earlier.

In order to understand whether the equilibrium can be efficient, we consider next whether there are any externalities among banks. To this end we analyze the impact of an increase in α_2 at bank 2 on the utility of the consumers at bank 1 (starting from a situation where $\alpha_1 > \alpha_2$). From equation (8) we have

$$\begin{aligned}\frac{\partial EU_1}{\partial \alpha_2} &= \int_0^{\lambda \overline{d}} \frac{\partial \widetilde{y}(x)}{\partial \alpha_2}(R - (v_1(\widetilde{y}(x)) - \lambda \overline{d}))f(\widetilde{y}(x))f(x)dx \\ &\quad - \int_{\lambda \overline{d}}^{\widetilde{x}(0)} \frac{\partial \widetilde{y}(x)}{\partial \alpha_2}(R - (v_2(\widetilde{y}(x)) - \lambda \overline{d}) + \lambda k)f(\widetilde{y}(x))f(x)dx. \qquad (14)\end{aligned}$$

Since $\frac{\partial \widetilde{y}(x)}{\partial \alpha_2} < 0$ for $x < \lambda \overline{d}$ and $\frac{\partial \widetilde{y}(x)}{\partial \alpha_2} > 0$ for $x > \lambda \overline{d}$, we have that $\frac{\partial EU_1}{\partial \alpha_2} < 0$. Thus, when bank 2 adjusts its portfolio to make it more similar to the one of bank 1, the utility of bank 1's consumers is reduced.

The reason for this externality can be appreciated from Figure 1b. If bank 2 invests less in asset X this will both cause a counterclockwise rotation of $\widetilde{y}(x)$ and $\widetilde{y}_2(x)$. The rotation of the latter increases the area where bank 1 fails jointly with bank 2. However, it can be shown that this does not reduce utility at bank 1 since previously bank 2 only had an infinitesimal surplus of liquidity and hence paid virtually zero for bank 1's assets. The change in $\widetilde{y}(x)$, however, reduces area C and increases area D. This reduces the utility of bank 1 since in area C it gains from fire sales and in area D it fails.

There is hence a negative externality if a bank (starting from a situation where banks are specialized in different assets) moves closer to the fully diversified portfolio (analogous to portfolio theory, we call this portfolio the market portfolio).[14] In equilibrium there may thus be a tendency for banks to invest too much in the market portfolio. The next proposition shows that there are indeed cases where this occurs (note that this is not trivial since the efficient outcome is a corner solution and hence it may coincide with the equilibrium outcome even in the presence of externalities).

Proposition 3 *The equilibrium may be inefficient. When this is the case banks invest a larger share in the market portfolio than is efficient.*

Proof. *We show that there are parameter constellations for which banks have an incentive to deviate from the efficient no-diversification outcome ($\alpha_1 = 1$ and $\alpha_2 = 0$). Without loss of generality we focus on bank 1. We obtain for the derivative $\frac{\partial EU_1}{\partial \alpha_1}$ at $\alpha_1 = 1$ and $\alpha_2 = 0$:*

$$\frac{\partial EU_1}{\partial \alpha_1} = \int_0^{\lambda\overline{d}} \int_{\lambda\overline{d}}^{2\lambda\overline{d}-x} (x-y) f(y) f(x) dy dx - 2\int_0^{\lambda\overline{d}} \frac{\partial \widetilde{y}(x)}{\partial \alpha_1} (\lambda\overline{d} - x) f(\widetilde{y}(x)) f(x) dx$$
$$+ \int_0^{\lambda\overline{d}} \left(\frac{\partial \widetilde{y}(x)}{\partial \alpha_1} f(\widetilde{y}(x)) - \frac{\partial \widetilde{y}_1(x)}{\partial \alpha_1} f(\widetilde{y}_1(x))\right)(2R + \lambda k) f(x) dx. \tag{15}$$

The first integral is negative since $x < y$ within the integration bounds. The second integral is positive since $\frac{\partial \widetilde{y}(x)}{\partial \alpha_1} < 0$ for $x < \lambda\overline{d}$. The third integral may also be positive since

[14] Note that the notion of a market portfolio is a different one from portfolio theory. Portfolio theory considers assets in fixed supply and the market portfolio refers to the set of assets that have already been invested in. Here assets are in infinite supply and the market portfolio refers to the combination of assets that minimizes variance among the potentially available assets (in our setup there is no trade-off between risk and return on the asset level since all assets have the same expected fundamental return).

$\frac{\partial \widetilde{y}(x)}{\partial \alpha_1} > \frac{\partial \widetilde{y}_1(x)}{\partial \alpha_1}$. The second integral can be made arbitrarily small by making $f(\widetilde{y}(x))$ small and the third integral can be made arbitrarily small by making $f(\widetilde{y}_1(x))$ or R and k small. Hence, there are parameter values for which $\frac{\partial EU_1}{\partial \alpha_1} < 0$. For these parameter values bank 1 profits from increasing its diversification (reducing α_1). No diversification is then not an equilibrium and banks will in equilibrium be more diversified than socially optimal. ■

It follows that if the social planer can influence portfolio allocations at banks, he may in certain situations improve welfare in the economy by reducing banks' investment in the market portfolio.

3 Liquidation By Consumers

In the model presented in the previous section, banks invested on behalf of consumers and offered them deposit contracts. As a result, liquidations in the economy were caused by bank runs. In this section we show that this is not a crucial requirement for the analysis: the non-diversification result also holds if consumers invest themselves, rather than delegating to banks.

We modify the economy as follows. There are now two consumers in the economy (denoted 1 and 2) and no banks. The consumers have the same preferences as in the previous section, but now have a consumption need at date 1 with certainty ($\lambda = 1$). This rules out a role for intermediaries in providing liquidity insurance. Furthermore, consumers can now themselves invest in the two assets at date 0. We again denote with α_i ($i = 1, 2$) the fraction of their funds invested in asset Y (effectively, the consumers replace the two banks in the investment process).

The following situations can arise at date 1. If a consumer has sufficient liquid resources ($v_i \geq \overline{d}$), there is no need to liquidate assets. He can then meet his consumption need and store any remaining goods for consumption at date 2. If he faces a deficit, he can sell assets to the other consumer. If he succeeds in raising sufficient liquidity in this way, he can meet the consumption need and will consume the returns of any remaining assets at date 2. If he cannot raise sufficient liquidity, he incurs costs k and consumes only at date 1.

The outcomes associated with the different cases at date 1 are like those in the previous section. If $\widetilde{v} \geq \overline{d}$, there are sufficient resources in the economy. Even if a consumer faces a deficit, he can then generate sufficient liquidity by selling assets to the other consumer at a price of $p = R$. The utility for a consumer is then $R + v_i$, regardless of whether he faces a deficit or not. If $\widetilde{v} < \overline{d}$ we again have to distinguish between two cases: either one of the consumers has a surplus, or none of the consumers has a surplus. In the first case, since there is an aggregate shortage, the deficit consumer will not be able to raise sufficient liquidity through asset sales. He hence will not be able to satisfy his consumption needs. He then incurs costs k and has to liquidate his assets. The equilibrium asset price will be $p = v_j - \overline{d}$ (where j denotes the surplus consumer) as in the previous section for $\lambda = 1$. The resulting utilities are $v_i + p - k = v_i + v_j - \overline{d} - k$ for the deficit consumer, and $2R + \overline{d}$ for the surplus consumer. In the second case neither consumer has sufficient liquidity. Hence they both incur costs k and can only consume at the intermediate date. Since the assets cannot be prematurely liquidated, they become worthless for the consumers. Each consumer then obtains a utility of $v_i - k$ $(i = 1, 2)$.

Using the threshold functions $\widetilde{y}_i(x)$ and $\widetilde{y}(x)$ defined in the previous section (equations (5) and (6), we can write the expected return of consumer 1:

$$\begin{aligned} EU_1 = & \int_0^{\widetilde{x}(0)} \int_{\widetilde{y}(x)}^{\infty} (R + v_1) f(y) f(x) dy dx + \int_{\widetilde{x}(0)}^{\infty} \int_0^{\infty} (R + v_1) f(y) f(x) dy dx \\ & + \int_0^{\overline{d}} \int_{\widetilde{y}_1(x)}^{\widetilde{y}(x)} (2R + \overline{d}) f(y) f(x) dy dx + \int_{\lambda \overline{d}}^{\widetilde{x}_2(0)} \int_{\widetilde{y}_2(x)}^{\widetilde{y}(x)} (v_1 + v_2 - \overline{d} - k) f(y) f(x) dy dx \\ & + \int_{\widetilde{x}_2(0)}^{\widetilde{x}(0)} \int_0^{\widetilde{y}(x)} (v_1 + v_2 - \overline{d} - k) f(y) f(x) dy dx + \int_0^{\overline{d}} \int_0^{\widetilde{y}_1(x)} (v_1 - k) f(y) f(x) dy dx \\ & + \int_{\overline{d}}^{\widetilde{x}(0)} \int_0^{\widetilde{y}_2(x)} (v_1 - k) f(y) f(x) dy dx. \end{aligned} \tag{16}$$

This expression is identical to the one for bank 1 (equation 7) for the special case of $\lambda = 1$. It follows that the economy faces the same optimization problems as in the previous section.

Therefore

Proposition 4 *If consumers invest themselves instead of banks, full diversification is still neither an equilibrium nor optimal.*

Remark 6 *As already emphasized, the non-diversification result is driven by a concavity of the shortfall costs (in the model arising because a marginal shortfall induces a fixed cost). An alternative way to produce this concavity (which does not build on asset sales among investors or banks) is the following. Suppose that at date* 1 *the investor (who could be a firm manager) can increase the final date output of his portfolio by exerting effort* $e \geq 0$*. Assume that effort* e *raises final output one-for-one (output is hence* $R+e$*) and induces private costs* $c(e)$ *that are convex in effort. Suppose that the investor can partially discontinue his assets at date* 1 *in order to finance any shortfalls (*$\overline{d} - v_i > 0$*). The value of a unit of the portfolio if discontinued is* R *since investor effort on this part of the portfolio will be zero. In order to finance a shortfall of* $\overline{d} - v_i$*, the investor then has to discontinue a share* $q \in [0,1]$ *of his portfolio. This share is determined by* $qR = \overline{d} - v_i$*, hence* $q = (\overline{d} - v_i)/R$*. The investor's utility is* $U_i(v_i) = \overline{d} + (1-q)(R+e^*) - c(e^*)$*, where* e^* *is determined by the first-order-condition* $(1-q) - c'(e^*) = 0$*. Using the envelope theorem we have* $U_i'(v_i) = 1 + e^*/R$ *and* $U_i''(v_i) = e^{*\prime}(v_i)/R > 0$*. Utility is hence convex in* v_i*, hence costs are concave in the shortfall* $\overline{d} - v_i$*. An increased heterogeneity of shortfalls thus increases utility, hence there is again a cost to diversification.*

4 Liquidation Due to Insolvency and Premature Liquidation

In the model of Section 2, banks failed because they could not pay out early depositors at the intermediate date. By assuming that asset returns at the final date are sufficiently high (equation 2), situations of insolvency were ruled out. This had the consequence that whether runs take place is exclusively driven by the aggregate resources in the economy. A shortfall at an individual bank could always be overcome using the interbank market. In this section we consider alternatively an economy in which banks fail due to insolvency. A bank's individual allocation then matters since interbank markets cannot resolve insolvency problems. This introduces a new rationale for diversification at the bank level because, by lowering the variance of a bank's returns, diversification reduces the likelihood of insolvency at a bank. However, we will see that this alteration does not reverse the main results regarding the desirability of diversification.

We first dispense with condition (2), which guaranteed the solvency of banks. Second, we assume that $f(.)$ only has support on $(\lambda\overline{d}, \infty)$ (while previously the support was on $[0, \infty)$). The date 1 return on a bank's portfolio hence never falls below $\lambda\overline{d}$. This ensures that a bank that offers a deposit contract $\overline{d}$ can always pay out early consumers, ruling out situations of illiquidity. As in Section 2, banks will find it optimal to offer such contracts since if they offer larger withdrawals at the intermediate date they will face more runs at no benefit, while if they offer lower withdrawals they provide no liquidity insurance.[15]

Another element of the model in Section 2 is that assets become worthless at date 1 without bank input. This had the consequence that joint bank failures are very costly as there is then no bank that can continue the assets. In this section we relax this assumption and allow banks to liquidate assets prematurely (that is, physically, as in Diamond and Dybvig, (1983)) at date 1. We assume that there is a proportional loss γ $(0 < \gamma < 1)$ associated with this, hence liquidation of a unit of an asset only fetches $(1 - \gamma)R$.

Runs at the intermediate date will now occur at a bank if the resources available to a late consumer at date 2 (after early consumers have withdrawn at date 1) fall short of the amount late consumers can withdraw at date 1. Given that withdrawals by early consumers

[15] Since we now consider situations of insolvency (and not illiquidity at date 1), it no longer matters whether uncertain part of asset returns realize at the intermediate or at the final date. For ease of comparison with the previous analysis, however, we continue to focus on uncertainty at the intermediate date.

sum to $\lambda\overline{d}$ and that there are $1-\lambda$ late consumers this condition writes $\frac{R+v_i-\lambda\overline{d}}{1-\lambda} < \overline{d}$. Rearranging gives us the condition for insolvency

$$R + v_i < \overline{d}. \tag{17}$$

Note that in contrast to Section 2, whether a run occurs is independent of the situation at the other bank. Diversification at a bank will thus have a benefit regardless of the level of aggregate diversification: by lowering the variance of v_i, diversification will reduce the likelihood that a bank's assets $(R+v_i)$ fall below its obligations $(\overline{d})$.

Again, three different situations can arise at the intermediate date. First, both banks are solvent. Second, both banks are insolvent. And third, one bank is solvent while the other is not. In the first situation both banks pay out $\overline{d}$ to early consumers at date 1 and the interbank market is not operative. Late consumers obtain the residual value $\frac{R+v_i-\lambda d}{1-\lambda}$ at date 2. In the second situation both banks face runs and have to liquidate their assets prematurely at a cost of γ. A bank thus has in total an amount $(1-\gamma)R+v_i$ of goods available to all its depositors. Since this amount is smaller than $\overline{d}$ by condition (17), early consumers cannot fulfill their consumption requirements and hence suffer the costs k.

The third situation is more complicated since the bank that faces the run now has two ways to generate liquidity: through asset sales or through premature liquidation. We have to distinguish between three cases. First, the liquidity that the solvent bank has left after paying its early consumers $(v_i-\lambda\overline{d})$ is less than the value of the assets of the insolvent bank if prematurely liquidated $((1-\gamma)R)$. Second, the surplus liquidity of the solvent bank is equal or higher to the liquidation value of the assets, but less than their continuation value (R). And third, the surplus liquidity is equal or higher than the continuation value.

In the first case $(v_i-\lambda\overline{d} < (1-\gamma)R)$ the price of an asset will be $p=(1-\gamma)R$ and hence equal to its value in a premature liquidation. To see this, note that we must have $p \geq (1-\gamma)R$ since otherwise the insolvent bank would be better off liquidating the asset. Suppose that the price were strictly larger than the liquidation value: $p > (1-\gamma)R$. Since $v_i-\lambda\overline{d} < (1-\gamma)R$, the solvent bank cannot purchase all assets at this price. Some assets hence have to be physically liquidated at the insolvent bank. However, this implies that p cannot form an equilibrium since the returns from doing so are lower than the returns from selling to the solvent bank (at given prices the insolvent bank would strictly prefer to do more asset sales). Thus we must have that $p = (1-\gamma)R$. At this price the solvent bank uses its entire surplus liquidity to purchase as much assets as possible of the insolvent bank, while the remaining assets are prematurely liquidated. Since both methods of liquidation incur a loss of γ, consumers at the insolvent bank obtain $(1-\gamma)R + v_j$. Since this is less than $\overline{d}$, early consumers additionally incur the costs k. Early consumers at the solvent bank get $\overline{d}$. Since the bank can make a return $\frac{R}{p} = \frac{1}{1-\gamma} > 1$ on its surplus liquidity, late consumers get $(R + \frac{v_i - \lambda\overline{d}}{1-\gamma})/(1-\lambda)$. This is more than in the case where the other bank is solvent.

In the second case ($(1-\gamma)R \leq v_i - \lambda\overline{d} < R$) the solvent bank has sufficient liquidity to purchase the assets of the insolvent bank at their liquidation value (but not at their continuation value). The equilibrium is then that all assets are sold to the solvent bank at a price of $p = v_i - \lambda\overline{d} < R$ (hence there is cash-in-the-market pricing as in Section 2). This is, first, because only at this price the demand for the asset by the solvent bank is one and, second, because of $p = v_i - \lambda\overline{d} \geq (1-\gamma)R$ physical liquidation is not preferred by the insolvent bank . Consumers at the insolvent bank then get $v_j + v_i - \lambda\overline{d}$; the early consumers additionally suffer costs k. Early consumers at the solvent bank get $\overline{d}$ and late consumers obtain $\frac{2R}{1-\lambda}$, which again is strictly larger than with a solvent second bank.

In the third case the solvent bank has sufficient liquidity to purchase all assets at their continuation value ($v_i - \lambda\overline{d} \geq R$). The equilibrium outcome is then that all assets are sold to the solvent bank at a price of $p = R$. Consumers at the insolvent bank obtain $R + v_j$, early consumers at this bank additionally suffer costs k. Early consumers at the solvent bank get $\overline{d}$, while late consumers get $\frac{R+v_i-\lambda\overline{d}}{1-\lambda}$ as in the case without an insolvent bank.

We define again a critical return function $\widetilde{y}_i(x)$ ($i = 1, 2$), which now gives the return on asset Y that avoids insolvency at bank i for a given realization on asset X. From equation

(17) and recalling that $v_i = \alpha_i y + (1-\alpha_i)x$ we obtain

$$\widetilde{y}_i(x) = \frac{\overline{d} - R}{\alpha_i} - \frac{1-\alpha_i}{\alpha_i}x. \tag{18}$$

From (18) we have that the corresponding $\widetilde{x}_i(0)$ is equal to $\frac{\overline{d}-R}{1-\alpha_i}$. Note that since a bank's failure is now independent of the situation at the other bank, there is no equivalent to the aggregate critical return function $\widetilde{y}(x)$ in Section 2.

We next derive the expected utility for consumers at bank 1. Doing this for an arbitrary degree of diversification is complicated by the fact that there are three different cases which can arise when one bank fails while the other not. We therefore focus in this section on deriving properties of full diversification only. For this it suffices to consider small deviations from full diversification. This simplifies the analysis since it can then be shown that for appropriate parameter constellations the second and the third insolvency case does not occur. The reason is that if banks are sufficiently close to full diversification, their portfolio values become arbitrarily close. It then cannot be the case that one bank is insolvent while the other has a liquidity surplus large enough to purchase all assets at their liquidation value.

Lemma 1 *Suppose that $|\alpha_i - \alpha_j| < \varepsilon$ and that*

$$R > \frac{1-\lambda}{2-\gamma}\overline{d}. \tag{19}$$

Then for sufficiently small ε we have that whenever bank i is solvent but bank j is not ($R + v_i \geq \overline{d}$ and $R + v_j < \overline{d}$) bank i cannot purchase all assets from bank j at their liquidation value ($v_i - \lambda\overline{d} < (1-\gamma)R$).

Proof. *Without loss of generality assume $i = 1$ and $j = 2$. We then have $R + v_1 \geq \overline{d}$ and $R + v_2 < \overline{d}$. From this follows that $v_1 > v_2$ and hence $y > x$. It follows then that $v_1 = v_2 + (\alpha_1 - \alpha_2)(y-x) < v_2 + \varepsilon(y-x)$. We then have for the surplus liquidity of bank 1*

$$v_1 - \lambda\overline{d} < v_2 + \varepsilon(y-x) - \lambda\overline{d} < (1-\lambda)\overline{d} - R + \varepsilon(y-x) < (1-\gamma)R + \varepsilon(y-x), \tag{20}$$

where for the second inequality we have used $R + v_2 < \overline{d}$ (insolvency of bank 2). The third inequality follows from (19). We hence have for sufficiently small ε that $v_1 - \lambda\overline{d} < (1-\gamma)R$.

■

We assume that in what follows that condition (19) is fulfilled. We can then summarize the expected utility for consumers at bank 1 (analogous for bank 2) as follows:

1. Both banks are solvent ($y \geq \widetilde{y}_1(x)$ and $y \geq \widetilde{y}_2(x)$). The total expected utility for a consumer of bank 1 is then $\lambda\overline{d} + (1-\lambda)(\frac{R+v_1-\lambda\overline{d}}{1-\lambda}) = R + v_1$.

2. Bank 1 is solvent, but bank 2 is not ($\widetilde{y}_1(x) \leq y < \widetilde{y}_2(x)$). The expected utility is then $\lambda\overline{d} + (1-\lambda)(R + \frac{v_1-\lambda\overline{d}}{1-\gamma})/(1-\lambda) = R + v_1 + \frac{\gamma}{1-\gamma}(v_1 - \lambda\overline{d})$. This situation can occur when $x < R - \overline{d}$.

3. Bank 1 is insolvent ($y < \widetilde{y}_1(x)$). The expected utility is then $(1-\gamma)R + v_1 - \lambda k$. Note that it does not matter whether bank 2 is solvent (in this case the asset price would be $p = (1-\gamma)R$ and hence identical to the proceeds from premature liquidation).

We can next write the total expected utility of a consumer of bank 1 (presuming that banks have chosen sufficiently similar α's) as follows

$$EU_1 = \int_0^{\overline{d}-R}\int_{\widetilde{y}_2(x)}^{\infty}(R+v_1)f(y)f(x)dydx + \int_{\overline{d}-R}^{\widetilde{x}_1(0)}\int_{\widetilde{y}_1(x)}^{\infty}(R+v_1)f(y)f(x)dydx$$
$$+\int_{\widetilde{x}_1(0)}^{\infty}\int_0^{\infty}(R+v_1)f(y)f(x)dydx + \int_0^{\overline{d}-R}\int_{\widetilde{y}_1(x)}^{\widetilde{y}_2(x)}(R+v_1+\frac{\gamma}{1-\gamma}(v_1-\lambda\overline{d}))f(y)f(x)dydx$$
$$+\int_0^{\widetilde{x}_1(0)}\int_0^{\widetilde{y}_1(x)}((1-\gamma)R+v_1-\lambda k)f(y)f(x)dydx. \quad (21)$$

This can be rearranged to

$$EU_1 = R+\mu+\int_0^{\overline{d}-R}\int_{\widetilde{y}_1(x)}^{\widetilde{y}_2(x)}\frac{\gamma}{1-\gamma}(v_1-\lambda\overline{d})f(y)f(x)dydx-\int_0^{\widetilde{x}_1(0)}\int_0^{\widetilde{y}_1(x)}(\gamma R+\lambda k)f(y)f(x)dydx. \quad (22)$$

The first integral is the gain bank 1 obtains in situation 2 above (expressed relative to situation 1 where no insolvency occurs). It arises because the bank can purchase assets from the other bank at discounted prices. This integral corresponds to area A in Figure 2a, where $\widetilde{y}_1(x) \leq y < \widetilde{y}_2(x)$. The second integral gives the losses associated with situation 3. In this situation the bank experiences a run and suffers the liquidation loss γR and early consumers incur costs k. This integral corresponds to areas B and C in the figure where $y < \widetilde{y}_1(x)$.

Proposition 5 *Full diversification at each bank is also not an equilibrium in the economy with insolvency.*

Proof. *We show that bank* 1 *has an incentive to deviate from a full diversification allocation. For this consider the derivative of the expected utility of consumers at bank* 1 *(equation (22)) with respect to* α_1*:*

$$\frac{\partial EU_1}{\partial \alpha_1} = \int_0^{\overline{d}-R} \int_{\widetilde{y}_1(x)}^{\widetilde{y}_2(x)} \frac{\gamma}{1-\gamma}(y-x)f(y)f(x)dydx$$

$$-\int_0^{\overline{d}-R} \frac{\partial \widetilde{y}_1(x)}{\partial \alpha_1} \frac{\gamma}{1-\gamma}(v_1(\widetilde{y}_1(x)) - \lambda\overline{d})f(\widetilde{y}_1(x))f(x)dx - \int_0^{\widetilde{x}_1(0)} \frac{\partial \widetilde{y}_1(x)}{\partial \alpha_1}(\gamma R + \lambda k)f(\widetilde{y}_1(x))f(x)dx. \quad (23)$$

Evaluating at full diversification gives

$$\frac{\partial EU_1}{\partial \alpha_1}\Big|_{\alpha_1=\alpha_2=\frac{1}{2}} = -\int_0^{\overline{d}-R} \frac{\partial \widetilde{y}_1(x)}{\partial \alpha_1} \frac{\gamma}{1-\gamma}(v_1(\widetilde{y}_1(x)) - \lambda\overline{d})f(\widetilde{y}_1(x))f(x)dx. \quad (24)$$

This expression is larger than zero since $\frac{\partial \widetilde{y}_1(x)}{\partial \alpha_1} < 0$ *for* $x < \overline{d} - R$ *and* $v_1 - \lambda\overline{d} > 0$ *(since* x,$y > \lambda\overline{d}$ *and hence* $v_1 > \lambda\overline{d}$*). Deviating from the full diversification allocation thus improves the utility of bank* 1*. It follows that full diversification cannot be an equilibrium.*

■

The intuition behind this result can be appreciated from Figure 2b. The solvency thresholds under full diversification are depicted by the bold line. If bank 1 now invests more in Y this results in a rotation of $\widetilde{y}_1(x)$, creating areas A and B. In area A the bank previously failed with the other bank, while it now survives. The gains from this are $R + \lambda k + \frac{\gamma}{1-\gamma}(v_1(\widetilde{y}_1(x)) - \lambda\overline{d})$. The last term arises because surplus liquidity can be used to purchase assets from the other bank at discounted values (note that we have $v_1(\widetilde{y}_1(x)) - \lambda\overline{d} > 0$ because of $x, y > \lambda\overline{d}$). In area B the bank now fails alone, while previously it survived with the other bank. The losses from this are $R + \lambda k$ and hence lower than the gains for a realization in area A. Since the probabilities associated with both areas are identical close to full diversification, the bank's consumers are better off when the bank deviates.

Proposition 6 *Full diversification at each bank is also inefficient in the economy with insolvency.*

Proof. *We show that reducing the amount of diversification at bank* 1 *improves welfare at the full diversification allocation. For this we first consider the impact of a reduction in diversification at bank* 2 *(a reduction in* α_2*) on bank* 1*. From (22) we have*

$$\frac{-\partial EU_1}{\partial \alpha_2} = -\int_0^{\overline{d}-R} \frac{\partial \widetilde{y}_2(x)}{\partial \alpha_2} \frac{\gamma}{1-\gamma}(v_1(\widetilde{y}_2(x)) - \lambda\overline{d}) f(\widetilde{y}_2(x)) f(x) dx. \tag{25}$$

This expression is larger than zero since $\frac{\partial \widetilde{y}_2(x)}{\partial \alpha_2} < 0$ *for* $x < \overline{d} - R$ *and since* $v_1 > \lambda\overline{d}$*. Thus, less diversification at bank* 2 *increases welfare at bank* 1*. From the symmetry of the problem it then follows that* $\frac{\partial EU_2}{\partial \alpha_1} > 0$*, that is, if bank* 1 *reduces its amount of diversification, bank* 2 *benefits. Taking this result together with* $\frac{\partial EU_1}{\partial \alpha_1}\mid_{\alpha_1=\alpha_2=\frac{1}{2}} > 0$ *(from Proposition 5) we then have that* $\frac{\partial (EU_1+EU_2)}{\partial \alpha_1}\mid_{\alpha_1=\alpha_2=\frac{1}{2}} > 0$*.* ■

The result that full diversification remains inefficient is noteworthy since in contrast to Section 2 diversification at each bank is now required to reduce a bank's likelihood of failure (in the previous sections it was sufficient to have diversification on the level of the economy). This can be seen by considering Figure 2b. Bank 1 fails in areas B and C when its solvency threshold is $y_1^1(x)$. It is easy to show that these areas are minimized if there is full diversification at the bank. The reason why the inefficiency of diversification remains is that diversification increases the likelihood of joint failures: area C (where we have $y < \widetilde{y}_1(x)$ and $y < \widetilde{y}_2(x)$) unambiguously increases with diversification at bank 1. This is costly for both banks because in an individual failure at least some assets of a failing bank can be continued at the other bank (the total costs of a joint failure are thus more than twice the costs of a single failure). Diversification at a bank hence poses a trade-off: it reduces the likelihood that a bank experiences a run (regardless of the other bank) but also increases the probability of joint runs occurring. As Proposition 6 has shown, close to full diversification the second effect always outweighs the first. Thus, incomplete diversification is beneficial.[16] Note also that as in Section 2 there is a negative externality associated with investing in the market portfolio (we have $\frac{\partial EU_2}{\partial \alpha_1} > 0$, see proof of Proposition 6). Hence, in equilibrium there may be an inefficiently high amount of investment in the market portfolio.

Remark 7 *In our analysis we have used a specific asset structure: there were only two assets, which were, moreover, uncorrelated. These features are not important for the results. First, if assets are correlated this on the one hand reduces the costs of diversification since portfolios will be already correlated if they are specialized. On the other hand, however, higher correlation at the same time also reduces the gains from diversification, which arise from reducing portfolio volatility. Full diversification remains undesirable as a result. The inefficiency of full diversification also holds if there are more than two assets (calculations available on request from the author). The reason is analogous to the one behind Propositions 2 and 6. Suppose there is an arbitrary number of assets instead of two. If banks are each diversified, they will both hold the market portfolio (in the case of a large number of assets, assets then need to be partially correlated in order for the market portfolio to be still risky). If one bank deviates from the market portfolio (for example, by holding a combination of the market portfolio and an individual asset), the bank will in certain situations face new runs but there will also be situations where it no longer experiences a run. The overall social costs of the former will again be lower. This is, first, because the bank's assets can in these situations be (partially) transferred to the other bank, instead of having*

[16] However, it is no longer necessarily the case that a complete lack of diversification is efficient. Depending on the parameter constellations it may either be efficient to have partial diversification or no diversification (calculations available on request from the author).

to be prematurely liquidated. And second, because in the case where the bank now survives, the assets of the other bank no longer have to be all prematurely liquidated.

4.1 Risk Aversion and Liquidity Choice

The analysis in the preceding sections has been facilitated by two features of the model that allowed us to focus exclusively on the diversification problem. First, consumers' risk aversion had a specific form: there was a discrete loss when consumption fell below $\overline{d}$ at date 1, but utility was otherwise linear. This made the problem of the optimal deposit contract trivial. Second, we assumed that the storage technology is dominated at date 0. As a result, the bank's portfolio problem boiled down to choosing a mix of risky assets. In this subsection we relax these assumptions.

We modify the model of the insolvency economy as follows. Consumers now have standard Diamond-Dybvig preferences. They have a period utility of $u(c)$, which is assumed to be twice differentiable, increasing, and strictly concave. Consumers are uncertain about when to consume: with probability λ they are early consumers (that is they can only consume at date 1), while with probability $1-\lambda$ they are late consumers (only consume at date 2). A consumer's utility can be summarized as follows:

$$U(c_1, c_2) = \begin{cases} \text{with probability } \lambda\text{:} & u(c_1) \\ \text{with probability } 1-\lambda\text{:} & u(c_2) \end{cases} \tag{26}$$

Uncertainty about preferences is again resolved at the beginning of date 1 and is private information. We also assume that the storage technology is now longer dominated at date 0. Banks will hence invest a positive share of their funds in the storage technology. We denote this share at bank i with $l_i \in (0,1]$. The remaining share of the funds $(1-l_i)$ is invested in the (risky) assets. Denoting with α_i the fraction of this share that is invested in asset Y, we have that the bank invests in total $(1-l_i)\alpha_i$ in asset Y and $(1-l_i)(1-\alpha_i)$ in asset X. We continue to define v_i by $v_i = \alpha_i y + (1-a_i)x$, thus v_i now gives the value of one unit of the asset portfolio at date 1. Moreover, we assume that the date 1 asset returns have full support on $[0, \infty)$ as in Section 2.[17]

At date 0 each bank now has to chose three variables: the deposit contract d_i, the amount of funds to be stored l_i, and the composition of its asset portfolio α_i. We focus our analysis on equilibria with symmetric and interior choices of d and l (that is, $d_1^* = d_2^* = d^*$ with $d^* \in [0, \infty)$ and $l_1^* = l_2^* = l^*$ with $l^* \in (0, 1)$). The next two propositions show that the previous results carry over.

Proposition 7 *Full diversification at each bank continues not to be an equilibrium.*

Proof. *See Appendix.* ■

The basic intuition for this result is as for Proposition 5: the net-effect of a deviation from full diversification is that it increases the likelihood of situations where the deviating bank survives, while the other bank fails. This is beneficial for the deviating bank due to the possibility of acquiring assets in fire-sales. There is now, however, an additional twist because diversification has also effects because it influences the variance of consumers' payoffs, which now matters since consumers are risk-averse. However, close to full diversification these effects are not of first order importance.

Proposition 8 *Full diversification at each bank continues to be inefficient.*

Proof. *See Appendix.* ■

The reason for this result is precisely the same as for Proposition 6: lowering diversification at full diversification is always beneficial since it reduces the likelihood of joint failures in which assets can no longer be continued at other banks and have to be prematurely liquidated.

[17] In the insolvency economy we previously constrained the date 1 asset returns to be larger than $\lambda \overline{d}$ in order to rule out illiquidity problems at banks. This restriction is no longer meaningful since d may now in principle take any value in equilibrium.

5 Conclusions

We have considered an economy where consumers face liquidity shocks and where the costs of liquidations are endogenous. We have shown that in this economy the efficiency of full diversification breaks down. The reason for this result is that joint shortages of resources at consumers (or institutions) carry an efficiency loss over and above individual shortages. Diversification is then costly since it implies that consumers hold similar portfolios and thus increases their likelihood of encountering joint shortages. We have also shown that the optimal degree of diversification in this economy is always incomplete, and may even be zero. Consumers, however, may fail to implement the optimal degree due to an externality. Because of this externality there is a tendency for consumers to hold in equilibrium a larger share of the market portfolio than is efficient. This creates a rationale for discouraging investment in the market portfolio.

The inefficiency result has a bearing on the interpretation of the ongoing financial crisis. Institutions around the world had extended their activities by investing in the same asset: U.S. subprime mortgages. From the perspective of most institutions (e.g., commercial banks that were not specialized in U.S. mortgage lending, or insurance companies and hedge funds) this investment amounted to diversification. However, it had the effect of making institutions that were previously diverse more similar to each other. The consequence was that, following a bad performance of these investments, many institutions had to liquidate assets at the same time and potential buyers were limited, thus inducing significant costs (it is argued that there are currently fire-sales prices in several asset classes, see for example Allen and Carletti (2008) and Longstaff (2008)). Our welfare results suggest that the expansion of activities was inefficient (from an ex-ante perspective) as individual institutions do not fully internalize the social benefits from making diverse investments.

Our analysis has a number of interesting implications. For one, it suggests that when evaluating the efficiency of portfolio allocations, one should not only study individual portfolios (which are typically undiversified) but also the economy's aggregate portfolio.

Efficiency requires the latter to be diversified, but not necessarily the former. Another implication of our analysis is that it can be optimal for otherwise identical consumers to hold different portfolios. This not only provides a rationale for a specialization in individual portfolios (such as the home bias) but also for the observed heterogeneity in investors' portfolios (e.g., Heaton and Lucas, 2000). Finally we observe that it is an interesting property of our economy that portfolio choices are interdependent. Due to the higher costs of joint liquidations, an investor obtains a lower utility from a portfolio that is correlated with those of other investors in the economy. An investors' optimal portfolio can hence not be determined in isolation from other portfolio allocations. This is in contrast to standard asset allocation models (such as the CAPM) where portfolio allocations are independent of each other.

Appendix: Proofs

Proof of Proposition 7

We first derive the conditions under which only insolvency runs but no illiquidity runs occur in the economy. A bank i ($i = 1, 2$) will face an insolvency run if the resources available to late consumers at date 2 are less than the amount promised to early withdrawals (d_i). The bank's resources at date 1 are $(1 - l_i)v_i + l_i$. After paying early consumers the bank is left with $(1 - l_i)v_i + l_i - \lambda d_i$ which it stores for consumption in the final period. The return from its portfolio at date 2 are $(1 - l_i)R$. The total resources available to a late consumer at date 2 are thus $\frac{(1-l_i)(R+v_i)+l_i-\lambda d_i}{1-\lambda}$. The condition for an insolvency runs hence is $\frac{(1-l_i)(R+v_i)+l_i-\lambda d_i}{1-\lambda} < d_i$. Rearranging gives

$$(1 - l_i)(R + v_i) + l_i < d_i. \tag{27}$$

In order to make sure that only insolvency runs occur, we have to assume that whenever the bank is solvent, it is also liquid. For this, the following condition suffices:

$$R \leq (1 - \lambda)\frac{d_i}{1 - l_i}. \tag{28}$$

To see this, assume that the bank is solvent, that is equation (27) does not hold. We then have $(1 - l_i)(R + v_i) + l_i \geq d_i$. Using (28) to substitute R we obtain $(1 - l_i)v_i + l_i \geq \lambda d_i$. Noting that $(1 - l_i)v_i + l_i$ is the bank's liquidity at date 1, it follows that the bank is liquid. We assume from now on that condition (28) is met. Note that there always exist parameters for which this is the case in equilibrium. For example, if we make R small the left-hand side of (28) can be made arbitrarily small. At the same time the right-hand side increases since a lower R makes it more attractive for the bank to hold liquidity (l_i will increase).

Again, three different situations can occur at date 1. First, both banks may be solvent (equation (27) is fulfilled at neither bank). There are then no runs, nor is there a need for asset sales or premature liquidation. Consequently, the payouts to early and late consumers at bank i are d_i and $\frac{(1-l_i)(R+v_i)+l_i-\lambda d_i}{1-\lambda}$, respectively. Second, both banks may be insolvent and face runs. The assets at both banks then have to be prematurely liquidated at cost γ and all consumers at bank i obtain $(1-l_i)((1-\gamma)R+v_i)+l_i$. Early consumers consume this at date 1, while late consumers store the goods for consumption at date 2.

Third, one bank may be solvent, while the other is not. The insolvent bank then faces a run, forcing it to prematurely liquidate its assets or to sell them to the solvent bank. As before, there are three cases that arise with respect to the liquidity of the solvent bank. The first case is that the solvent bank (denoted with i) cannot purchase all assets from the insolvent bank at the liquidation value. Noting that the available excess liquidity at the solvent bank is $(1-l_i)v_i+l_i-\lambda d_i$, this condition writes $(1-l_i)v_i+l_i-\lambda d_i<(1-l_j)(1-\gamma)R$. The price of the asset is then $p=(1-\gamma)R$. Consumers at the insolvent bank get $(1-l_j)((1-\gamma)R+v_j)+l_j$, regardless of their type. Early consumers at the solvent bank get d_i. Since the bank makes a return of $\frac{1}{1-\gamma}$ on its surplus liquidity, late consumers get $((1-l_i)R+\frac{(1-l_i)v_i+l_i-\lambda d_i}{1-\gamma})/(1-\lambda)$. The second case is where the solvent bank can purchase the assets at liquidation value, but not at their full value: $(1-l_j)(1-\gamma)R\leq(1-l_i)v_i+l_i-\lambda d_i<(1-l_j)R$. The asset price is then determined by cash-in-the-market pricing, that is, the price adjusts such that the surplus liquidity of the solvent bank is just enough to purchase all assets from the insolvent bank. The equilibrium price is hence determined by the condition $(1-l_i)v_i+l_i-\lambda d_i=p(1-l_j)$. Rearranging gives $p=\frac{(1-l_i)v_i+l_i-\lambda d_i}{1-l_j}$. All consumers at the insolvent bank get $(1-l_j)v_j+l_j+p(1-l_j)$, which is equal to $(1-l_j)v_j+l_j+(1-l_i)v_i+l_i-\lambda d_i$. Early consumers at the solvent bank still get d_i, while late consumers get $\frac{(1-l_i)R+(1-l_j)R}{1-\lambda}$ since the solvent bank now also has the assets of the insolvent bank. The third case is where the solvent bank can purchase the assets of the insolvent bank at their full price. The condition for this is $(1-l_i)v_i+l_i-\lambda d_i\geq(1-l_j)R$. The asset price in this case is $p=R$. All consumers at the insolvent bank get $(1-l_j)(v_j+R)+l_j$, while early and late consumers at the solvent bank get d_i and $\frac{(1-l_i)(v_i+R)+l_i-\lambda d_i}{1-\lambda}$, respectively.

The critical return needed for solvency is now determined by $(1-l_i)(R+v_i)+l_i=d_i$

(from equation 28). Using the definition of v_i, this can be solved for y:

$$\widetilde{y}_i(x) = \frac{d_i - l_i - (1-l_i)R}{(1-l_i)\alpha_i} - \frac{1-\alpha_i}{\alpha_i}x. \tag{29}$$

We next derive the expected utility for consumers at bank 1. It again suffices for the proof to consider small deviations from full diversification. This, as before, simplifies the analysis since for appropriate parameter constellations the second and third insolvency case do not occur. In fact, the condition which guarantees that only case one exists for symmetric deposit contracts and liquidity choices ($l = l_1 = l_2$ and $d = d_1 = d_2$) is:

$$R > \frac{1-\lambda}{2-\gamma}\frac{d}{1-l}. \tag{30}$$

The proof is analog to Lemma 1. Suppose that $|\alpha_i - \alpha_j| < \varepsilon$. We need to show that for sufficiently small ε we have that whenever bank i is solvent but bank j is not ($(1-l)(R+v_i)+l \geq d$ and $(1-l)(R+v_j)+l < d$), bank i cannot purchase all assets from bank j at their liquidation value ($(1-l)v_i + l - \lambda d < (1-l)(1-\gamma)R$). Without loss of generality we set $i = 1$ and $j = 2$. An in Lemma 1 we have that $v_1 = v_2 + (\alpha_1 - \alpha_2)(y-x) < v_2 + \varepsilon(y-x)$. We can then write for the surplus liquidity of bank 1:

$$\begin{aligned} &(1-l)v_1 + l - \lambda d < (1-l)(v_2 + \varepsilon(y-x)) + l - \lambda d \\ &< (1-\lambda)d + (1-l)(-R + \varepsilon(y-x)) < (1-l)(1-\gamma)R + \varepsilon(1-l)(y-x), \end{aligned} \tag{31}$$

where the second inequality is implied by the insolvency of bank 2, and the third inequality follows from condition (30). From (31) it follows that for small enough ε we have that $(1-l)v_1 + l - \lambda d < (1-l)(1-\gamma)R$. We assume from now on that condition (30) is fulfilled (note that this condition does not conflict with condition (28) because of $\gamma < 1$).

We can summarize the expected utility for consumers at bank 1 as follows:

1. Both banks are solvent ($y \geq \widetilde{y}_1(x)$ and $y \geq \widetilde{y}_2(x)$). The total expected utility for a consumer of bank 1 is $\lambda u(d_1) + (1-\lambda)u(\frac{(1-l_1)(R+v_1)+l_1-\lambda d_1}{1-\lambda})$.

2. Bank 1 is solvent, but bank 2 is not ($\widetilde{y}_1(x) \leq y < \widetilde{y}_2(x)$). The total expected utility is then $\lambda u(d_1) + (1-\lambda)u(\frac{(1-l_1)R + \frac{(1-l_1)v_1+l_1-\lambda d_1}{1-\gamma}}{1-\lambda})$.

3. Bank 1 is insolvent ($y < \widetilde{y}_1(x)$). The expected utility is then $u((1-l_1)(1-\gamma)R+v_1+l_1)$. Note, again, that in this case it does not matter whether bank 2 is solvent.

With this we can write the expected utility at bank 1 for symmetric deposit contracts and liquidity choices, presuming again that α_1 and α_2 are sufficiently similar:

$$\begin{aligned}
EU_1 = &\int_0^{\frac{d-l-(1-l)R}{1-l}} \int_{\widetilde{y}_2(x)}^{\infty} (\lambda u(d) + (1-\lambda)u(\frac{(1-l)(R+v_1)+l-\lambda d}{1-\lambda}))f(y)f(x)dydx \\
&+ \int_{\frac{d-l-(1-l)R}{1-l}}^{\widetilde{x}_1(0)} \int_{\widetilde{y}_1(x)}^{\infty} (\lambda u(d) + (1-\lambda)u(\frac{(1-l)(R+v_1)+l-\lambda d}{1-\lambda}))f(y)f(x)dydx \\
&+ \int_{\widetilde{x}_1(0)}^{\infty} \int_0^{\infty} (\lambda u(d) + (1-\lambda)u(\frac{(1-l)(R+v_1)+l-\lambda d}{1-\lambda})f(y)f(x)dydx \\
&+ \int_0^{\frac{d-l-(1-l)R}{1-l}} \int_{\widetilde{y}_1(x)}^{\widetilde{y}_2(x)} (\lambda u(d) + (1-\lambda)u(\frac{(1-l)R + \frac{(1-l)v_1+l-\lambda d}{1-\gamma}}{1-\lambda})f(y)f(x)dydx \\
&+ \int_0^{\widetilde{x}_1(0)} \int_0^{\widetilde{y}_1(x)} (u((1-l)((1-\gamma)R+v_1)+l))f(y)f(x)dydx. \qquad (32)
\end{aligned}$$

We can rearrange this to

$$\begin{aligned}
EU_1 = \lambda u(d) + (1-\lambda) &\int_0^{\infty} \int_0^{\infty} u(\frac{(1-l)(R+v_1)+l-\lambda d}{1-\lambda})f(y)f(x)dydx \\
&+ \int_0^{\frac{d-l-(1-l)R}{1-l}} \int_{\widetilde{y}_1(x)}^{\widetilde{y}_2(x)} (1-\lambda)(u(\frac{(1-l)R + \frac{(1-l)v_1+l-\lambda d}{1-\gamma}}{1-\lambda} \\
&\quad -u(\frac{(1-l)(R+v_1)+l-\lambda d}{1-\lambda}))f(y)f(x)dydx \\
&- \int_0^{\widetilde{x}_1(0)} \int_0^{\widetilde{y}_1(x)} (\lambda u(d) + (1-\lambda)u(\frac{(1-l)(R+v_1)+l-\lambda d}{1-\lambda}) \\
&\quad -u((1-l)((1-\gamma)R+v_1)+l))f(y)f(x)dydx. \qquad (33)
\end{aligned}$$

The first two terms in this equation correspond to the first two terms in equation (22) and give a consumer's expected utility in the absence of runs. The following term (which is the counterpart of the first integral in (22)) gives the gains from being solvent when the other bank fails, arising because assets can be purchased at a price below their continuation value. The final term (corresponding to the last integral in (22)) expresses the losses when the bank faces a run.

In order to show that full diversification cannot be an equilibrium, we show that bank 1 has an incentive to deviate from any full diversification allocation (that is, it has an incentive to deviate from any allocation with $\alpha_1 = \alpha_2 = \frac{1}{2}$ and arbitrary $d \in [0, \infty)$ and $l \in (0, 1)$). From (33) we have for $\frac{\partial EU_1}{\partial \alpha_1}$ evaluated at full diversification:

$$\frac{\partial EU_1}{\partial \alpha_1}_{\alpha_1=\alpha_2=\frac{1}{2}} = -\int_0^{\frac{d-l-(1-l)R}{1-l}} \frac{\partial \widetilde{y}_1(x)}{\partial \alpha_1}(1-\lambda)(u(\frac{(1-l)R+\frac{l+(1-l)v_1(\widetilde{y}_1(x))-\lambda d}{1-\gamma}}{1-\lambda}$$
$$-u(\frac{(1-l)(R+v_1(\widetilde{y}_1(x)))+l-\lambda d}{1-\lambda}))f(\widetilde{y}_1(x))f(x)dx \tag{34}$$

This terms is larger than zero since $\frac{\partial \widetilde{y}_1(x)}{\partial \alpha_1} < 0$ for $x < \frac{d-l-(1-l)R}{1-l}$ and $\frac{(1-l)R+\frac{l+(1-l)v_1(\widetilde{y}_1(x))-\lambda d}{1-\gamma}}{1-\lambda} - \frac{(1-l)(R+v_1(\widetilde{y}_1(x)))+l-\lambda d}{1-\lambda} = \frac{\gamma}{1-\gamma}\frac{l+(1-l)v_1(\widetilde{y}_1(x))-\lambda d}{1-\gamma} > 0$ (these are the benefits from purchasing assets at fire-sales from the insolvent bank). It follows that we have $\frac{\partial EU_1}{\partial \alpha_1} |_{\alpha_1=\alpha_2=\frac{1}{2}} > 0$, hence full diversification is not an equilibrium.

Proof of Proposition 8

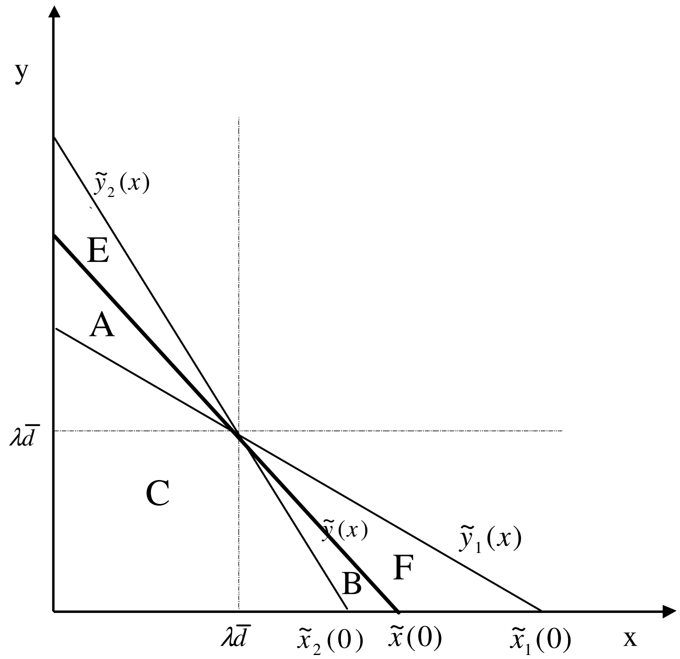
We first show that if bank 2 reduce its diversification (by reducing α_2), bank 1 benefits. From (33) we have for the derivative of EU_1 with respect to $-\alpha_1$ at $\alpha_1 = \alpha_2 = \frac{1}{2}$:

$$\frac{-\partial EU_1}{\partial \alpha_2} \mid_{\alpha_1=\alpha_2=\frac{1}{2}} = \int_0^{\frac{d-l-(1-l)R}{1-l}} \frac{-\partial \widetilde{y}_2(x)}{\partial \alpha_2}(1-\lambda)(u(\frac{(1-l)R+\frac{l+(1-l)v_1(\widetilde{y}_2(x))-\lambda d}{1-\gamma}}{1-\lambda}$$
$$-u(\frac{(1-l)(R+v_1(\widetilde{y}_2(x)))+l-\lambda d}{1-\lambda}))f(\widetilde{y}_2(x))f(x)dx. \tag{35}$$

Since we again have $\frac{(1-l)R+\frac{l+(1-l)v_1(\widetilde{y}_2(x))-\lambda d}{1-\gamma}}{1-\lambda} > \frac{(1-l)(R+v_1(\widetilde{y}_2(x)))+l-\lambda d}{1-\lambda}$ and $\frac{-\partial \widetilde{y}_2(x)}{\partial \alpha_2} > 0$ for $x < \frac{d-l-(1-l)R}{1-l}$, it follows that $\frac{-\partial EU_1}{\partial \alpha_2}_{\alpha_1=\alpha_2=\frac{1}{2}} > 0$. Using the symmetry of the problem we then also have that $\frac{\partial EU_2}{\partial \alpha_1}_{\alpha_1=\alpha_2=\frac{1}{2}} > 0$. From the proof of Proposition 7 we already know that $\frac{\partial EU_1}{\partial \alpha_1}_{\alpha_1=\alpha_2=\frac{1}{2}} > 0$. Hence we have that $\frac{\partial (EU_1+EU_2)}{\partial \alpha_1}_{\alpha_1=\alpha_2=\frac{1}{2}} > 0$, which proves the inefficiency of full diversification.

Figure 1: The Illiquidity Economy

(a)



(b)

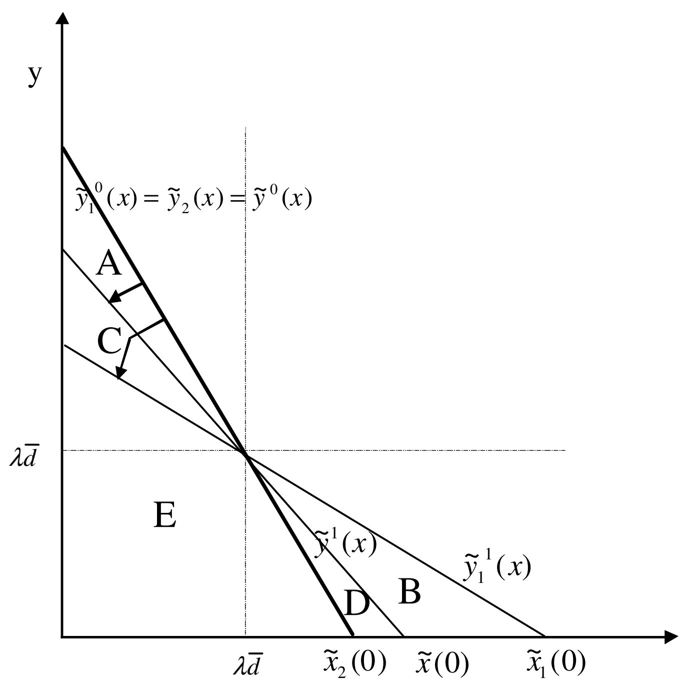
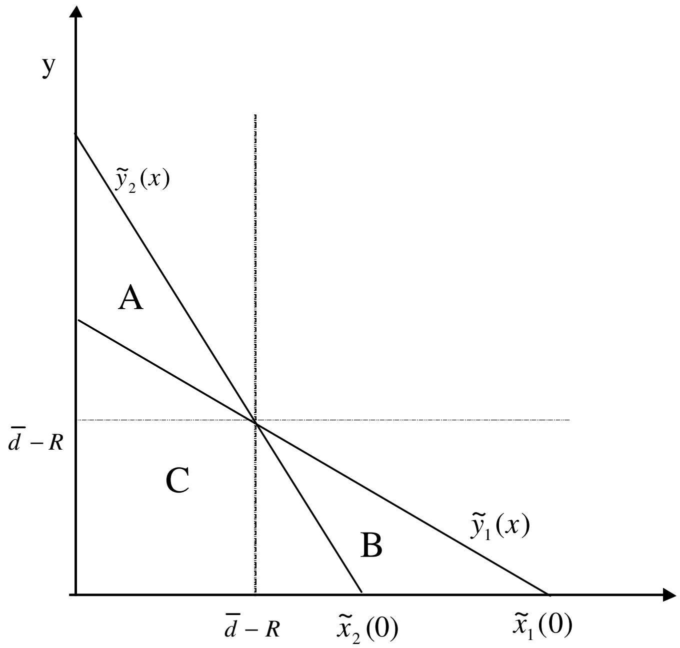
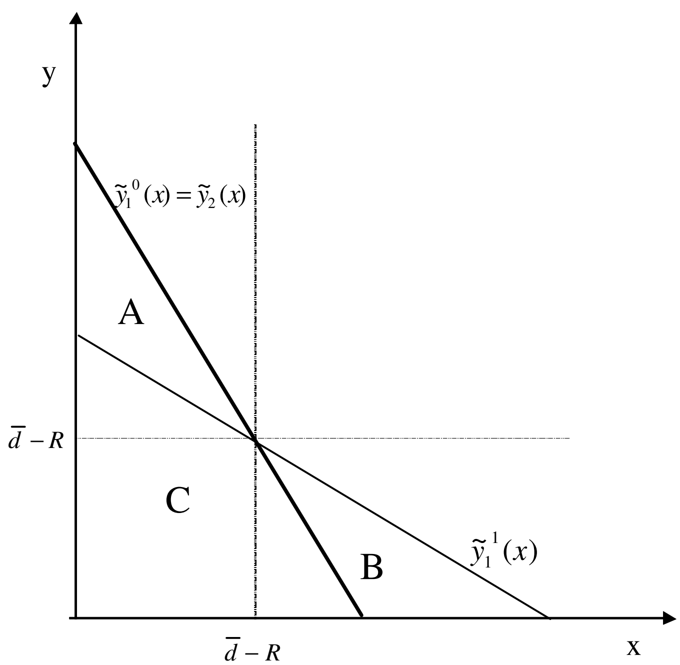


Figure 2: The Insolvency Economy

(a)



(b)



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Derivations (Not For Publication)

Derivation of Equation (7)

In equation (7) realizations with zero density are already excluded, that is the integrals only sum up over outcomes where $x, y \geq 0$. The first two integral terms then refer to case 1 and hence cover the area where $y \geq \widetilde{y}(x)$. At $\widetilde{x}(0)$ we have that $\widetilde{y}(x) = 0$, hence for x larger than $\widetilde{x}(0)$ we obtain this case for all values of y (this is the second integral term). The third integral term covers case 2a, where we have $y \geq \widetilde{y}_1(x)$ and $y < \widetilde{y}(x)$. This implies that $\widetilde{y}(x) > \widetilde{y}_1(x)$, which can only occur for $x < \lambda\overline{d}$, hence the upper integration bound of the outer integral is $\lambda\overline{d}$. The fourth and fifth integral term cover case 2b where $y \geq \widetilde{y}_2(x)$ and $y < \widetilde{y}(x)$. We then have $\widetilde{y}(x) > \widetilde{y}_2(x)$, which can only occur for $x > \lambda\overline{d}$, hence the lower integration bound of the outer integral of the fourth integral term is $\lambda\overline{d}$. At $x_2(0)$ we have that $\widetilde{y}_2(x) = 0$, hence for $x > x_2(0)$ we can start integrating from zero for y (fifth integral term). The last two terms relate to case 2c, where we both have $y > \widetilde{y}_1(x)$ and $y > \widetilde{y}_2(x)$. For $x < \lambda\overline{d}$ these conditions simplify to $y > \widetilde{y}_1(x)$ (sixth integral term), while for $x > \lambda\overline{d}$ they simplify to $y > \widetilde{y}_2(x)$ (last integral).

Derivation of Equation (8)

Subtracting $R + v_1$ from the argument in each integral term in (7) and adding the arising term ($\int_0^\infty \int_0^\infty (R + v_1) f(y) f(x) dy dx$) separately, we obtain

$$\begin{aligned} E \quad U_1(\alpha_1, \alpha_2) = & \int_0^\infty \int_0^\infty (R + v_1) f(y) f(x) dy dx + \int_0^{\lambda\overline{d}} \int_{\widetilde{y}_1(x)}^{\widetilde{y}(x)} (R - (v_1 - \lambda\overline{d})) f(y) f(x) dy dx \\ & - \int_{\lambda\overline{d}}^{\widetilde{x}_2(0)} \int_{\widetilde{y}_2(x)}^{\widetilde{y}(x)} (R - (v_2 - \lambda\overline{d}) + \lambda k) f(y) f(x) dy dx \\ & - \int_{\widetilde{x}_2(0)}^{\widetilde{x}(0)} \int_0^{\widetilde{y}(x)} (R - (v_2 - \lambda\overline{d}) + \lambda k) f(y) f(x) dy dx \\ & - \int_0^{\lambda\overline{d}} \int_0^{\widetilde{y}_1(x)} (R + \lambda k) f(y) f(x) dy dx - \int_{\lambda\overline{d}}^{\widetilde{x}_2(0)} \int_0^{\widetilde{y}_2(x)} (R + \lambda k) f(y) f(x) dy dx. \end{aligned} \tag{36}$$

Observing that $\int_0^\infty \int_0^\infty (R+v_1)f(y)f(x)dydx$ is the mean of $(R+v_1)$, we have that the first integral equals $R+\mu$, which gives us equation (8).

Derivation of Equation (9)

For the derivation note that the expressions that arise because a variation in α_1 changes $\widetilde{x}_2(0)$ in the second and the third integral term in (8) add to zero. Furthermore, the expressions that arise because $\widetilde{x}(0)$ changes in the third integral term and $\widetilde{x}_2(0)$ changes in the last integral term are zero because of $\widetilde{y}(\widetilde{x}(0)) = 0$ and $\widetilde{y}_2(\widetilde{x}_2(0)) = 0$ (by the definition of $\widetilde{x}(0)$ and $\widetilde{x}_2(0)$). Note furthermore that $v_2(\widetilde{y}_2(x)) = \lambda d$. Hence the expressions that arise through changes in $\widetilde{y}_2(x)$ (second and last integral term) also add up to zero.

Derivation of Equation (12)

Note that as for the derivation of (9), all expressions that arise because a change in α_1 influences $\widetilde{x}_2(0)$ and $\widetilde{x}(0)$ add to zero.

Derivation of Equation (13)

Here we use again that $\frac{\partial \widetilde{y}}{\partial \alpha_1} f(\widetilde{y}(x))f(x) \mid_{x=b} = -\frac{\partial \widetilde{y}}{\partial \alpha_1} f(\widetilde{y}(x))f(x) \mid_{x=2\lambda\bar{d}-b}$. Hence the first and the second integral sum to zero.

Derivation of Equation (14)

Note again that all expressions that arise because a variation in α_2 changes $\widetilde{x}_2(0)$ and $\widetilde{x}(0)$ add to zero. We hence have for the derivative

$$\begin{aligned} \frac{\partial U_1}{\partial \alpha_2} &= \int_0^{\lambda\bar{d}} \frac{\partial \widetilde{y}(x)}{\partial \alpha_2}(R-(v_1(\widetilde{y}(x))-\lambda\bar{d}))f(\widetilde{y}(x))f(x)dx \\ &\quad - \int_{\lambda\bar{d}}^{\widetilde{x}(0)} \frac{\partial \widetilde{y}(x)}{\partial \alpha_2}(R-(v_2(\widetilde{y}(x))-\lambda\bar{d})+\lambda k)f(\widetilde{y}(x))f(x)dx \\ &\quad - \int_{\lambda\bar{d}}^{\widetilde{x}_2(0)} \frac{\partial \widetilde{y}_2(x)}{\partial \alpha_2}(v_2(\widetilde{y}_2(x))-\lambda\bar{d}))f(\widetilde{y}_2(x))f(x)dx. \end{aligned} \tag{37}$$

Noting that $v_2(\widetilde{y}_2(x)) = \lambda\overline{d}$ the third integrals vanishes and we obtain (14).

Derivation of Equation (15)

Evaluating equation (9) at $\alpha_1 = 1$ and $\alpha_2 = 0$ gives

$$\begin{aligned}\frac{\partial EU_1}{\partial \alpha_1} = &-\int_0^{\lambda\overline{d}}\int_{\lambda\overline{d}}^{2\lambda\overline{d}-x}(y-x)f(y)f(x)dydx + \int_0^{\lambda\overline{d}}\frac{\partial\widetilde{y}(x)}{\partial\alpha_1}(R-(\lambda\overline{d}-x))f(\widetilde{y}(x))f(x)dx\\ &-\int_{\lambda\overline{d}}^{2\lambda\overline{d}}\frac{\partial\widetilde{y}(x)}{\partial\alpha_1}(R-(x-\lambda\overline{d})+\lambda k)f(\widetilde{y}(x))f(x)dx\\ &-\int_0^{\lambda\overline{d}}\frac{\partial\widetilde{y}_1(x)}{\partial\alpha_1}(2R+\lambda k)f(\widetilde{y}_1(x))f(x)dx, \end{aligned} \tag{38}$$

where we have used $\widetilde{y}_1(x) = \lambda\overline{d}$, $\widetilde{x}(0) = 2\lambda\overline{d}$, $v_1(\widetilde{y}(x)) = 2\lambda\overline{d} - x$, $v_2(\widetilde{y}(x)) = x$ and $v_1(\widetilde{y}_1(x)) = \lambda\overline{d}$. The second and the third integral can be combined into a single integral: $\int_0^{\lambda\overline{d}}\frac{\partial\widetilde{y}(x)}{\partial\alpha_1}(2R - 2(\lambda\overline{d} - x) + \lambda k)f(\widetilde{y}(x))f(x)dx$. We can than combine the $2R + \lambda k$ part of the argument of this integral with the last integral to obtain (15).

Derivation of Equation (21)

The derivation is similar to equation (7). First, we exclude realizations with zero densities, that is the integrals only sum up over outcomes for which $x, y \geq 0$. The first three integral terms then refer to situation 1 where $y \geq \widetilde{y}_1(x), \widetilde{y}_2(x)$. The first integral covers realization for x until $\overline{d} - R$, for which we have $\widetilde{y}_2(x) > \widetilde{y}_1(x)$ and hence the situation is ensured by $y \geq \widetilde{y}_2(x)$. The second integral covers realizations until $\widetilde{x}_1(0)$, where we have $\widetilde{y}_1(x) > \widetilde{y}_2(x)$ and hence the situation occurs for $y \geq \widetilde{y}_1(x)$. At $\widetilde{x}_1(0)$ we have that $\widetilde{y}_1(x) = 0$, hence for x larger than $\widetilde{x}_1(0)$ we obtain situation 1 for all values of y (this is the third integral term). The fourth integral term covers situation 2, where we have $\widetilde{y}_1(x) \leq y < \widetilde{y}_2(x)$. This situation can only occur for $x < \lambda\overline{d}$. The final terms covers the situation 3 where realizations are $y < \widetilde{y}_1(x)$.

Derivation of Equation (22)

Analogous to equation (8).

Derivation of Equation (23)

For the derivation note that the expression that arises because a variation in α_1 changes $\widetilde{x}_1(0)$ in the last integral is zero because of $\widetilde{y}_1(\widetilde{x}_1(0)) = 0$.

Derivation of Equation (24)

At full diversification we have $\widetilde{y}_1(x) = \widetilde{y}_2(x)$ and hence the first integral term vanishes. The third integral also vanishes because of $\frac{\partial \widetilde{y}_1}{\partial \alpha_1} f(\widetilde{y}_1(x)) f(x) \mid_{x=b} = -\frac{\partial \widetilde{y}_1}{\partial \alpha_1} f(\widetilde{y}_1(x)) f(x) \mid_{x=2(\overline{d}-R)-b}$ and $\widetilde{x}_1(0) = 2(\overline{d} - R)$.

Derivation of Equation (32)

Analogous to equation (21) (each of the integrals in (32) corresponds to an integral in (21)).

Derivation of Equation (33)

Analogous to equation (8).

Derivation of Equation (34)

Note first again that the effects that arise through changes in $\widetilde{x}_1(0)$ sum to zero. We hence have

$$
\begin{aligned}
\frac{\partial EU_1}{\partial \alpha_1}_{\alpha_1=\alpha_2=\frac{1}{2}} &= (1-\lambda)\int_0^\infty\int_0^\infty (1-l)(y-x)u'(\frac{(1-l)(R+\frac{x+y}{2})+l-\lambda d}{1-\lambda})f(y)f(x)dydx \\
&+\int_0^{\frac{d-l-(1-l)R}{1-l}}\int_{\widetilde{y}_1(x)}^{\widetilde{y}_2(x)}(1-l)(y-x)(\frac{1}{1-\gamma}u'(\frac{(1-l)R+\frac{(1-l)\frac{x+y}{2}+l-\lambda d}{1-\gamma}}{1-\lambda} \\
&-u'(\frac{(1-l)(R+\frac{x+y}{2})+l-\lambda d}{1-\lambda}))f(y)f(x)dydx \\
&-\int_0^{2\frac{d-l-(1-l)R}{1-l}}\int_0^{2\frac{d-l-(1-l)R}{1-l}-x}(1-l)(y-x)(u'(\frac{(1-l)(R+\frac{x+y}{2})+l-\lambda d}{1-\lambda})- \\
&u'((1-l)(\frac{x+y}{2}+(1-\gamma)R)+l)))f(y)f(x)dydx \\
&-\int_0^{2\frac{d-l-(1-l)R}{1-l}}\frac{\partial\widetilde{y}_1(x)}{\partial\alpha_1}(\lambda u(d)+(1-\lambda)u(\frac{(1-l)(R+v_1(\widetilde{y}_1(x)))+l-\lambda d}{1-\lambda} \\
&-u((1-l)(v_1(\widetilde{y}_1(x))+(1-\gamma)R)))f(\widetilde{y}_1(x))f(x)dx \\
&-\int_0^{\frac{d-l-(1-l)R}{1-l}}\frac{\partial\widetilde{y}_1(x)}{\partial\alpha_1}(1-\lambda)(u(\frac{(1-l)R+\frac{l+(1-l)v_1(\widetilde{y}_1(x))-\lambda d}{1-\gamma}}{1-\lambda} \\
&-u(\frac{(1-l)(R+v_1(\widetilde{y}_1(x)))+l-\lambda d}{1-\lambda}))f(\widetilde{y}_1(x))f(x)dx. \qquad (39)
\end{aligned}
$$

Due to symmetry the first, the third and the fourth integral term are zero. Since at full diversification $\widetilde{y}_1(x)=\widetilde{y}_2(x)$, the second integral term is zero as well. This leaves us with the last term, which is equation (34).