Dynamic Bond Portfolio Choice with Macroeconomic Information[*]

Alexandros Kostakis[†]and Peter Spencer[‡]

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Abstract

This study examines the optimal portfolio choice of a long-term bond investor, who faces a set of macroeconomic risk factors, both observable (inflation and output gap) and latent ones (real interest rate, inflation central tendency and real interest rate central tendency). It makes use of the essentially affine macro-finance term structure model of Dewachter, Lyrio and Maes (2006) that allows for time-varying risk premia, capturing the failure of the expectations hypothesis. Employing this setup, the investment as well as the hedging opportunities provided by consistently priced zero-coupon bonds for a power utility agent are examined. Moreover, real bonds are introduced and their role for investment and hedging purposes is considered. This study also serves as an evaluation of the employed macro-finance term structure model from an asset allocation perspective, revealing that more attention should be paid to the covariance structure of the bonds' returns.

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1 Introduction

Recent studies on asset allocation, starting from Kim and Omberg (1996), Brennan, Schwartz and Lagnado (1997) and Campbell and Viceira (1999), have examined the impact of horizon effects on risky assets' demands. Since the work of Merton (1971, 1973), it has been well understood that a multi-period investor forms a rather different portfolio as compared to the static investor examined by Markowitz (1952). The difference arises due to hedging demands employed to offset undesirable shocks in the underlying opportunity set. Following this strand of the literature, the present study examines the optimal portfolio choice of a long-term bond investor, who faces a set of macroeconomic risk factors, both observable (inflation and output gap) and latent ones (real interest rate, inflation central tendency and real interest rate central tendency). It makes use of the essentially affine macro-finance term structure model of Dewachter, Lyrio and Maes (2006) that allows for time-varying risk premia and consistently estimates the central tendencies simultaneously with the term structure. Employing this setup, the investment as well as the hedging opportunities provided by consistently priced zero-coupon bonds for a power utility agent are examined.

There are a series of ways in which this study contributes to the literature. Firstly, it incorporates macroeconomic information in an asset allocation context and shows how this can be of significant use for both a myopic and a long-term investor. Despite the conclusion of Cochrane (2007, p. 242) that finance has a lot to say about macroeconomics, macroeconomic information has been relatively neglected in the dynamic asset allocation literature. The stochastic factors affecting the investment opportunity set are commonly assumed to be financial variables, such as the dividend yield (Barberis, 2000) and the Sharpe ratio (Wachter, 2002) that exhibit only a modest degree of predictive ability. On the other hand, macroeconomic variables have been shown to be significant predictors of future bond returns (Ang and Piazzesi, 2003).

Secondly, it focuses on bond portfolio choice that is relatively unexplored in the literature, since the majority of the studies on dynamic asset allocation examine stock-only portfolios. The notable exceptions are the studies of Campbell and Viceira (2001), Brennan and Xia (2002) and Sangvinatsos and Wachter (2005). The degree of the bond yields' predictability as well as the bonds' no-arbitrage pricing, based on the underlying stochastic factors, imply that a bond portfolio setting provides an even more robust framework to examine, in contrast to most of the literature that

uses ad hoc assumptions for the asset returns' dynamics (see Brennan et al., 1997 for an early example). Furthermore, bonds-only portfolios are extremely important for the fund management industry and for central banks.

The related work on dynamic bond portfolio choice includes the studies of Brennan and Xia (2000) and Sorensen (1999), who assume that interest rates are as in Vasicek (1977). A similar framework has been employed in Xia (2002), who examines the impact of limited access to nominal bonds on an investor's welfare. Munk, Sorensen and Vinther (2004) examine the stock-bond mix of a long-term power utility investor in the presence of mean-reverting returns, stochastic interest rates à la Vasicek and inflation uncertainty. This framework has been further exploited for the derivation of the optimal portfolio-consumption policy of a long-term investor by Munk and Sorensen (2004). In the latter study, apart from a Vasicek interest rate model, a three-factor, non-Markovian, Heath-Jarrow-Morton term structure model is also employed. The main conclusion is that hedging demands are very sensitive to the choice of the term structure model. Theoretical approaches to bond-only portfolio selection problems have been considered by Schroder and Skiadas (1999), Tehranchi and Ringer (2004) as well as Liu (2007).

Our study allows for time-varying bond premia, in contrast to Campbell and Viceira (2001) and Brennan and Xia (2002) who assume them to be constant. Therefore, we can capture the failure of the expectations hypothesis. Most importantly, we use a macro-finance term structure model rather than the purely latent factor term structure model that has been commonly used in the finance literature (see Duffee, 2002 for an excellent review of the literature) and has been employed by Sangvinatsos and Wachter (2005) for dynamic bond portfolio choice. Unlike the latent factor framework that lacks a clear economic interpretation, the macro-finance model of Dewachter et al. (2006) allows us to reach more robust conclusions for bond portfolio choice. It additionally allows the central tendencies of inflation and the real interest rate to affect bond premia and hence the optimal portfolio choice; this issue has not been considered before in the literature to the best of our knowledge. Furthermore, the existence of five underlying risk factors enables us to examine the implications of portfolio selection among multiple bonds.

In addition to the previous issues, this study also serves as an evaluation of the employed term structure model from an asset allocation perspective. The term structure literature has mainly focused on fitting past and predicting future yields, while

the asset allocation perspective allows us to examine a series of more general issues. In particular, the covariance and correlation structure of the bonds' returns is extremely important for the formation of optimal portfolios. Equally important are the implied market prices of risk that affect the sensitivities of an investor's wealth to the underlying macroeconomic factors. In other words, the implied risk premia contain significant information for the degree of risk aversion and the horizon of market participants.

Finally, we can derive conclusions with respect to what a long-term investor regards as a riskless asset. This very important issue has been repeatedly examined in the literature (see Modigliani and Sutch, 1965, Stiglitz, 1970 and Fischer, 1975, for early discussions), but it was only recently that Wachter (2003) provided a formal theoretical treatment. In particular, we are able to introduce real bonds in the asset menu of the investor that helps us investigate how the definition of the riskless asset is modified when the investor's horizon changes. The introduction of real bonds allows us also to examine their role for investment and hedging purposes.

The rest of this study is organized as follows. Section 2 outlines the employed term structure model, while Section 3 discusses the data issues and the implications of the estimated model. Section 4 derives the optimal portfolio choice for the long-term investor in complete markets and Section 5 derives the corresponding portfolio choice in an incomplete market setting. Section 6 discusses the results for the portfolio choice in both complete and incomplete bond markets, while Section 7 concludes.

2 The term structure model

2.1 Risk factors

We employ the setup of Dewachter et al. (2006). There are five stochastically time-varying risk factors: the output gap y, the inflation rate π, the real interest rate ρ, the inflation central tendency π^* and the central tendency of the real interest rate ρ^*.[1]

The dynamics of the risk factors, $(y, \pi, \rho, \pi^*, \rho^*)$, are characterized by the following

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[1]For precision, the real instantaneously risk-free rate prevailing in an economy with stochastic price level, as derived in Section 4.2, is slightly different from the definition of Dewachter et al. (2006), but we keep the same notation for ease of reference.

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Stochastic Differential Equations (SDEs):

$$dy = [\kappa_{yy}y + \kappa_{y\pi}(\pi - \pi^*) + \kappa_{y\rho}(\rho - \rho^*)]dt + \sigma_y dw_y \tag{1}$$

$$d\pi = [\kappa_{\pi y}y + \kappa_{\pi\pi}(\pi - \pi^*) + \kappa_{\pi\rho}(\rho - \rho^*)]dt + \sigma_\pi dw_\pi \tag{2}$$

$$d\rho = [\kappa_{\rho y}y + \kappa_{\rho\pi}(\pi - \pi^*) + \kappa_{\rho\rho}(\rho - \rho^*)]dt + \sigma_\rho dw_\rho \tag{3}$$

$$d\pi^* = \kappa_{\pi^*\pi^*}(\pi^* - \theta_{\pi^*})dt + \sigma_{\pi^*}dw_{\pi^*} \tag{4}$$

$$d\rho^* = \kappa_{\rho^*\rho^*}(\rho^* - \theta_{\rho^*})dt + \sigma_{\rho^*}dw_{\rho^*} \tag{5}$$

where $w_j(t)$, $j = \{y, \pi, \rho, \pi^*, \rho^*\}$ denote independent standard Brownian motions defined on the probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathcal{F}$ and time set $[0, T]$, $0 \leq T < \infty$.

The assumed dynamics of the output gap (y), inflation (π) and the real interest rate (ρ) imply that these are affected by the short-run deviations from their central tendencies (the central tendency of the output gap is zero). Moreover, this structure implies that the central bank follows a feedback rule for the real interest rate. The central tendencies of the inflation (π^*) and the real interest rate (ρ^*) are allowed to be mean-reverting processes, capturing possible inertia in the adjustment process.

Vector X contains these risk factors:

$$X = (y, \ \pi, \ \rho, \ \pi^*, \ \rho^*)^T \tag{6}$$

Re-writing the dynamics of the risk factors in a vector form, we have:

$$dX = [\bar{\psi} + KX]dt + Sdw \tag{7}$$

where $\bar{\psi} = (0, 0, 0, -\kappa_{\pi^*\pi^*}\theta_{\pi^*}, -\kappa_{\rho^*\rho^*}\theta_{\rho^*})^T$ is a 5x1 vector, dw is a 5x1 vector:

$$dw = (dw_y, \ dw_\pi, \ dw_\rho, \ dw_{\pi^*}, dw_{\rho^*})^T \tag{8}$$

K is a 5x5 matrix with elements

$$K = \begin{bmatrix} \kappa_{yy} & \kappa_{y\pi} & \kappa_{y\rho} & -\kappa_{y\pi} & -\kappa_{y\rho} \\ \kappa_{\pi y} & \kappa_{\pi\pi} & \kappa_{\pi\rho} & -\kappa_{\pi\pi} & -\kappa_{\pi\rho} \\ \kappa_{\rho y} & \kappa_{\rho\pi} & \kappa_{\rho\rho} & -\kappa_{\rho\pi} & -\kappa_{\rho\rho} \\ 0 & 0 & 0 & \kappa_{\pi^*\pi^*} & 0 \\ 0 & 0 & 0 & 0 & \kappa_{\rho^*\rho^*} \end{bmatrix} \tag{9}$$

and $S = \mathrm{diag}(\sigma_y, \sigma_\pi, \sigma_\rho, \sigma_{\pi^*}, \sigma_{\rho^*})$ is a diagonal 5x5 matrix.

The last assumption essentially implies that there is no interrelationship in the volatility structure of the macro factors. Following Duffee (2002), the market price of risk is assumed to be time-varying and affine in the risk factors, given by $\xi = S\Lambda + S^{-1}\Xi X$, where Λ is a 5x1 vector and Ξ is a 5x5 matrix, the elements of which are estimated from the joint model. The matrix Ξ contains the sensitivities of the prices of risk to the macroeconomic factors. In particular, to avoid identification problems, only a restricted set of the market prices of risk is estimated (see Dai and Singleton, 2000 for a discussion). This term structure model can be classified as $A_0(5)$ within the family of the essentially affine models.

2.2   Bond returns

The price of a zero-coupon default-free nominal bond at time t maturing at time $t + \tau \equiv T$ is assumed to be given by $P(X, t) = \exp(-a(\tau) - b(\tau)^T X)$, where $a(\tau)$ is a scalar and $b(\tau)$ is a 5x1 vector, with initial conditions $a(0) = 0$ and $b(0) = \mathbf{0}_{5x1}$. The no-arbitrage condition under the risk neutral measure Q requires that the expected excess returns of this bond are zero, since the market price of risk is zero under Q.

Given the market price of risk ξ, it holds that:

$$d\tilde{w} = dw + \xi dt \tag{10}$$

where $\tilde{w}(t) = (\tilde{w}_y, \tilde{w}_\pi, \tilde{w}_\rho, \tilde{w}_{\pi^*}, \tilde{w}_{\rho^*})^T$ is a vector of independent standard Brownian motions under the risk neutral measure.

The dynamics of X under Q are given by:

$$dX = [\bar{\psi} - S^2\Lambda + (K - \Xi)X]dt + Sd\tilde{w} \tag{11}$$

Therefore, the no-arbitrage condition for the bond price under Q implies that the following Partial Differential Equation (PDE) should hold:

$$(\bar{\psi} - S^2\Lambda + (K - \Xi)X)^T \frac{\partial P}{\partial X} + \frac{1}{2}\frac{\partial^2 P}{\partial X \partial X^T}S^2 + \frac{\partial P}{\partial t} = rP \tag{12}$$

where r is the instantaneously nominal risk-free rate, given by $r \equiv \pi + \rho = \delta_1^T X$, where $\delta_1 = (0\ 1\ 1\ 0\ 0)^T$.

Substituting the partial derivatives of P into the PDE and using the method of undetermined coefficients, we end up with a system of Ordinary Differential Equations (ODEs) in $a(\tau)$ and $b(\tau)$:

$$\frac{\partial a}{\partial \tau} = \bar{\psi}^T b - \frac{1}{2}Tr(b^T S^2 b) - \Lambda^T S^2 b \tag{13}$$

and

$$\frac{\partial b}{\partial \tau} = \delta_1 + (K - \Xi)^T b \tag{14}$$

along with the corresponding initial conditions. These ODEs are solved numerically using a Runge-Kutta scheme.[2]

Under the risk-neutral measure Q, the absence of arbitrage opportunities dictates that the returns' dynamics of the zero-coupon nominal bond i in this setup are given by:

$$\frac{dP_i}{P_i} = rdt - b(\tau)^T S d\tilde{w} \tag{15}$$

Switching from the risk-neutral measure Q to the physical measure P, the bond returns' dynamics under the physical measure are given by the SDE:

$$\begin{aligned} \frac{dP_i}{P_i} &= (r - b(\tau)^T S\xi)dt - b(\tau)^T S dw \\ &= (r - b(\tau)^T S^2\Lambda - b(\tau)^T \Xi X)dt - b(\tau)^T S dw \end{aligned} \tag{16}$$

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[2]The ODEs in this study have been numerically solved in Mathematica 6 using the function **ndsolve**.

3 Data, estimation and implications

3.1 Data

In order to estimate the term structure model, Dewachter et al. (2006) make use of data for the yields on zero-coupon US Treasury bonds with maturities of 3 and 6 months and 1, 2, 5 and 10 years.[3] A quarterly frequency is adopted for the period 1964:Q1 to 1998:Q4. Inflation was constructed by taking the annual percentage change in the CPI index provided by the IMF and the output gap series were constructed based on data provided by the Congressional Budget Office. The filtered time series for the five factors are depicted in Figure 1 and Figure 2.[4] It is important to observe that there is variation in the underlying macroeconomic variables, so we expect that the investment opportunity set considerably varies through time.

-Figures 1 and 2 about here-

3.2 Estimation

In order to estimate both the dynamics of the underlying risk factors and the prices of risk in an efficient way, Dewachter et al. (2006) employ a Kalman filter algorithm.[5] We refer to Section 3 of their study for the proper definition of the measurement and the transition equation of the model as well as for its implications for the macroeconomic dynamics. The estimates of the parameters and their standard errors are given in Table 1. The results show that the market prices of risk are time-varying indeed.[6] Moreover, the volatilities of the risk factors are estimated to be relatively low, an important observation for the rest of the analysis.

-Table 1 about here-

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[3] The data are available on Gregory Duffee's website.

[4] We would like to thank Marco Lyrio for the provision of the data.

[5] The presence of unobservable factors dictates the use of a filtering procedure. Duffee and Stanton (2008) argue that for Gaussian models the Kalman filter yields more efficient estimates in comparison to the Efficient Method of Moments when the time series are highly persistent.

[6] It is important to note that, following the dynamic asset allocation literature, we use the point estimates of the coefficients. Even though the unrestricted coefficients of the matrix Ξ are significantly different from zero, this is not true for the constant component of the inflation and the output gap risk premia, i.e. Λ_y and Λ_π. Setting either parameter to zero could affect the analysis.

3.3 Implications for the bonds' returns

3.3.1 Nominal bonds' sensitivities to the risk factors

Given the estimates of the term structure model and the market prices of risk, we firstly examine the sensitivities of the bond returns to the underlying risk factors. This will help us understand how the expected returns and the volatilities of the bonds for various maturities are formed. In particular, we solve the ODEs in $b(\tau)$ given by equation (14) in order to derive the factor loadings. Figure 3 presents these loadings adjusted for the corresponding bond maturity, $(b(\tau)/\tau)$.

-Figure 3 about here-

The inflation and the real rate affect only the very short maturities. Practically, their impact on the nominal bonds with maturity beyond one year is negligible. Negligible is also the impact of the output gap, regardless of the bonds' maturity. On the other hand, the central tendency of inflation has a dominant impact on the prices of bonds with maturities longer than one year. The impact of the real rate central tendency is also significant for similar maturities.

It should be noted that Figure 3 presents the factor loadings adjusted for the corresponding maturity $(b(\tau)/\tau)$, while for the bond returns and their volatilities, it is the plain values of $b(\tau)$ that matter. As a result, it is mainly the inflation central tendency π^* that affects the nominal bond returns and volatilities for maturities longer than one year.

3.3.2 Expected excess bond returns

An extremely important input for any portfolio choice problem is the expected excess returns offered by the available assets, so it is interesting to examine the implications of this model for the expected excess returns of the nominal bonds for various maturities. In particular, given (16), these are given by $-b(\tau)^T S^2 \Lambda - b(\tau)^T \Xi X$. Figure 4 plots the expected excess returns of the nominal zero-coupon bonds with maturities 1, 3, 5, 7 and 10 years for the whole sample period.

-Figure 4 about here-

There are a series of observations to make: Firstly, there is considerable time-variation in the expected excess returns. This finding underlines the importance of the

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returns' predictability through a set of underlying risk factors that induces hedging demands in an investor's optimal portfolio choice. The expectations hypothesis that assumes constant term premia and has been used by Campbell and Viceira (2001) and Brennan and Xia (2002) for dynamic bond portfolio choice is clearly rejected, given the statistical significance of the time-varying risk coefficients in Ξ.

Secondly, this term structure model implies much more "reasonable" risk premia in comparison to a series of previously used models.[7] Thirdly, another attractive feature of this essentially affine term structure model is that the variation in the expected excess returns is significantly higher than their average level. As Duffee (2002) notes, this is a necessary qualification for a model to be consistent with the variety of yield curve shapes observed in the data. The commonly used completely affine models fail to meet this requirement and, as a result, they are less fit for the selection of optimal portfolios.

Furthermore, it should be noted that the excess returns tend to strongly co-move through time, despite the quite diverse economic conditions encountered in the sample period. We conjecture that this is true because the expected excess returns are mainly driven by the inflation central tendency.[8] For example, variations in the output gap have almost no effect on the bonds' returns. As a final comment it should be stressed that for the largest part of the sample period the 1-year zero-coupon bond has a relatively high expected excess return (higher than 0.5%). This fact crucially affects the formation of optimal portfolios.

3.3.3 The Covariance and Correlation structure of bonds' returns

The term structure literature usually neglects the models' implications for the covariance and correlation structure of the bonds' returns. However, this is an extremely important input for portfolio selection that is traditionally examined within a mean-variance framework. Given the coefficient estimates and the factor loadings $b(\tau)$, we report in Table 2 the covariances and correlations for the 1, 2, 3, 5, 7 and 10-year

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[7]We refer to the extreme premia reported in Sangvinatsos and Wachter (2005) as well as the ones implied by the model of Dewachter and Lyrio (2006).

[8]This observation is analogous to the result of Cochrane and Piazzesi (2005) that a single factor can explain much of the time variation in expected excess bond returns. It is also related to the known result of Litterman and Scheinkmann (1991) that the parallel shifts of the yield curve can be captured by a single factor.

nominal zero-coupon bonds' returns.

-Table 2 about here-

The volatilities of the returns are extremely low, especially for short maturities. An explanation for this result is that the factor that has a dominant role for bond pricing, the inflation central tendency, is estimated to have an extremely low volatility (see Table 1). Furthermore, this $A_0(5)$ term structure model does not allow for time-varying bond returns' volatilities and it ignores possible covariances between the various risk factors. According to Duffee (2002), the limited ability of the essentially affine models to capture the time variation in conditional variances is the price to pay for their superior forecasting power. This trade off is critically taken into account when we report our results.

With respect to the correlations of the returns, these are found to be extremely high. In particular, for bonds with similar maturities the correlation is higher than 0.95. Again, this characteristic significantly affects the formation of optimal portfolios, since including bonds of similar maturities in the asset menu will lead to a nearly singular covariance matrix, leading to extreme results.

3.3.4   The maximal Sharpe ratio

Given the market price of risk $\xi = S\Lambda + S^{-1}\Xi X$, an interesting way to examine the implications of the present term structure model for an investor is to derive the Maximal Sharpe Ratio (MSR), given by the norm of ξ, i.e. $MSR = \sqrt{\xi^T \xi}$. Figure 5 depicts the MSR for the sample period. The MSR is always positive, because it is assumed that the investor can take both long and short positions in any asset. There are two very important observations to make: Firstly, there are a series of periods where the (perceived) MSR is extremely high. As we discussed in the previous subsections, this finding is due to the combination of the relatively high expected excess returns and the very low volatilities implied by the estimated term structure model. This finding is essentially equivalent to a "bond premium puzzle", drawing the analogy to the equity premium puzzle of Mehra and Prescott (1985), since a high reward for bearing risk in the bond market implies highly volatile marginal utilities for the investors. Secondly, there is considerable time-variation in the investment opportunities. This fact underlines the importance of horizon effects and market timing in portfolio selection.

4 Portfolio choice in complete markets

4.1 Real wealth and access to nominal bonds

This section examines the optimal portfolio choice in a complete nominal bond market for a long-term investor, who takes into account the stochastic evolution of the underlying risk factors. Formally, the investor has to allocate his wealth W at time t among 5 zero-coupon default-free bonds with different maturities $(T_1, T_2,..., T_5)$, the returns of which are given in equation (16), and cash yielding the nominal instantaneously risk-free rate, $\frac{dP_0}{P_0} = rdt$. The portions of wealth allocated to each of the zero-coupon bonds are collected in the 5x1 vector $\phi = (\phi_1, \phi_2, ..., \phi_5)^T$, while the portion of the wealth invested in the instantaneously riskless asset is $\phi_0 = 1 - i^T\phi$, where i is a vector of ones.

We examine here the case where the investor's utility is defined over terminal real wealth. Given the bond returns' dynamics, the wealth dynamics of the investor are given by the following SDE:

$$dW = W(\phi^T \frac{dP}{P} + (1 - i^T\phi)\frac{dP_0}{P_0}) =$$
$$= W(\phi^T[-B(\tau)^T S^2\Lambda - B(\tau)^T\Xi X)] + r)dt - W\phi^T B(\tau)^T Sdw \qquad (17)$$

where now $\frac{dP}{P}$ is the vector of the 5 zero-coupon bond returns. Hence, $B(\tau)$ is now a matrix 5x5, since this stacks the column vectors $b(\tau)$ for each of the 5 bonds in a matrix form. In other words, $B(\tau) = (b(\tau_1), b(\tau_2), .., b(\tau_5))$, where $(\tau_1, \tau_2, ..., \tau_5)$ are the durations of the 5 zero-coupon bonds.

Since the vector of the shocks affecting the returns of the risky assets dw is the same as the vector of shocks affecting the underlying risk factor dynamics, we can use the Martingale approach of Cox and Huang (1989) and Karatzas, Lehoczky and Shreve (1987) to solve the optimal portfolio choice problem. The main observation is that there exists a unique pricing kernel m. This pricing kernel is a Stochastic Discount Factor (SDF) that converts the risky asset dynamics into a martingale process and it can be interpreted as a system of Arrow-Debreu prices. Given that

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ξ is the market price of risk in the complete nominal market, the dynamics of the nominal SDF are given by:

$$\frac{dm}{m} = -rdt - \xi^T dw \tag{18}$$

with initial value $m_{t_0} = 1$. The crucial observation for the martingale methodology is that the process $m_t W_t$ is a martingale too. As a result, $E_{t_0}[m_T W_T] = W_{t_0}$.

The long-term investor seeks to maximize

$$\max E_{t_0} \left\{ \frac{(\frac{W_T}{\Pi_T})^{1-\gamma}}{1-\gamma} \right\} \tag{19}$$

subject to the constraint

$$E_{t_0}[m_T W_T] = W_{t_0} \tag{20}$$

In order to examine this problem, we need to specify the dynamics of the price level Π_t. Following Brennan and Xia (2002), the price level dynamics are given by the SDE:

$$\frac{d\Pi}{\Pi} = \pi dt + \sigma_\Pi^T dw \tag{21}$$

where σ_Π is a 5x1 vector. In general, the shocks affecting the stochastic price level need not be perfectly correlated with the shocks to the underlying risk factors in X.[9] In our setup, however, we get this perfect correlation since one of the underlying risk factors is the inflation rate π. In other words, we can specify the vector σ_Π as $\sigma_\Pi = (0 \; \sigma_\pi \; 0 \; 0 \; 0)^T$, where σ_π is the diffusion coefficient of the inflation rate process specified in Section 2.1. As a result, we are in the "complete markets case" as termed by Brennan and Xia (2002), in the sense that the shocks to the bonds' returns are perfectly correlated with the shocks affecting the underlying risk factors including the price level.[10] Therefore, we can use the martingale methodology for this case too.

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[9]This case, which is equivalent to an incomplete market case, is examined by Sangvinatsos and Wachter (2005), where the underlying risk factors are latent factors with no clear macroeconomic interpretation and hence their shocks cannot be perfectly correlated with the shocks to the price level dynamics.

[10]As Brennan and Xia (2002, p. 1206) note, this is the case when "the expected rate of inflation π is not observable but must be inferred from the observation of the price level itself". This is exactly true in the Dewachter et al. (2006) term structure model that we are using, since the rate

Consequently, the dynamic portfolio choice problem becomes a static one. Forming the Lagrangian function:

$$L(W_T, l) = E_{t_0} \left\{ \frac{(\frac{W_T}{\Pi_T})^{1-\gamma}}{1-\gamma} \right\} - l[E_{t_0}[m_T W_T] - W_{t_0}] \tag{22}$$

and taking the First Order Condition (FOC) with respect to the terminal wealth we have:

$$\frac{\partial(.)}{\partial W_T} = 0 \Rightarrow W_T = (\Pi_T^{1-\gamma} l m_T)^{-\frac{1}{\gamma}} \Rightarrow \frac{W_T}{\Pi_T} = (l m_T \Pi_T)^{-\frac{1}{\gamma}} \tag{23}$$

where l is the Lagrange multiplier. Defining the variable $Z_t \equiv (l m_t \Pi_t)^{-1}$, real wealth at time t can be written as:

$$\frac{W_t}{\Pi_t} = (l m_t \Pi_t)^{-\frac{1}{\gamma}} = Z_t E_t[(Z_T)^{\frac{1}{\gamma}-1}] \tag{24}$$

As a result, wealth W is a function of Z, Π, X and t. Consequently, we can write it as $G(Z, \Pi, X, t) \equiv F(X, t) \Pi Z^{\frac{1}{\gamma}} = W(t)$.

Applying Ito's lemma, the dynamics of the variable Z are given by:

$$\frac{dZ}{Z} = (r - \pi + \sigma_\Pi^T \sigma_\Pi + \xi^T \xi - \sigma_\Pi^T \xi)dt + (\xi - \sigma_\Pi)^T dw \tag{25}$$

while under the risk neutral measure Q they are given by:

$$\frac{dZ}{Z} = (r - \pi + \sigma_\Pi^T \sigma_\Pi)dt + (\xi - \sigma_\Pi)^T d\tilde{w} \tag{26}$$

It should be also noted that the price level dynamics under the risk neutral measure Q are given by:

$$\frac{d\Pi}{\Pi} = (\pi - \sigma_\Pi^T \xi)dt + \sigma_\Pi^T d\tilde{w} \tag{27}$$

and the dynamics of X under Q are given by (11).

As Wachter (2002) argues, we can interpret wealth as a zero-coupon bond that pays a final amount W_T at time T. The no-arbitrage condition for a zero-coupon bond $G(Z, \Pi, X, t)$ implies that under the physical measure P, its instantaneous expected excess returns should be equal to the market price of risk multiplied by the diffusion of the wealth process G. It actually proves easier to work under the risk-neutral

of inflation is calculated as the annual percentage change in the CPI.

measure Q. In this case, the instantaneous expected excess returns should be equal to zero, since the market price of risk is zero under Q.

This argument implies that the following PDE should hold:

$$\frac{\partial G}{\partial Z}Z(r-\pi+\sigma_\Pi^T\sigma_\Pi)+[\bar{\psi}-S^2\Lambda+(K-\Xi)X]^T\frac{\partial G}{\partial X}+\frac{\partial G}{\partial \Pi}\Pi(\pi-\sigma_\Pi^T\xi)+$$
$$+\frac{\partial G}{\partial t}+\frac{1}{2}Tr(SS^T\frac{\partial^2 G}{\partial X\partial X^T})+\frac{1}{2}\frac{\partial^2 G}{\partial Z^2}Z^2(\xi-\sigma_\Pi)^T(\xi-\sigma_\Pi)+\frac{1}{2}\frac{\partial^2 G}{\partial \Pi^2}\Pi^2\sigma_\Pi^T\sigma_\Pi+$$
$$+(S\sigma_\Pi)^T\Pi\frac{\partial^2 G}{\partial \Pi\partial X}+\frac{\partial^2 G}{\partial Z\partial \Pi}\Pi Z(\xi-\sigma_\Pi)^T\sigma_\Pi+Z(\xi-\sigma_\Pi)^T S\frac{\partial^2 G}{\partial Z\partial X}=rG \qquad (28)$$

along with the terminal condition $G(Z_T,X_T,\Pi_T,T)=\Pi_T Z_T^{\frac{1}{\gamma}}$.

It is also crucial to note that the optimally invested wealth should have a diffusion term identical to the diffusion term of the wealth process G. Therefore, for the optimal portfolio choice ϕ, it should hold that:

$$G(\phi^T(-B^T S)=\frac{\partial G}{\partial Z}Z(\xi-\sigma_\Pi)^T+\frac{\partial G}{\partial \Pi}\Pi\sigma_\Pi^T+(S\frac{\partial G}{\partial X})^T \qquad (29)$$

Proposition 3.1

Let us conjecture the following form for the function $G(Z,X,\Pi,t)$:

$$G(Z,X,\Pi,t)=\Pi_t Z_t^{\frac{1}{\gamma}}F(X,t)=\Pi_t Z_t^{\frac{1}{\gamma}}\exp[\frac{1}{\gamma}(\frac{1}{2}X^T Q(t)X+d(t)^T X+c(t))] \qquad (30)$$

where $Q(t)$, $d(t)$, $c(t)$ are 5x5 matrix, 5x1 vector and scalar functions of time correspondingly with terminal conditions $Q(T)=\mathbf{0}_{5x5}$, $d(T)=\mathbf{0}_{5x1}$ and $c(T)=0$.

For an investor who has a power utility over real terminal wealth with coefficient of relative risk aversion $\gamma\neq 1$ and has access to nominal bonds, his optimal portfolio choice is given by:

$$\phi_t=\frac{1}{\gamma}(B^T S^2 B)^{-1}(-B^T S^2\Lambda-B^T\Xi X_t)+(1-\frac{1}{\gamma})(B^T S^2 B)^{-1}(-B^T S)\sigma_\Pi$$
$$+\frac{1}{\gamma}(B^T S^2 B)^{-1}(-B^T S)S[d(t)+\frac{1}{2}(Q(t)+Q(t)^T)X_t] \qquad (31)$$

with the functions $d(t)$ and $Q(t)$ satisfying the system of ODEs (80) - (81) given in the Appendix. The remainder $\phi_0=1-i^T\phi$ is invested in the nominal instantaneously riskless asset (see Appendix for the proof).

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The first two terms of the optimal portfolio choice expression (31) compose the myopic term *à la* Markowitz (1952). In particular, the second term arises because the investor seeks to maximize his utility over real wealth having access to nominal bonds that are priced under the nominal SDF. The third term provides the hedging demand component *à la* Merton (1973). There are two interesting observations to make:

i) The long-term power utility investor has a hedging demand apart from the standard mean-variance myopic one. This depends on the diffusion coefficients of the bond returns' dynamics as well as on the sensitivity of the investor's wealth to the risk factors, represented here by the functions $d(t)$ and $Q(t)$. It is straightforward to observe that, if the investor is not sensitive to any shifts in the risk factors, then there is no intertemporal hedging demand component, since $\frac{\partial G}{\partial X} = \mathbf{0}$.

ii) Both the myopic and the hedging bond demand components are characterized by market timing, i.e. the optimal portfolio choice depends on the current level of the risk factors. The reason for this behaviour is that bond returns are predictable through the risk factors in X; as a result, the investor needs to take this information into account for investment as well as for hedging purposes.

4.2 Introducing real bonds

If the investor is interested in maximizing the expected utility of his real wealth, then there is the scope of adding real bonds in the asset space. The term structure model we have employed allows us to introduce and price real bonds in a convenient way. In this subsection we price real zero-coupon bonds using the real SDF. Starting from the dynamics of the nominal SDF m:

$$\frac{dm}{m} = -rdt - \xi^T dw \tag{32}$$

and applying Ito's lemma to the function $M = m\Pi$, we can find the dynamics of the real SDF M. In particular, these are given by the SDE:

$$\frac{dM}{M} = \frac{d(m\Pi)}{m\Pi} = -(r - \pi + \sigma_\Pi^T \xi)dt - (\xi - \sigma_\Pi)^T dw \tag{33}$$

There are two observations to make, given the dynamics of the real SDF. Firstly, the real instantaneously risk free rate is given by $r - \pi + \sigma_\Pi^T \xi$. This is distinct from the conventionally characterized as real rate $\rho = r - \pi$. The difference is the extra

term, $\sigma_\Pi^T \xi$, that arises due to the stochastic evolution of the price level. Secondly, the market price of risk under the real SDF is also modified and it is given by $\xi - \sigma_\Pi$.

It is assumed that the price of a real zero-coupon bond, P^R, is an affine function of the underlying risk factors in X and time t. In particular, the price of the real bond is given by:

$$P^R(t, X) = \exp(-a^R(\tau) - b^R(\tau)^T X) \tag{34}$$

where $a^R(\tau)$ is a scalar and $b^R(\tau)$ is a 5x1 vector, with initial conditions $a^R(0) = 0$ and $b^R(0) = \mathbf{0}_{5x1}$.

Under the risk neutral measure Q, the no-arbitrage condition for a real zero-coupon bond states that its expected excess returns over the real instantaneously risk free rate, $r - \pi + \sigma_\Pi^T \xi$, should be equal to zero. It should be noted that under the real SDF, switching from the risk-neutral measure Q to the physical measure P, it holds that $d\tilde{w} = dw + (\xi - \sigma_\Pi)dt$. Therefore, the dynamics of X under Q are now given by:

$$dX = [\bar{\psi} - S^2\Lambda + S\sigma_\Pi + (K - \Xi)X]dt + Sd\tilde{w} \tag{35}$$

Therefore, the no-arbitrage condition for the real bond prices under Q leads to the following equation:

$$(\bar{\psi} - S^2\Lambda + S\sigma_\Pi + (K - \Xi)X)^T \frac{\partial P^R}{\partial X} + \frac{1}{2}\frac{\partial^2 P^R}{\partial X \partial X^T}S^2 + \frac{\partial P^R}{\partial t} = (r - \pi + \sigma_\Pi^T \xi)P^R \tag{36}$$

Substituting the partial derivatives of P^R in the PDE and using the method of undetermined coefficients, we get a system of ODEs in $a^R(\tau)$ and $b^R(\tau)$:

$$\frac{\partial a^R}{\partial \tau} = \bar{\psi}^T b^R - \frac{1}{2}Tr(b^{RT}S^2 b^R) - \Lambda^T S^2 b^R + \sigma_\Pi^T S\Lambda + \sigma_\Pi^T S^T b^R \tag{37}$$

and

$$\frac{\partial b^R}{\partial \tau} = (\delta_1 - \delta_2) + (K - \Xi)^T b^R + \Xi^T S^{-1}\sigma_\Pi \tag{38}$$

where $\delta_2 = (0\ 1\ 0\ 0\ 0)^T$, along with the corresponding initial conditions.

Under the risk-neutral measure Q, the absence of arbitrage opportunities dictates that the returns' dynamics of the real zero-coupon bond i in this setup are given by:

$$\frac{dP_i^R}{P_i^R} = (r - \pi + \sigma_\Pi^T \xi)dt - b^R(\tau)^T S d\tilde{w} \tag{39}$$

Consequently, under the physical measure P, the bond returns dynamics are given by:

$$\frac{dP_i^R}{P_i^R} = (r - \pi + \sigma_\Pi^T \xi - b^R(\tau)^T S\xi + b^R(\tau)^T S\sigma_\Pi)dt - b^R(\tau)^T S dw \tag{40}$$

Figure 6 shows the factor loadings $b^R(\tau)$, adjusted for the corresponding maturity τ, i.e. $b^R(\tau)/\tau$. In comparison to the corresponding factor loadings of the nominal bonds, the main difference is that both the inflation and the output gap have an important effect on the real bond prices and returns for short maturities. The real rate ρ affects only the very short maturities. As in the case of nominal bonds, the inflation central tendency is the dominant factor for the real bond prices. Nevertheless, its impact is now less pronounced for longer maturities that are also slightly more affected by the real rate central tendency.

-Figure 6 about here-

Figure 7 shows the expected excess returns of the real bonds over the real risk free rate, given by $-b^R(\tau)^T S^2 \Lambda - b^R(\tau)^T \Xi X + b^R(\tau)^T S\sigma_\Pi$. It is interesting to observe that the real bonds with long maturities exhibit negative expected excess returns for a portion of the examined period. In general, for a large portion of the examined period, the 1-year and the 3-year real zero-coupon bonds are characterized by higher expected excess returns in comparison to the 7-year and the 10-year bonds. Table 3 presents the covariance and the correlation structure of the real bonds' returns. The results document the very low volatilities and the very high correlations of the returns.

-Table 3 and Figure 7 about here-

Some of the cases that we examine in the subsequent empirical results include both nominal and real bonds in the asset menu of the long-term investor. For consistency, both types of bonds should be expressed under the same SDF (either nominal or real). When we deal with this case, we choose to express the real bonds' returns dynamics under the nominal SDF. To this end, we need to apply Ito's lemma to the function

$(P^R \Pi)$. It should be reminded that the real bonds' returns dynamics under the real SDF are given by equation (40). Applying Ito's lemma to the function $(P^R \Pi)$, we find the dynamics of the real bonds' returns under the nominal SDF. In particular, these are given by:

$$\frac{d(P_i^R \Pi)}{(P_i^R \Pi)} = (r + \sigma_\Pi^T \xi - b^R(\tau)^T S \xi)dt - (b^R(\tau)^T S - \sigma_\Pi^T)dw \tag{41}$$

Consequently, when we examine the optimal portfolio choice of the long-term investor in the presence of both real and nominal zero-coupon bonds, the previous expression for the real bond returns' dynamics under the nominal SDF is used. For example, the diffusion term of a real bond's returns under the nominal SDF, given by $-(b^R(\tau)^T S - \sigma_\Pi^T)$, is employed in the expression of the optimal portfolio choice given in equation (31) instead of the corresponding diffusion term of the nominal bond's returns dynamics, given by $-b(\tau)^T S$.

4.3 Real wealth and access to real bonds

In this subsection, we examine the case of an investor who maximizes utility over terminal real wealth, when he has access to a complete real bond market. The real wealth dynamics of this investor are given by:

$$d(\frac{W}{\Pi}) = \frac{W}{\Pi}(\phi^{RT}[-B^R(\tau)^T S(\xi - \sigma_\Pi)] + r - \pi + \sigma_\Pi^T \xi)dt + \frac{W}{\Pi}\phi^{RT}(-B^R(\tau)^T S)dw \tag{42}$$

where ϕ^R is the vector of the portions of his real wealth $\frac{W}{\Pi}$ allocated to each of the 5 available real zero-coupon bonds and $B^R(\tau) = (b^R(\tau_1), b^R(\tau_2), .., b^R(\tau_5))$.

The crucial observation is that using the real pricing kernel M, the process $M_t \frac{W_t}{\Pi_t}$ is a martingale too. As a result, we get the static budget constraint $E_{t_0}[M_T \frac{W_T}{\Pi_T}] = \frac{W_{t_0}}{\Pi_{t_0}}$. The investor seeks to maximize (19) subject to this constraint. Forming the corresponding Lagrangian function and taking the FOC with respect to the terminal real wealth we have:

$$\frac{\partial(.)}{\partial(W_T/\Pi_T)} = 0 \Rightarrow \frac{W_T}{\Pi_T} = (lM_T)^{-\frac{1}{\gamma}} \tag{43}$$

where l is the corresponding Lagrange multiplier.

We define the variable $Z_t^R \equiv (lM_t)^{-1}$, hence real wealth at time t can be written as:

$$\frac{W_t}{\Pi_t} = (lM_t)^{-\frac{1}{\gamma}} = Z_t^R E_t[(Z_T^R)^{\frac{1}{\gamma}-1}] \tag{44}$$

Given the definition of the real SDF, $M = m\Pi$, one may note that $Z_t^R = Z_t$. Therefore, the dynamics of Z_t^R are given by (25) under the physical measure P. However, since the market price of risk under the real SDF is $(\xi - \sigma_\Pi)$, the dynamics of Z_t^R under the risk neutral measure Q are now given by:

$$\frac{dZ^R}{Z^R} = (r - \pi + \sigma_\Pi^T \xi)dt + (\xi - \sigma_\Pi)^T d\tilde{w} \tag{45}$$

and the dynamics of X under Q are given by (35).

Real wealth $\frac{W}{\Pi}$ is a function of Z^R, X and t. Consequently, we can define $G^R(Z^R, X, t) \equiv F^R(X, t)(Z^R)^{\frac{1}{\gamma}} = \frac{W_t}{\Pi_t}$, with the terminal condition $G^R(Z_T^R, X_T, T) = (Z_T^R)^{\frac{1}{\gamma}}$. Consequently, real wealth in this case can be interpreted as a real zero-coupon bond paying a final amount at time T. This means that under the risk neutral measure, its expected excess returns (over the real risk free rate) should be equal to zero. In other words, G^R should satisfy under Q:

$$\frac{\partial G^R}{\partial Z^R} Z^R(r - \pi + \sigma_\Pi^T \xi) + [\bar{\psi} - S^2\Lambda + S\sigma_\Pi + (K - \Xi)X]^T \frac{\partial G^R}{\partial X} +$$
$$+\frac{\partial G^R}{\partial t} + \frac{1}{2}Tr(SS^T \frac{\partial^2 G^R}{\partial X \partial X^T}) + \frac{1}{2}\frac{\partial^2 G^R}{(\partial Z^R)^2}(Z^R)^2(\xi - \sigma_\Pi)^T(\xi - \sigma_\Pi) +$$
$$+Z^R(\xi - \sigma_\Pi)^T S \frac{\partial^2 G^R}{\partial Z^R \partial X} = (r - \pi + \sigma_\Pi^T \xi)G^R \tag{46}$$

Moreover, the optimally invested real wealth should have a diffusion term identical to the diffusion term of the real wealth process G^R. Therefore, the optimal portfolio choice ϕ^R should satisfy:

$$G^R \phi^{RT}(-B^{RT}S) = \frac{\partial G^R}{\partial Z^R} Z^R(\xi - \sigma_\Pi)^T + (S\frac{\partial G^R}{\partial X})^T \tag{47}$$

Proposition 3.2

Let us conjecture the following form for the function $G^R(Z^R, X, t)$:

$$G^R(Z^R, X, t) = (Z_t^R)^{\frac{1}{\gamma}} F^R(X, t) = (Z_t^R)^{\frac{1}{\gamma}} \exp[\frac{1}{\gamma}(\frac{1}{2}X^T Q^R(t)X + d^R(t)^T X + c^R(t))] \tag{48}$$

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where $Q^R(t)$, $d^R(t)$, $c^R(t)$ are 5x5 matrix, 5x1 vector and scalar functions of time correspondingly with terminal conditions $Q^R(T) = \mathbf{0}_{5x5}$, $d^R(T) = \mathbf{0}_{5x1}$ and $c^R(T) = 0$.

For an investor who has a power utility over real terminal wealth with coefficient of relative risk aversion $\gamma \neq 1$ and has access to real bonds, his optimal portfolio choice is given by:

$$\phi_t^R = \frac{1}{\gamma}(B^{RT}S^2B^R)^{-1}(-B^{RT}S^2\Lambda - B^{RT}\Xi X_t + B^{RT}S\sigma_\Pi)$$

$$+\frac{1}{\gamma}(B^{RT}S^2B^R)^{-1}(-B^{RT}S)S[d^R(t) + \frac{1}{2}(Q^R(t) + Q^R(t)^T)X_t] \qquad (49)$$

with the functions $d^R(t)$ and $Q^R(t)$ satisfying the system of ODEs (80) - (81). The remainder $\phi_0^R = 1 - i^T\phi^R$ is invested in the instantaneously real riskless asset (see Appendix for the proof).

5 Portfolio choice in incomplete markets

In this section, we examine the optimal portfolio choice of the long-term investor with a power utility function over terminal wealth in an incomplete bond market. The market incompleteness arises in our study only when the investor has access to fewer bonds in comparison to the number of the underlying risk factors. Therefore, in an incomplete nominal bond market, $B(\tau)$ is a 5xn matrix with $n < 5$ and in an incomplete real bond market, $B^R(\tau)$ is a 5xn matrix with $n < 5$.

5.1 Access to nominal bonds

In an incomplete nominal bond market, any price of risk $\bar{\xi}$ can be written as:

$$\bar{\xi} = (-B^TS)^T(B^TS^2B)^{-1}(-B^TS)\bar{\xi} + (\bar{\xi} - (-B^TS)^T(B^TS^2B)^{-1}(-B^TS)\bar{\xi}) \qquad (50)$$

The first term of this expression, $(-B^TS)^T(B^TS^2B)^{-1}(-B^TS)\bar{\xi} = \xi$ is the unique price of risk that both prices and it is spanned by the available assets, i.e. it is the price of risk in the complete market case that we examined in the previous section. The second term $\bar{\xi} - (-B^TS)^T(B^TS^2B)^{-1}(-B^TS)\bar{\xi} \equiv v$ is the residual of the projection

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of $\bar{\xi}$ onto the available assets and lies in the null space of $(-B^T S)$, i.e. it holds that $(-B^T S)v = \mathbf{0}$. Note that in a complete market $v = \mathbf{0}$.

Consequently, in the incomplete market case, the nominal SDF m^v is associated with the price of risk $\bar{\xi} = \xi + v$. The dynamics of m^v are given by:

$$\frac{dm^v}{m^v} = -rdt - (\xi + v)^T dw \tag{51}$$

In general, the budget constraint $E_{t_0}[m_T^v W_T] = W_{t_0}$ should be satisfied by any pricing kernel m^v. However, we cannot optimize with respect to this budget constraint because it is not possible to replicate the resulting process for wealth by trading in the underlying assets, since we are in an incomplete market. He and Pearson (1991) argue that it is sufficient to verify this budget constraint for a single pricing kernel m^{v*}. This is termed as the minimax pricing kernel because it is the kernel that minimizes the agent's maximized utility. In other words, we can solve the portfolio choice problem in the incomplete market case as in the complete market case by "adding" the assets that are necessary to complete the market and at the same time modifying their returns' process in a way that their optimal portfolio weights are zero. This is accomplished by the specific minimax kernel m^{v*}, because it guarantees that the "added" assets have such properties that the investor does not want to trade them.

Hence, the pricing kernel we use follows the SDE:

$$\frac{dm^{v*}}{m^{v*}} = -rdt - (\xi + v^*)^T dw \tag{52}$$

The solution of the optimal portfolio choice problem follows the same steps as in the previous section. The agent seeks to maximize:

$$\max E_{t_0} \left\{ \frac{(\frac{W_T}{\Pi_T})^{1-\gamma}}{1-\gamma} \right\} \tag{53}$$

subject to the constraint

$$E_{t_0}[m_T^{v*} W_T] = W_{t_0} \tag{54}$$

The nominal wealth dynamics are the same as in (17). The FOC of the problem is:

$$\frac{\partial(.)}{\partial W_T} = 0 \Rightarrow W_T = (\Pi_T^{1-\gamma} lm_T^{v^*})^{-\frac{1}{\gamma}} \Rightarrow \frac{W_T}{\Pi_T} = (lm_T^{v^*}\Pi_T)^{-\frac{1}{\gamma}} \tag{55}$$

where l is the corresponding Lagrange multiplier.

Defining the variable $Z_t^{v^*} \equiv (lm_t^{v^*}\Pi_t)^{-1}$, real wealth at time t can be written as:

$$\frac{W_t}{\Pi_t} = (lm_t^{v^*}\Pi_t)^{-\frac{1}{\gamma}} = Z_t^{v^*} E_t[(Z_T^{v^*})^{\frac{1}{\gamma}-1}] \tag{56}$$

As a result, wealth W_t is a function of $Z_t^{v^*}$, Π, X and t. Consequently, we can define the function $G^I(Z^{v^*},\Pi, X,t) \equiv F^I(X,t)\Pi(Z^{v^*})^{\frac{1}{\gamma}} = W(t)$.

Applying Ito's lemma, the dynamics of the variable Z^{v^*} are given by:

$$\frac{dZ^{v^*}}{Z^{v^*}} = (r - \pi + \sigma_\Pi^T\sigma_\Pi + (\xi + v^*)^T(\xi + v^*) - \sigma_\Pi^T(\xi + v^*))dt + (\xi + v^* - \sigma_\Pi)^T dw \tag{57}$$

while, under the risk neutral measure Q, they are given by:

$$\frac{dZ^{v^*}}{Z^{v^*}} = (r - \pi + \sigma_\Pi^T\sigma_\Pi)dt + (\xi + v^* - \sigma_\Pi)^T d\tilde{w} \tag{58}$$

It should be also noted that the price level dynamics under the risk neutral measure Q are given by:

$$\frac{d\Pi}{\Pi} = (\pi - \sigma_\Pi^T(\xi + v^*))dt + \sigma_\Pi^T d\tilde{w} \tag{59}$$

while the corresponding dynamics of X are given by:

$$dX = [\bar{\psi} + KX - S(\xi + v^*)]dt + S d\tilde{w} \tag{60}$$

We can interpret again nominal wealth as a zero-coupon bond that pays a final amount W_T at time T. The no-arbitrage condition for a zero-coupon bond $G^I(Z^{v^*}, X, \Pi, t)$ under the risk-neutral measure implies that its instantaneous expected excess returns should be equal to zero. Following this observation, we get the

PDE:

$$\frac{\partial G^I}{\partial Z^{v^*}}Z^{v^*}(r-\pi+\sigma_\Pi^T\sigma_\Pi)+[\bar{\psi}+KX-S(\xi+v^*)]^T\frac{\partial G^I}{\partial X}$$
$$+\frac{\partial G^I}{\partial \Pi}\Pi(\pi-\sigma_\Pi^T(\xi+v^*))+\frac{\partial G^I}{\partial t}+\frac{1}{2}Tr(SS^T\frac{\partial^2 G^I}{\partial X\partial X^T})+\frac{1}{2}\frac{\partial^2 G^I}{\partial \Pi^2}\Pi^2\sigma_\Pi^T\sigma_\Pi$$
$$+\frac{1}{2}\frac{\partial^2 G^I}{(\partial Z^{v^*})^2}(Z^{v^*})^2(\xi+v^*-\sigma_\Pi)^T(\xi+v^*-\sigma_\Pi)+(S\sigma_\Pi)^T\Pi\frac{\partial^2 G^I}{\partial \Pi\partial X}$$
$$+\frac{\partial^2 G^I}{\partial Z^{v^*}\partial \Pi}\Pi Z^{v^*}(\xi+v^*-\sigma_\Pi)^T\sigma_\Pi+Z^{v^*}(\xi+v^*-\sigma_\Pi)^TS\frac{\partial^2 G^I}{\partial Z^{v^*}\partial X}=rG^I \qquad (61)$$

with the terminal condition $G^I(Z_T^{v^*},X_T,\Pi_T,T)=\Pi_T(Z_T^{v^*})^{\frac{1}{\gamma}}$.

Moreover, the optimally invested wealth should have a diffusion term equal to the diffusion term of the wealth process G^I. Therefore, the optimal portfolio choice ϕ^I should satisfy:

$$G^I(\phi^I)^T(-B^TS)=\frac{\partial G^I}{\partial Z^{v^*}}Z^{v^*}(\xi+v^*-\sigma_\Pi)^T+\frac{\partial G^I}{\partial \Pi}\Pi\sigma_\Pi^T+(S\frac{\partial G^I}{\partial X})^T \qquad (62)$$

Proposition 3.3

Let us conjecture the following form for the function $G^I(Z^{v^*},X,\Pi,t)$:

$$G^I(Z^{v^*},X,\ \Pi,\ t)=\Pi_t(Z_t^{v^*})^{\frac{1}{\gamma}}F^I(X,t)=\Pi_t(Z_t^{v^*})^{\frac{1}{\gamma}}\exp[\frac{1}{\gamma}(\frac{1}{2}X^TQ^I(t)X+d^I(t)^TX+c^I(t))] \qquad (63)$$

where $Q^I(t)$, $d^I(t)$, $c^I(t)$ are 5x5 matrix, 5x1 vector and scalar functions of time correspondingly, with terminal conditions $Q^I(T)=\mathbf{0}_{5x5}$, $d^I(T)=\mathbf{0}_{5x1}$ and $c^I(T)=0$.

For an investor who has a power utility over real terminal wealth with coefficient of relative risk aversion $\gamma\neq 1$ and has access to an incomplete nominal bond market, the minimax pricing kernel is given by:

$$v^*=(1-\gamma)[(\sigma_\Pi^T)^\perp]^T-(S^\perp)^T[d^I+\frac{1}{2}(Q^I+(Q^I)^T)X] \qquad (64)$$

where $(\sigma_\Pi^T)^\perp$ is the residual of the projection of σ_Π^T onto the traded assets, $(\sigma_\Pi^T)^\perp=\sigma_\Pi^T-\sigma_\Pi^T(-B^TS)^T(B^TS^2B)^{-1}(-B^TS)$ and $(S^\perp)$ is the residual of the projection of S onto the traded assets, $S^\perp=S-S(-B^TS)^T(B^TS^2B)^{-1}(-B^TS)$.

Moreover, his optimal portfolio choice is given by:

$$\phi_t^I = \frac{1}{\gamma}(B^T S^2 B)^{-1}(-B^T S^2 \Lambda - B^T \Xi X_t) + (1 - \frac{1}{\gamma})(B^T S^2 B)^{-1}(-B^T S)\sigma_\Pi$$
$$+ \frac{1}{\gamma}(B^T S^2 B)^{-1}(-B^T S)S[d^I(t) + \frac{1}{2}(Q^I(t) + Q^I(t)^T)X_t] \qquad (65)$$

with the functions $d^I(t)$ and $Q^I(t)$ satisfying the system of ODEs in (88)-(89) given in the Appendix. The remainder $\phi_0^I = 1 - i^T \phi^I$ is invested in the nominal instantaneously riskless asset (see Appendix for the proof).

5.2   Access to real bonds

The derivation of the optimal asset allocation for a long-term investor with power utility over terminal wealth in an incomplete real bond market follows the same steps. Any price of risk $\overline{\xi - \sigma_\Pi}$ can be written as:

$$\overline{\xi - \sigma_\Pi} = (-B^{RT} S)^T (B^{RT} S^2 B^R)^{-1}(-B^{RT} S)(\overline{\xi - \sigma_\Pi}) +$$
$$+ ((\overline{\xi - \sigma_\Pi}) - (-B^{RT} S)^T (B^{RT} S^2 B^R)^{-1}(-B^{RT} S)(\overline{\xi - \sigma_\Pi})) \qquad (66)$$

The first term of this expression:

$$(-B^{RT} S)^T (B^{RT} S^2 B^R)^{-1}(-B^{RT} S)(\overline{\xi - \sigma_\Pi}) = \xi - \sigma_\Pi$$

is the unique price of risk that both prices and it is spanned by the available real assets, i.e. it is the price of risk in the complete real market case that we examined in the previous section. The second term:

$$(\overline{\xi - \sigma_\Pi}) - (-B^{RT} S)^T (B^{RT} S^2 B^R)^{-1}(-B^{RT} S)(\overline{\xi - \sigma_\Pi}) \equiv V$$

is the residual of the projection of the price of risk onto the available real bonds and lies in the null space of $(-B^{RT} S)$, i.e. it holds that $(-B^{RT} S)V = \mathbf{0}$.

Consequently, in the incomplete market case, the real SDF, M^V, is associated with the price of risk $(\overline{\xi - \sigma_\Pi}) = \xi - \sigma_\Pi + V$. Following the same argument of He and Pearson (1991), we seek to find the minimax pricing kernel M^{V^*}, which guarantees that the real bonds "added" to complete the market have such properties that the investor does not want to trade them.

The pricing kernel we use follows the SDE:

$$\frac{dM^{V^*}}{M^{V^*}} = -(r - \pi + \sigma_\Pi^T(\xi + V^*))dt - (\xi + V^* - \sigma_\Pi)^T dw \qquad (67)$$

The agent seeks to maximize:

$$\max E_{t_0}\left\{ \frac{(\frac{W_T}{\Pi_T})^{1-\gamma}}{1-\gamma} \right\} \qquad (68)$$

subject to the constraint $E_{t_0}[M_T^{V^*}\frac{W_T}{\Pi_T}] = \frac{W_{t_0}}{\Pi_{t_0}}$. Forming the corresponding Lagrangian function and taking the FOC with respect to terminal real wealth, we have:

$$\frac{\partial(.)}{\partial(W_T/\Pi_T)} = 0 \Rightarrow \frac{W_T}{\Pi_T} = (lM_T^{V^*})^{-\frac{1}{\gamma}} \qquad (69)$$

where l is the corresponding Lagrange multiplier. We define the variable $Z_t^{V^*} \equiv (lM_t^{V^*})^{-1}$, hence real wealth at time t can be written as:

$$\frac{W_t}{\Pi_t} = (lM_t^{V^*})^{-\frac{1}{\gamma}} = Z_t^{V^*} E_t[(Z_T^{V^*})^{\frac{1}{\gamma}-1}] \qquad (70)$$

The dynamics of $Z_t^{V^*}$ are given by (57) under the physical measure P. However, since the market price of risk under the real SDF is $(\xi + V^* - \sigma_\Pi)$, the dynamics of $Z_t^{V^*}$ under the risk neutral measure Q are now given by:

$$\frac{dZ_t^{V^*}}{Z_t^{V^*}} = (r - \pi + \sigma_\Pi^T(\xi + V^*))dt + (\xi + V^* - \sigma_\Pi)^T d\tilde{w} \qquad (71)$$

and the dynamics of X under Q are given by:

$$dX = [\bar{\psi} + KX - S(\xi + V^* - \sigma_\Pi)]dt + Sd\tilde{w} \qquad (72)$$

Real wealth $\frac{W}{\Pi}$ is a function of $Z_t^{V^*}$, X and t. Consequently, we can define

$$G^{IR}(Z^{V^*}, X, t) \equiv F^{IR}(X, t)(Z^{V^*})^{\frac{1}{\gamma}} = \frac{W_t}{\Pi_t},$$

with the terminal condition $G^{IR}(Z_T^{V^*}, X_T, T) = (Z_T^{V^*})^{\frac{1}{\gamma}}$. Consequently, real wealth in this case can be interpreted as a real zero-coupon bond paying a final amount at time T. This interpretation implies that under the risk neutral measure, its expected excess returns (over the real risk free rate) should be equal to zero. In other words, G^{IR} should satisfy under Q:

$$\frac{\partial G^{IR}}{\partial Z^{V^*}} Z^{V^*}(r - \pi + \sigma_\Pi^T(\xi + V^*)) + [\bar{\psi} + KX - S(\xi + V^* - \sigma_\Pi)]^T \frac{\partial G^{IR}}{\partial X} +$$

$$+ \frac{\partial G^{IR}}{\partial t} + \frac{1}{2} Tr(SS^T \frac{\partial^2 G^{IR}}{\partial X \partial X^T}) + \frac{1}{2} \frac{\partial^2 G^{IR}}{(\partial Z^{V^*})^2} (Z^{V^*})^2 (\xi + V^* - \sigma_\Pi)^T(\xi + V^* - \sigma_\Pi)$$

$$+ Z^{V^*}(\xi + V^* - \sigma_\Pi)^T S \frac{\partial^2 G^{IR}}{\partial Z^{V^*} \partial X} = (r - \pi + \sigma_\Pi^T(\xi + V^*))G^{IR} \qquad (73)$$

Moreover, the optimally invested wealth should have a diffusion term equal to the diffusion term of the real wealth process G^{IR}. Therefore, the optimal portfolio choice ϕ^{IR} should satisfy:

$$G^{IR}(\phi^{IR})^T(-B^{RT}S) = \frac{\partial G^{IR}}{\partial Z^{V^*}} Z^{V^*}(\xi + V^* - \sigma_\Pi)^T + (S \frac{\partial G^{IR}}{\partial X})^T \qquad (74)$$

Proposition 3.4

Let us conjecture the following form for the function $G^{IR}(Z^{V^*}, X, t)$:

$$G^{IR}(Z^{V^*}, X, t) = (Z^{V^*})^{\frac{1}{\gamma}} F^{IR}(X, t) = (Z^{V^*})^{\frac{1}{\gamma}} \exp[\frac{1}{\gamma}(\frac{1}{2} X^T Q^{IR}(t)X + d^{IR}(t)^T X + c^{IR}(t))] \qquad (75)$$

where $Q^{IR}(t)$, $d^{IR}(t)$, $c^{IR}(t)$ are 5x5 matrix, 5x1 vector and scalar functions of time correspondingly, with terminal conditions $Q^{IR}(T) = \mathbf{0}_{5x5}$, $d^{IR}(T) = \mathbf{0}_{5x1}$ and $c^{IR}(T) = 0$.

For an investor who has a power utility over real terminal wealth with coefficient of relative risk aversion $\gamma \neq 1$ and has access to an incomplete real bond market, the minimax pricing kernel is given by:

$$V^* = -(S^\perp)^T[d^{IR} + \frac{1}{2}(Q^{IR} + (Q^{IR})^T)X] \qquad (76)$$

where $(S^\perp)$ is the residual of the projection of S onto the traded real bonds, $S^\perp = S - S(-B^{RT}S)^T(B^{RT}S^2 B^R)^{-1}(-B^{RT}S)$.

Moreover, his optimal portfolio choice is given by:

$$\phi_t^{IR} = \frac{1}{\gamma}(B^{RT}S^2 B^R)^{-1}(-B^{RT}S^2\Lambda - B^{RT}\Xi X_t + B^{RT}S\sigma_\Pi)$$

$$+ \frac{1}{\gamma}(B^{RT}S^2 B^R)^{-1}(-B^{RT}S)S[d^{IR}(t) + \frac{1}{2}(Q^{IR}(t) + Q^{IR}(t)^T)X_t] \qquad (77)$$

with the functions $d^{IR}(t)$ and $Q^{IR}(t)$ satisfying the system of ODEs (94)-(95). The remainder, $\phi_0^{IR} = 1 - i^T \phi^{IR}$, is invested in the instantaneously real riskless asset (see Appendix for the proof).

6 Results

This section reports the optimal portfolio choices of an investor who maximizes his utility over real terminal wealth for various degrees of relative risk aversion and investment horizons. The first subsection refers to the case of a complete bond market while the second subsection refers to the case of an incomplete bond market, where the incompleteness arises because the available bonds are fewer than the number of the underlying risk factors.

Following Sangvinatsos and Wachter (2005), the results that we report refer to a specific combination of the underlying risk factors. Nevertheless, instead of setting ad hoc values for the risk factors, we employ the values of the macroeconomic variables that were actually experienced in 1975:Q1.[11] This helps us derive an economic interpretation for our results. This specific choice is made by taking into account the implications of the term structure model for the bonds' expected excess returns and their covariances, as discussed in Section 3.3. In particular, we have chosen this date because it yields reasonable expected excess returns for the bonds relative to their volatilities. Given the bond premium puzzle implication of this model, other periods of our sample would yield extreme results, obscuring the analysis. We discuss further this issue in Section 6.3.

6.1 Portfolio choice in complete markets

In a complete market, an investor may form a portfolio with multiple bonds. In particular, the existence of five risk factors in the term structure model of Dewachter et al. (2006) allows us to examine the investor's bond portfolio choice among five nominal bonds. Panel A of Table 5 presents these choices for $\gamma = 4$ and $\gamma = 10$, as well as when the investment horizon increases up to $T = 10$ years. The results show that, in case five bonds are available, the investor should take extreme long and

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[11]These refer to an output gap of $y = -5.89\%$, inflation $\pi = 10.43\%$, real interest rate $\rho = -5.21\%$, inflation central tendency $\pi^* = 4.55\%$ and real rate central tendency $\rho^* = 0.67\%$.

short positions. The reason for this behaviour is the extremely high correlation of the bonds' returns, mentioned in Section 3.3. Even very small differences in the risk-return trade off of the various bonds motivate the investor to hold highly leveraged positions.

- Tables 4 and 5 about here-

This may be a quite puzzling observation, but it is a direct consequence of the estimated term structure model. There are various ways that these extreme positions could be much smoother in reality. Firstly, some types of institutional investors, such as pension and bond funds, are not allowed to take short positions. Such a constraint would lead to zero positions in the bonds that theoretically appear to be sold short. Secondly, transactions costs could actually make the perceived differences in the risk-return trade off disappear, neutralizing the incentive of holding so extreme positions. Furthermore, one should be very cautious when interpreting these results, because they refer to ex post estimates of the risk factors' and bond returns' volatilities. In particular, the volatility of the inflation central tendency that mainly affects bond returns was estimated to be extremely low over the sample period. However, the perceived volatility in the long-run inflation expectations may be much higher ex ante, so an investor may regard the bond returns' volatilities to be much higher than what this model implies. Consequently, if the level of risk is increased, the ex ante maximal Sharpe ratio would be much lower and the small differences in the bonds' premia would not lead to major demand shifts. In other words, if parameter uncertainty is taken into account as in Barberis (2000), Xia (2001) and Garlappi et al. (2007), bond demands may not be as extreme as the results in Table 5 suggest.

The next issue to examine is how real bonds can be mixed with nominal bonds in a portfolio. This exercise is motivated by the significantly lower correlations of the returns of these two different types of bonds, as Table 4 shows.[12] In particular, Panel B of Table 5 reports the results of bond portfolio selection among five bonds, i.e. nominal 1, 5 and 10-year bonds and real 3 and 7-year bonds, when the investor

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[12]It should be noted that in this case the expected excess returns as well as the volatilities and correlations of the real bonds are derived using their dynamics under the nominal SDF. We have shown how to derive the real bonds' returns dynamics in (41). Therefore, when we derive the optimal portfolio choice of the long-term investor from equation (31) in the presence of both real and nominal bonds, the dynamics of the real bonds' returns given by the SDE in (41) are appropriately used.

derives utility from his terminal wealth. The long and short positions are relatively large for low degrees of risk aversion in the myopic case ($T = 0$), but they are not as extreme as in the case where only nominal or only real bonds are available. Small differences in the risk-return trade offs are not extremely magnified in this case, because each bond also plays a significant diversification role in the formation of the optimal portfolios, since the correlations of the bonds' returns are now considerably less than perfect.

With respect to the hedging demands, their magnitude depends on the investor's horizon as well as on his degree of relative risk aversion. In particular, hedging demands tend to increase in absolute value as the horizon increases and tend to decrease as the degree of risk aversion increases. It is interesting to observe that the investor tries to match the duration of the bonds with his horizon. In other words, he tries to optimally combine the bonds so as to create a synthetic bond portfolio that has a maturity corresponding to his horizon. These results verify the argument of Brennan and Xia (2002).

An interesting issue to examine is how the definition of the riskless asset is modified, as the horizon of the investor increases. It is well known that an infinitely risk-averse myopic investor ($\gamma \to \infty$, $T = 0$) with utility over nominal wealth would assign a zero portfolio weight to the available risky bonds, investing his wealth only in the instantaneously nominal riskless asset that yields the nominal risk-free rate r. On the other hand, for the myopic investor with utility over real terminal wealth, the portion of wealth invested in the instantaneously nominal riskless asset is not exactly equal to one. This can be seen by setting $\gamma \to \infty$ and $T = 0$ in (31). This substitution yields $\phi_0 = 1 - i^T (B^T S^2 B)^{-1} (-B^T S) \sigma_{\Pi}$, which is equal to 1.05 in the particular case we examine, as reported in Panel A of Table 6. The explanation for this result is that the available assets are nominal, while the myopic investor we examine seeks to form a mean-variance efficient portfolio in real terms. Therefore, he should take into account the inflation risk loadings of the various bonds. In particular, while the total demand for the nominal bonds is $i^T \phi = -0.05$, the internal allocation exhibits significant long and short positions, exploiting the differences in the inflation risk loadings.

-Table 6 about here-

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Examining the optimal portfolio choice of a long-term investor $(T > 0)$, we set $\gamma = 100,000$ to approximate infinite relative risk aversion. Panel A of Table 6 reports the portfolio choices among five nominal bonds as the investor's horizon increases up to $T = 10$ years. These results show that the instantaneously nominal riskless asset is not risk-free for a long-term investor, due to the reinvestment risk that it generates, as explained in Stiglitz (1970). Moreover, the results show that none of the nominal zero-coupon bonds can play the role of the riskless asset, even when the maturity of the bond is identical to the horizon of the investor. This finding can be explained by the fact that the investor cares about his real terminal wealth but has access only to nominal bonds and it reflects the argument that a nominal bond cannot be a perfect hedging instrument for shocks in the real wealth process. This is true even when the inflation risk is relatively low, because the infinitely risk averse long-term investor would not like to be exposed to any unhedgeable shock affecting his real wealth. Consequently, the investor attempts to proxy the non-existing riskless asset by taking positions in the available nominal bonds and the optimal mix depends on his horizon as well as on the available bond maturities.

The introduction of a real bond market allows us to examine how this infinitely risk-averse investor would behave, if he could allocate his wealth among real zero-coupon bonds, priced under the real SDF. For the myopic case $(T = 0)$, the risk-free asset is the instantaneously real riskless asset that yields $r - \pi + \sigma_\Pi^T \xi$. Setting $\gamma \to \infty$ and $T = 0$ in (49), we get $\phi^R = \mathbf{0}$ and $\phi_0^R = 1$. Panel B of Table 6 reports this case as well as the optimal bond portfolio choice for non-myopic investment horizons. The striking result is that the infinitely risk averse long-term investor, who cares about his real wealth at a terminal date, should allocate his wealth to a single zero-coupon bond that has a duration equal to his horizon.

The previous result has been proved theoretically by Wachter (2003) and it is also stated in Liu (2007), but it has not been shown empirically in the literature, to the best of our knowledge.[13] Moreover, this finding underlines the importance of introducing real bonds in an economy (see Viard, 1993 inter alia for a discussion). This result offers an explanation for the existence of assets with negative expected excess returns, as the real bonds exhibited in a large portion of the examined period (see

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[13]This is because Campbell and Viceira (2001) consider utility over consumption and access to zero-coupon bonds, while Sangvinatsos and Wachter (2005) consider utility over real terminal wealth but access only to nominal zero-coupon bonds.

Figure 7). In particular, these bonds are held by long-term investors who are attracted by the significant hedging value that these assets incorporate. The traditional myopic framework could not have justified their existence.

The framework we have been using allows us also to examine the sensitivities of the investor's wealth with respect to shifts in the underlying macroeconomic factors. In particular, these sensitivities are affected by the agent's degree of risk aversion as well as his investment horizon. In the case of an investor who derives utility from his real terminal wealth and faces a complete nominal bond market, his wealth elasticities are given by $\frac{\partial G}{\partial X}\frac{1}{G}$, where G is defined in (30). For the current analysis, it proves more informative to report the norm of these elasticities, given by $\sqrt{(\frac{\partial G}{\partial X}\frac{1}{G})^T(\frac{\partial G}{\partial X}\frac{1}{G})}$.

Figure 8 reports this norm as the horizon increases for various levels of relative risk aversion. The results show that, for low levels of relative risk aversion ($\gamma < 10$) and long investment horizons, these elasticities are extremely high. Only a combination of short horizons and very high levels of risk aversion would yield a low norm of elasticities. This effect explains the large shifts in the hedging demands that we report in our portfolio choice results when the investment horizon or the degree of relative risk aversion is modified. Unreported results show that the wealth elasticity with respect to the inflation central tendency is the dominant one.[14]

-Figure 8 about here-

6.2   Portfolio choice in incomplete markets

The multi-factor term structure model of Dewachter et al. (2006) allowed us to examine the formation of portfolios with five zero-coupon bonds, within a complete market setting. Nevertheless, there are a series of reasons why an investor may actually allocate his wealth to a restricted set of bonds. In particular, this section examines the portfolio choice among two or three zero-coupon bonds and the corresponding

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[14]We have also examined the case of interim consumption under complete markets. The results, available upon request, show that the introduction of interim consumption essentially reduces the effective investment horizon relative to the terminal wealth case, confirming the argument of Wachter (2002). Moreover, no zero-coupon bond is regarded as a risk-free asset. In the presence of interim consumption, an infinitely risk-averse investor would seek to hold a coupon bond that matches his consumption stream. It should be noted that, as mentioned by Sangvinatsos and Wachter (2005), the analogy between the case of utility over terminal wealth and the case of utility over interim consumption does not carry through in the incomplete market case.

riskless asset. Within our setup, this leads to an incomplete market setting, analyzed in Section 5.

More specifically, it is common for institutional investors to face short selling constraints, disabling them to fully exploit the differences in the bonds' risk-return profiles by holding leveraged positions. Most importantly, transaction costs may actually make these risk-returns differences disappear for bonds of similar maturities. As Figure 4 and Figure 7 show, the expected excess returns are very similar for bonds with close maturities and they tend to strongly co-move through time. For example, the expected excess return of the 5-year nominal zero-coupon bond was, on average, only 0.34% higher than the expected excess return offered by the 3-year nominal zero-coupon bond for our sample period. Hence, transaction costs of the order of 0.2% could make bonds of very close maturities practically redundant.

Liquidity considerations would offer an additional reason why an investor may be willing to hold positions in a restricted set of bonds. While our portfolio choice exercise implicitly assumes that the zero-coupon bonds with prespecfied maturities are always available, this may not be true in practice. In particular, bonds of specific maturities may not be liquid enough, so an investor may not be able to fully capture the corresponding perceived expected returns that are implied by the estimated term structure model.[15] Consequently, this investor may prefer to hold a restricted set of highly liquid bonds and avoid loading illiquidity risk to his portfolio. Finally, it should be reminded that most of the dynamic bond portfolio studies make use of two or three risk factors to price bonds (see Campbell and Viceira, 2001 and Sangvinatsos and Wachter, 2005 correspondingly). The low dimension of these models simplifies the asset allocation problem, indicating that investors should form portfolios that are composed of only two or three bonds.

We firstly examine the portfolio choice among a 3-year and a 10-year nominal zero-coupon bond and the nominal instantaneously riskless asset. Panel A of Table 7 presents the optimal allocations for $\gamma = 4$ and $\gamma = 10$, as the horizon increases from $T = 0$ to $T = 10$ years. For the specific macroeconomic conditions that we have selected, the corresponding excess returns and their covariances imply that the myopic demand for the 10-year bond is much higher than the demand for the 3-year bond. Furthermore, the demand for the nominal instantaneously riskless asset is

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[15]See Amihud and Mendelson (1991) for the impact of transaction costs and liquidity on bond yields.

quite high. Nevertheless, long-term investors have significant hedging bond demands that dominate the corresponding total demands as the horizon increases. Moreover, the magnitude of the hedging demand is greater for low levels of relative risk aversion and the investor should actually borrow at the instantaneously riskless rate.

-Table 7 about here-

The next issue to examine in this subsection is the impact of macroeconomic shifts on optimal portfolio choice. This is an attractive feature of our study with respect to the rest of the literature, since we employ an essentially affine term structure model with a clear macroeconomic interpretation. Unlike Campbell and Viceira (2001) and Brennan and Xia (2002), who assume constant term premia, both myopic and hedging demands in our model are affected by the level of the underlying risk factors. In other words, the long-term investor is involved in market timing both in his myopic and his hedging demand, as it is evident from (31). Moreover, the modifications of the underlying risk factors are due to specific macroeconomic effects, in contrast to the latent factor model used by Sangvinatsos and Wachter (2005).

As it was mentioned in Section 3.3, the central tendency of inflation, π^*, has a dominant effect on bond returns. Consequently, we examine the impact on portfolio choice when this tendency increases or decreases by one standard deviation, keeping the rest of the factors constant. The portfolio choices reported in Panel B of Table 7 show that the increase in this central tendency implies a significant increase in the excess returns of the 3-year and the 10-year bond. As a result, the risk-return trade off of these bonds is modified, making the 3-year bond much more attractive now in comparison to the benchmark case. Not only the myopic demand for this bond is higher, but this is also true for the corresponding hedging demands. The total demand for the 10-year bond is lower for short horizons and it significantly increases only when the investor's horizon approaches the ten years. The total hedging demand still decreases as the degree of relative risk aversion increases, but its magnitude is now larger in comparison to the benchmark case.

On the other hand, a decrease in the central tendency of inflation has the opposite effect, as the results in panel C of Table 7 show. The reduction in the bonds' premia modifies their risk-return trade off in such a way that the myopic investor sells short the 3-year bond. There is some hedging role for this bond, but this has become very limited and only for investment horizons close to the bond's maturity. For long

horizons, the investor has a significantly negative hedging demand for this bond, since he mainly makes use of the 10-year bond as a hedging instrument. The magnitude of the total bond demand is now quite a lot smaller in comparison to the benchmark macroeconomic conditions, prevailing in 1975:Q1.

Pricing real bonds within our setup, we concluded that they exhibit a different behaviour in comparison to nominal bonds. This observation motivates us to examine also the optimal allocation of real wealth when a 3-year and a 10-year real bond are available, apart from the instantaneously real risk free rate. Panel A of Table 8 shows the myopic as well as the total bond demands, as the investor's horizon increases for the macroeconomic conditions prevailing in 1975:Q1. Even though both bonds have negative expected excess returns, the risk-return trade off that they exhibit motivates the myopic investor to hold a significant long position in the 3-year bond by selling short the 10-year real bond. As the horizon increases, the hedging demand for the 10-year bond becomes increasingly positive. In general, the investor attempts to combine the bonds' maturities so as to match his horizon.

-Table 8 about here-

Since the factor loadings of the macroeconomic risks are modified in the case of real bonds, it is interesting to examine the impact of macroeconomic change in this case too. Panel B of Table 8 shows the optimal portfolio choice when the inflation central tendency is increased by one standard deviation, while Panel C of Table 8 presents the case of the corresponding reduction in the inflation central tendency. In the first case, the increase in the premium of the 3-year real bond significantly increases both the myopic and the hedging demand for this bond for short horizons. On the other hand, the reduction in the central tendency of inflation reduces the reward for holding both the 3-year and the 10-year real bond. Consequently, the magnitude of both the myopic and the hedging bond demands becomes lower in comparison to the benchmark case.

Expanding the asset space to include a third nominal bond leads to larger long and short positions. Panel A of Table 9 shows the allocation of wealth to the nominal zero-coupon bonds with 1, 5 and 10-year maturities for various investment horizons when the investor maximizes utility over real terminal wealth. The magnitude of the hedging demands crucially depends on the horizon and it tends to decrease as the investor becomes more risk averse. The inclusion of another bond gives the investor

more flexibility in his attempt to create a hedging bond portfolio with duration that matches his horizon. The 10-year nominal bond plays a significant hedging role mainly for investors with long horizons.

-Table 9 about here-

Panel B of Table 9 reports the corresponding optimal portfolio choices when an investor can allocate his wealth among real zero-coupon bonds with 1, 5 and 10-year maturities. The 1-year bond offers a significantly higher premium in comparison to the negative premia offered by the 5-year and 10-year bonds. Consequently, the myopic investor should take a significant long position in the 1-year bond, selling short the 5-year bond. On the other hand, the 10-year bond incorporates significant hedging value for a long-term investor, especially when his horizon is longer than 5 years. These results also show that access to multiple real bonds may offer the required flexibility to long-term investors who seek to hedge away shocks to their real wealth process in an optimal way.

Subsequently, we examine the optimal asset allocation to one real and two nominal bonds. This is an interesting combination to examine because the investor can actually improve the diversification of his portfolio, extract the premia from nominal bonds and make use of the real bond's hedging value. It should be reminded that when deriving the optimal portfolio choice for this case, the real bond's SDE under the nominal SDF, given by (41), is appropriately used in equation (65). Panel C of Table 9 shows the portfolio choice results. Bond demands are much lower in magnitude and less sensitive to shifts in the investment horizon and the degree of relative risk aversion. This is due to the much lower correlation in the returns of the available bonds. The 5-year real bond plays a hedging role, especially for an investor with a horizon close to five years. The 1-year bond is used mainly for its attractive risk-return profile, while the 10-year nominal bond plays a very significant hedging role for horizons longer than five years.

6.3 Sensitivity analysis

The last issue to examine is how sensitive are the portfolio choices that we previously presented to the choice of the benchmark date 1975:Q1. The extreme swings in the maximal Sharpe ratio, illustrated in Figure 5, indicate that both the myopic and the

hedging demands will be subject to extreme swings too. This is due to the fact that under time-varying risk premia, the investor should be involved in market timing for investment as well as for hedging purposes. The large shifts in the macroeconomic factors as well as the very high wealth elasticities of the power utility investor with respect to these factors explain these extreme swings in portfolio choices.

We plot in Figure 9 the total myopic bond demand for a power utility investor with $\gamma = 10$, who has access to a 3-year and a 10-year nominal zero-coupon bond for the period 1964:Q1 to 1998:Q4 (solid line). The vector of the myopic bond demands is given by the first two terms in equation (65). The total myopic bond demand exhibits large shifts through time, verifying the argument that the time-variation in the bond premia is a very important issue that even a myopic investor should take into account. The evolution of the total myopic demand resembles the evolution of the maximal Sharpe ratio that is illustrated in Figure 5.

- Figure 9 about here-

More impressive are the extreme shifts in the total hedging bond demand. Figure 9 illustrates the evolution of the sum of the hedging demands for these two nominal bonds. The vector of the hedging demands is given by the third term in equation (65) for a power utility investor with $\gamma = 10$, who has a horizon of $T = 3$ and $T = 10$ years correspondingly. For both cases, the shifts are extreme for even small changes in the macroeconomic factors. Furthermore, the magnitude of the total hedging demand is very high and overwhelmingly dominates the corresponding total myopic bond demand for every single period. It is also very interesting to observe that the total hedging bond demand is almost the same for both investment horizons, to the extent that the two lines are indistinguishable. As we have previously analyzed, the investment horizon plays a crucial role for the internal composition of the hedging bond portfolio, not for its total magnitude. The magnitude as well as the extreme shifts in the total hedging bond demand are consequences of the huge wealth elasticities of the long-term power utility investor, illustrated in Figure 8, as well as the considerable variation in the investment opportunity set. Therefore, these results underline our previous conclusions, showing that the time-variation in bond premia is an extremely important issue that a long-term investor should not neglect in his attempt to hedge away undesirable shocks.

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We repeat the previous sensitivity analysis for the case of a power utility investor with $\gamma = 10$, who has access to a 3-year and a 10-year real zero-coupon bond for the period 1964:Q1 to 1998:Q4. Figure 10 illustrates the total myopic bond demand (solid line), the total hedging bond demand for an investor with a horizon of $T = 3$ years (triangle-marked line) as well as the corresponding total hedging bond demand when the investment horizon is $T = 10$ years (dashed line). The vector of the myopic bond demands is given by the first term in equation (77), while the vector of the hedging bond demands is given by the second term in (77).

-Figure 10 about here-

The results show that the shifts in the total myopic bond demand as well as in the total hedging bond demand are of great magnitude for the case of real bonds too. A visual inspection of Figure 10 shows that the evolution of the total myopic and total hedging bond demands follow closely the evolution of the maximal Sharpe ratio. This finding confirms the argument that the power utility investor should rebalance his bond portfolio according to the shifts in the macroeconomy. Moreover, the magnitude of the total hedging bond demand of the power utility investor with $\gamma = 10$ is greater than the corresponding total myopic bond demand for every period in our sample. Nevertheless, the difference in their magnitudes is less pronounced as compared to the case of nominal zero-coupon bonds. Given this evidence, it should be noted that the period we employed as our benchmark in this study, i.e. 1975:Q1, was selected on the basis that it yielded a low magnitude of myopic and hedging demands, facilitating our analysis.

7   Conclusion

This study examined the dynamic bond portfolio choice of a long-term, power utility investor. Using the macro-finance term structure model of Dewachter et al. (2006), we were able to provide a clear macroeconomic interpretation for the formation of bond premia and to examine how shifts in the macroeconomy affect the formation of optimal portfolios. We have also documented how the concept of the riskless asset is defined in the case of utility over terminal real wealth, when the investor has access either to nominal or real bonds. Until now, this issue has been explored in the literature only separately for each case.

Our results are of significant importance for institutional investors, such as pension funds. Matching the duration of a nominal bond with the investment horizon is a legitimate practice for infinitely risk averse agents who have utility over nominal wealth at a terminal date. However, if the investors are interested in real wealth, this practice does not provide a safe strategy, because the nominal bond is risky in real terms. In this case, a real bond with the appropriate duration becomes the riskless asset. This framework has also allowed us to provide reconciliation for the existence of assets with low or even negative expected excess returns, such as long-term real bonds. Standard mean-variance theory cannot provide an explanation why an investor should hold these bonds. Our results provide support for the popular perception that these bonds are mainly held for hedging purposes, especially when investors care for real wealth and consumption. Moreover, the real bonds could be included in a broader portfolio because their returns exhibit relatively low correlation with the returns of nominal bonds, so they may be useful instruments for diversification.

With respect to the term structure literature, we provide an evaluation of the essentially affine models from an asset allocation perspective. While the focus of the literature is on fitting past and predicting future yields, the covariance and correlation structure that estimated models imply for the bond returns is relatively neglected. It is shown that this is a major concern if one wishes to implement these models for portfolio choice, because the estimated volatilities are extremely low and the correlations of the returns are extremely high. A potential solution is to use a term structure model that allows for time-varying conditional volatilities, as in Spencer (2008), or to adopt a Bayesian approach for portfolio choice, as in Garlappi et al. (2007), assuming parameter uncertainty that effectively increases the returns' volatilities.

Finally, the high market prices of risk yield extreme wealth sensitivities to movements in the underlying risk factors, generating hedging demands of large magnitude. This effect becomes moderate only if we assume very short horizons and very high degrees of relative risk aversion. It is hard to reconcile these large sensitivities within the commonly used power utility framework in the presence of horizon effects. Consequently, we document a premium puzzle in the bond market. Future research could examine whether the loss aversion framework of Benartzi and Thaler (1995), as recently examined by Berkelaar, Kouwenberg and Post (2006), may yield more realistic conclusions.

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Appendix

Proof of Proposition 3.1

Employing the conjectured functional form for G given in (30), we can substitute the corresponding terms into the PDE (28) and recalling the definition of the price of risk ξ as well as the fact that the nominal risk-free rate is given by $r = \delta_1^T X$ and $r - \pi = (\delta_1 - \delta_2)^T X$, where $\delta_1 = (0\ 1\ 1\ 0\ 0)^T$ and $\delta_2 = (0\ 1\ 0\ 0\ 0)^T$, this PDE can be written as:

$$
\begin{aligned}
&(\frac{1}{\gamma} - 1)(\delta_1 - \delta_2)^T X + \frac{1}{2\gamma}\bar{\psi}^T(Q + Q^T)X + \frac{1}{\gamma}\bar{\psi}^T d + \frac{1}{2\gamma}X^T K^T(Q + Q^T)X + \\
&+ \frac{1}{\gamma}d^T KX + \frac{1-\gamma}{2\gamma^2}\Lambda^T S^T S\Lambda + \frac{1-\gamma}{\gamma^2}\Lambda^T \Xi X + \frac{1-\gamma}{2\gamma^2}X^T\Xi^T S^{-2}\Xi X - \\
&- \frac{1-\gamma}{\gamma^2}\sigma_\Pi^T S\Lambda - \frac{1-\gamma}{\gamma^2}\sigma_\Pi^T S^{-1}\Xi X + (\frac{1}{\gamma} - 1)\sigma_\Pi^T S\Lambda + (\frac{1}{\gamma} - 1)\sigma_\Pi^T S^{-1}\Xi X + \\
&+ \frac{1}{\gamma}(1 - \frac{1}{\gamma})\sigma_\Pi^T S[\frac{1}{2}(Q + Q^T)X + d] + \frac{1-\gamma}{2\gamma^2}\sigma_\Pi^T\sigma_\Pi + \frac{1}{2\gamma^2}d^T SS^T d + \\
&+ \frac{1}{8\gamma^2}X^T(Q + Q^T)SS^T(Q + Q^T)X + \frac{1}{2\gamma^2}d^T SS^T(Q + Q^T)X + \\
&+ \frac{1}{4\gamma}Tr(SS^T(Q + Q^T)) + \frac{1-\gamma}{2\gamma^2}\Lambda^T S^T S(Q + Q^T)X + \frac{1-\gamma}{2\gamma^2}X^T\Xi^T(Q + Q^T)X + \\
&+ \frac{1-\gamma}{\gamma^2}\Lambda^T S^T Sd + \frac{1-\gamma}{\gamma^2}d^T\Xi X + \frac{1}{2\gamma}X^T\dot{Q}X + \frac{1}{\gamma}\dot{d}^T X + \frac{1}{\gamma}\dot{c} = 0 \quad (78)
\end{aligned}
$$

where $\dot{c}$, $\dot{d}$, $\dot{Q}$ stand for the derivatives of c, d and Q with respect to time t.

From the previous expression, gathering terms in $X^T[.]X$, X and the scalar term, we get the following system of ODEs:

$$
\begin{aligned}
&\dot{c} + \bar{\psi}^T d + \frac{1-\gamma}{2\gamma}\Lambda^T S^T S\Lambda + \frac{1}{2\gamma}d^T SS^T d + \frac{1}{4}Tr(SS^T(Q + Q^T)) + \\
&+ \frac{1-\gamma}{\gamma}\Lambda^T S^T Sd - \frac{1-\gamma}{\gamma}\sigma_\Pi^T S\Lambda + (1 - \gamma)\sigma_\Pi^T S\Lambda + (1 - \frac{1}{\gamma})\sigma_\Pi^T Sd + \frac{1-\gamma}{2\gamma}\sigma_\Pi^T\sigma_\Pi = 0
\end{aligned}
$$

$$(79)$$

$$
\begin{aligned}
&\dot{d}^T + (1 - \gamma)(\delta_1 - \delta_2)^T + \frac{1}{2}\bar{\psi}^T(Q + Q^T) + d^T K + \frac{1-\gamma}{\gamma}\Lambda^T\Xi - \frac{1-\gamma}{\gamma}\sigma_\Pi^T S^{-1}\Xi + \\
&+ (1 - \gamma)\sigma_\Pi^T S^{-1}\Xi + \frac{1}{2\gamma}d^T SS^T(Q + Q^T) + \frac{1-\gamma}{2\gamma}\Lambda^T S^T S(Q + Q^T) + \\
&+ \frac{1}{2}(1 - \frac{1}{\gamma})\sigma_\Pi^T S(Q + Q^T) + \frac{1-\gamma}{\gamma}d^T\Xi = 0 \quad (80)
\end{aligned}
$$

$$\dot{Q} + K^T(Q+Q^T) + \frac{1-\gamma}{\gamma}\Xi^T S^{-2}\Xi + \frac{1}{4\gamma}(Q+Q^T)SS^T(Q+Q^T) + \frac{1-\gamma}{\gamma}\Xi^T(Q+Q^T) = 0$$

$$(81)$$

along with the corresponding terminal conditions. This is a system of 31 ODEs.

Furthermore, substituting the functional form of G into (29), we get the optimal portfolio choice in (31).

Proof of Proposition 3.2

Substituting the conjectured functional form for G^R into the PDE (46), we end up with exactly the same equation as in (78) with respect to $Q^R(t)$, $d^R(t)$ and $c^R(t)$. As a result, the ODEs that $Q^R(t)$, $d^R(t)$ and $c^R(t)$ should satisfy are of the same form as in (79) - (81).

Substituting the conjectured form of the function $G^R(Z^R, X, t)$ into (47), we get the optimal portfolio choice in (49).

Proof of Proposition 3.3

Substituting the conjectured form for G^I into (62), the optimal portfolio choice should satisfy:

$$(\phi^I)^T(-B^T S) = \frac{1}{\gamma}\xi^T + \frac{1}{\gamma}(v^*)^T + (1-\frac{1}{\gamma})\sigma_\Pi^T + \frac{1}{\gamma}(\frac{1}{2}X^T(Q^I + Q^{IT}) + d^{IT})S \quad (82)$$

According to the argument of He and Pearson (1991), v^* should guarantee that the unhedgeable parts of σ_Π^T and S should drop out. Note that σ_Π^T can be written as:

$$\sigma_\Pi^T = \sigma_\Pi^T(-B^T S)^T(B^T S^2 B)^{-1}(-B^T S) + [\sigma_\Pi^T - \sigma_\Pi^T(-B^T S)^T(B^T S^2 B)^{-1}(-B^T S)] \quad (83)$$

where the first component is the projection of σ_Π^T onto the available assets and the second component is the residual of the projection, $(\sigma_\Pi^T)^\perp$. Similarly, S can be written as:

$$S = S(-B^T S)^T(B^T S^2 B)^{-1}(-B^T S) + [S - S(-B^T S)^T(B^T S^2 B)^{-1}(-B^T S)] \quad (84)$$

where $S^\perp = S - S(-B^T S)^T(B^T S^2 B)^{-1}(-B^T S)$ is the residual of the projection of S onto the available assets. It should be noted that under complete markets, $(\sigma_\Pi^T)^\perp = \mathbf{0}$ and $S^\perp = \mathbf{0}$.

So, v^* should satisfy the following condition:

$$v^* = (1-\gamma)[(\sigma_\Pi^T)^\perp]^T - (S^\perp)^T[(\frac{1}{2}(Q^I + Q^{IT})X + d^I)] \tag{85}$$

Substituting this expression into the optimal portfolio choice (82), we get:

$$(\phi^I)^T(-B^T S) = \frac{1}{\gamma}\xi^T + (1-\frac{1}{\gamma})(\sigma_\Pi^T - (\sigma_\Pi^T)^\perp) + \frac{1}{\gamma}(\frac{1}{2}X^T(Q^I + Q^{IT}) + d^{IT})(S - S^\perp) \Rightarrow$$

$$(\phi^I)^T(-B^T S) = \frac{1}{\gamma}\xi^T + (1-\frac{1}{\gamma})\sigma_\Pi^T(-B^T S)^T(B^T S^2 B)^{-1}(-B^T S)+$$

$$+\frac{1}{\gamma}(\frac{1}{2}X^T(Q^I + Q^{IT}) + d^{IT})S(-B^T S)^T(B^T S^2 B)^{-1}(-B^T S) \tag{86}$$

Multiplying this expression by $(-B^T S)^T(B^T S^2 B)^{-1}$ and taking the transpose, we derive the optimal portfolio choice expression in (65).

Moreover, substituting the conjectured form for G^I and the previously derived equation for v^* into the PDE (61), we can derive an equation from which, if we collect the terms in $X^T[.]X$, X and the scalar correspondingly, we get the following system of ODEs:

$$\dot{c}^I + \bar{\psi}^T d^I + \frac{1-\gamma}{2\gamma}\Lambda^T S^T S\Lambda + \frac{1}{2\gamma}d^{IT}SS^T d^I + \frac{1}{4}Tr(SS^T(Q^I + Q^{IT}))+$$

$$+\frac{1-\gamma}{\gamma}\Lambda^T S^T S d^I - \frac{1-\gamma}{\gamma}\sigma_\Pi^T S\Lambda + (1-\gamma)\sigma_\Pi^T S\Lambda + (1-\frac{1}{\gamma})\sigma_\Pi^T S d^I + \frac{(1-\gamma)^2}{\gamma}(\sigma_\Pi^T)^\perp S d^I+$$

$$+\frac{1-\gamma}{2\gamma}\sigma_\Pi^T\sigma_\Pi - \frac{1-\gamma}{2\gamma}d^{IT}S^\perp S^T d^I + (1-\gamma)^2(1-\frac{1}{\gamma})(\sigma_\Pi^T)^\perp\sigma_\Pi + (1-\gamma)(\frac{1}{\gamma} - 1)d^{IT}S^\perp\sigma_\Pi$$

$$+\frac{(1-\gamma)^3}{2\gamma}(\sigma_\Pi^T)^\perp[(\sigma_\Pi^T)^\perp]^T - \frac{(1-\gamma)^2}{\gamma}(\sigma_\Pi^T)(S^\perp)^T d^I + \frac{1-\gamma}{2\gamma}d^{IT}S^\perp(S^\perp)^T d^I = 0 \quad (87)$$

$$\dot{d}^{IT} + (1-\gamma)(\delta_1 - \delta_2)^T + \frac{1}{2}\bar{\psi}^T(Q^I + Q^{IT}) + d^{IT}K + \frac{1-\gamma}{\gamma}\Lambda^T\Xi - \frac{1-\gamma}{\gamma}\sigma_\Pi^T S^{-1}\Xi+$$

$$+(1-\gamma)\sigma_\Pi^T S^{-1}\Xi + \frac{1}{2\gamma}d^{IT}SS^T(Q^I + Q^{IT}) + \frac{1-\gamma}{2\gamma}\Lambda^T S^T S(Q^I + Q^{IT})+$$

$$+\frac{1}{2}(1-\frac{1}{\gamma})\sigma_\Pi^T S(Q^I + Q^{IT}) + \frac{1-\gamma}{\gamma}d^{IT}\Xi + \frac{(1-\gamma)^2}{2\gamma}\sigma_\Pi^T S(Q^I + Q^{IT})-$$

$$-\frac{1-\gamma}{\gamma}d^{IT}S^\perp S(Q^I + Q^{IT}) + \frac{1-\gamma}{2}(\frac{1}{\gamma} - 1)\sigma_\Pi^T(S^\perp)^T(Q^I + Q^{IT})-$$

$$-\frac{1-\gamma}{2\gamma}(\sigma_\Pi^T)^\perp(S^\perp)^T(Q^I + Q^{IT}) + \frac{1-\gamma}{2\gamma}d^{IT}S^\perp(S^\perp)^T(Q^I + Q^{IT}) = 0 \tag{88}$$

46

$$\dot{Q}^I + K^T(Q^I + Q^{IT}) + \frac{1-\gamma}{\gamma}\Xi^T S^{-2}\Xi +$$

$$+\frac{1}{4\gamma}(Q^I + Q^{IT})SS^T(Q^I + Q^{IT}) + \frac{1-\gamma}{\gamma}\Xi^T(Q^I + Q^{IT}) +$$

$$+\frac{1-\gamma}{4\gamma}(Q^I + Q^{IT})S^\perp(S^\perp)^T(Q^I + Q^{IT}) - \frac{1-\gamma}{2\gamma}(Q^I + Q^{IT})S^\perp S(Q^I + Q^{IT}) = 0$$

$$(89)$$

along with the corresponding terminal conditions.

Proof of Proposition 3.4

Substituting the conjectured form for G^{IR} into (74), the optimal portfolio choice satisfies:

$$(\phi^{IR})^T(-B^{RT}S) = \frac{1}{\gamma}(\xi - \sigma_\Pi)^T + \frac{1}{\gamma}(V^*)^T + \frac{1}{\gamma}(\frac{1}{2}X^T(Q^{IR} + Q^{IRT}) + d^{IRT})S \quad (90)$$

Again, V^* should guarantee that the unhedgeable part of S should drop out. Hence, V^* should satisfy the following condition:

$$V^* = -(S^\perp)^T[(\frac{1}{2}(Q^{IR} + Q^{IRT})X + d^{IR})] \quad (91)$$

where $S^\perp = S - S(-B^{RT}S)^T(B^{RT}S^2B^R)^{-1}(-B^{RT}S)$ is the residual of the projection of S on the available real bonds.

Substituting the expression for V^* into (90), we get:

$$(\phi^{IR})^T(-B^{RT}S) = \frac{1}{\gamma}(\xi - \sigma_\Pi)^T + \frac{1}{\gamma}(\frac{1}{2}X^T(Q^{IR} + Q^{IRT}) + d^{IRT})(S - S^\perp) =$$

$$= \frac{1}{\gamma}(\xi - \sigma_\Pi)^T + \frac{1}{\gamma}(\frac{1}{2}X^T(Q^{IR} + Q^{IRT}) + d^{IRT})S(-B^{RT}S)^T(B^{RT}S^2B^R)^{-1}(-B^{RT}S)$$

$$(92)$$

Multiplying this expression by $(-B^{RT}S)^T(B^{RT}S^2B^R)^{-1}$ and taking the transpose, we derive the optimal portfolio choice expression in (77).

Substituting the conjectured form for G^{IR} and the previously derived expression for V^* into the PDE (73), we can derive an expression from which, if we gather the terms in $X^T[.]X$, X and the scalar, we get the following system of ODEs:

$$\dot{c}^{IR} + \bar{\psi}^T d^{IR} + \frac{1-\gamma}{2\gamma}\Lambda^T S^T S\Lambda + \frac{1}{2\gamma}d^{IRT}SS^T d^{IR} + \frac{1}{4}Tr(SS^T(Q^{IR} + Q^{IRT})) +$$

$$+\frac{1-\gamma}{\gamma}\Lambda^T S^T S d^{IR} - \frac{1-\gamma}{\gamma}\sigma_\Pi^T S\Lambda + (1-\gamma)\sigma_\Pi^T S\Lambda + (1-\frac{1}{\gamma})\sigma_\Pi^T S d^{IR} + \frac{1-\gamma}{2\gamma}\sigma_\Pi^T\sigma_\Pi$$

$$+\frac{\gamma-1}{\gamma}d^{IRT}S^\perp S d^{IR} + (1-\gamma)(\frac{1}{\gamma}-1)d^{IRT}S^\perp\sigma_\Pi + \frac{1-\gamma}{2\gamma}d^{IRT}S^\perp(S^\perp)^T d^{IR} = 0 \quad (93)$$

$$\dot{d}^{IRT} + (1-\gamma)(\delta_1 - \delta_2)^T + \frac{1}{2}\bar{\psi}^T(Q^{IR} + Q^{IRT}) + d^{IRT}K + \frac{1-\gamma}{\gamma}\Lambda^T\Xi - \frac{1-\gamma}{\gamma}\sigma_\Pi^T S^{-1}\Xi +$$

$$+(1-\gamma)\sigma_\Pi^T S^{-1}\Xi + \frac{1}{2\gamma}d^{IRT}SS^T(Q^{IR} + Q^{IRT}) + \frac{1-\gamma}{2\gamma}\Lambda^T S^T S(Q^{IR} + Q^{IRT}) +$$

$$+\frac{1}{2}(1-\frac{1}{\gamma})\sigma_\Pi^T S(Q^{IR} + Q^{IRT}) + \frac{1-\gamma}{\gamma}d^{IRT}\Xi + \frac{1-\gamma}{2\gamma}d^{IRT}S^\perp(S^\perp)^T(Q^{IR} + Q^{IRT})$$

$$+\frac{\gamma-1}{\gamma}d^{IRT}S^\perp S(Q^{IR} + Q^{IRT}) + \frac{1-\gamma}{2}(\frac{1}{\gamma}-1)\sigma_\Pi^T(S^\perp)^T(Q^{IR} + Q^{IRT}) = 0 \quad (94)$$

$$\dot{Q}^{IR} + K^T(Q^{IR} + Q^{IRT}) + \frac{1-\gamma}{\gamma}\Xi^T S^{-2}\Xi +$$

$$+\frac{1}{4\gamma}(Q^{IR} + Q^{IRT})SS^T(Q^{IR} + Q^{IRT}) + \frac{1-\gamma}{4\gamma}(Q^{IR} + Q^{IRT})S^\perp(S^\perp)^T(Q^{IR} + Q^{IRT})$$

$$+\frac{1-\gamma}{\gamma}\Xi^T(Q^{IR} + Q^{IRT}) - \frac{1-\gamma}{2\gamma}(Q^{IR} + Q^{IRT})S^\perp S(Q^{IR} + Q^{IRT}) = 0 \quad (95)$$

along with the corresponding terminal conditions.

Table 1: Estimated coefficients

	y	π	ρ	π^*	ρ^*
$\kappa_{y,.}$	-0.3146 (0.0638)**	-1.0748 (0.1765)**	-0.4555 (0.1711)**		
$\kappa_{\pi,.}$	0.3854 (0.064)**	-0.2452 (0.0446)**	-0.1319 (0.0249)**		
$\kappa_{\rho,.}$	-0.0685 (0.0197)**	-5.1575 (0.4009)**	-5.3035 (0.3718)**		
$\kappa_{\pi^*,.}$				-0.0036 (0.0012)**	
$\kappa_{\rho^*,.}$					-0.4849 (0.0347)**
$\theta_.$				0.0224 (0.0367)	0.0137 (0.0057)**
$\Lambda_.$	-63.9951 (65.1136)	34.8309 (63.3615)	32.5392 (14.2244)**	-21.9563 (11.0275)**	
$\Xi_{\rho,.}$	0.0082 (0.003)**	-0.3672 (0.1068)**	-0.9849 (0.2204)**	-1.1884 (0.3716)**	-1.5318 (0.4773)**
$\sigma^2_.$	0.000279 (0.000043)**	0.000146 (0.000019)**	0.001545 (0.000042)**	0.000067 (0.000010)**	0.000253 (0.000043)**

Notes: This table shows the estimated parameters for the dynamics in X and the market price of risk ξ as reported in Table II of Dewachter et al. (2006). Robust standard errors are given in the parentheses. ** and * indicate statistical significance at the 5% and the 10% level correspondingly.

Table 2: Covariances and Correlations of Nominal bonds

	1-yr	2-yr	3-yr	5-yr	7-yr	10-yr
Panel A: Covariance Matrix						
1-yr	0.0004					
2-yr	0.0007	0.0014				
3-yr	0.0009	0.0019	0.0026			
5-yr	0.0013	0.0026	0.0037	0.0054		
7-yr	0.0016	0.0032	0.0046	0.0069	0.0088	
10-yr	0.0019	0.0041	0.0058	0.0088	0.0115	0.0153
Panel B: Correlation Matrix						
1-yr	1					
2-yr	0.974	1				
3-yr	0.947	0.994	1			
5-yr	0.897	0.964	0.987	1		
7-yr	0.848	0.927	0.961	0.993	1	
10-yr	0.782	0.871	0.916	0.968	0.991	1

Notes: Panel A shows the covariance matrix of the nominal zero-coupon bond returns for various maturities. Panel B shows the corresponding correlation matrix.

Table 3: Covariances and Correlations of Real bonds (real SDF)

	Panel A: Covariance Matrix					
	1-yr	2-yr	3-yr	5-yr	7-yr	10-yr
1-yr	0.0005					
2-yr	0.0009	0.0018				
3-yr	0.0013	0.0025	0.0035			
5-yr	0.0017	0.0034	0.0048	0.0066		
7-yr	0.0019	0.0038	0.0066	0.0075	0.0087	
10-yr	0.0021	0.0041	0.0075	0.0081	0.0094	0.0104
	Panel B: Correlation Matrix					
	1-yr	2-yr	3-yr	5-yr	7-yr	10-yr
1-yr	1					
2-yr	0.982	1				
3-yr	0.964	0.995	1			
5-yr	0.933	0.976	0.991	1		
7-yr	0.91	0.956	0.975	0.995	1	
10-yr	0.885	0.932	0.951	0.975	0.99	1

Notes: Panel A shows the covariance matrix of the real zero-coupon bond returns for various maturities. Panel B shows the corresponding correlation matrix.

Table 4: Mixture of nominal and real bonds (nominal SDF)

	Panel A: Covariance Matrix				
	1-yr (N)	3-yr (R)	5-yr (N)	7-yr (R)	10-yr (N)
1-yr (N)	0.0004				
3-yr (R)	0.0009	0.0042			
5-yr (N)	0.0013	0.0037	0.0054		
7-yr (R)	0.0016	0.0061	0.0061	0.0093	
10-yr (N)	0.0019	0.0054	0.0088	0.0091	0.0153
	Panel B: Correlation Matrix				
	1-yr (N)	3-yr (R)	5-yr (N)	7-yr (R)	10-yr (N)
1-yr (N)	1				
3-yr (R)	0.741	1			
5-yr (N)	0.897	0.773	1		
7-yr (R)	0.819	0.965	0.859	1	
10-yr (N)	0.782	0.673	0.968	0.766	1

Notes: Panel A shows the covariance matrix of a mixture of nominal and real zero-coupon bond returns for various maturities. For consistency, we have used the dynamics of the real bonds' returns under the nominal SDF, as they are given by the SDE (41). (N) indicates a nominal bond while (R) indicates a real bond. Panel B shows the corresponding correlation matrix.

Table 5: Portfolio choice among five bonds

Panel A: 5 Nominal bonds								
		$\gamma = 4$				$\gamma = 10$		
	Premia	T=0	T=1	T=5	T=10	T=0	T=1	T=5	T=10
1-yr (N)	0.20%	20.88	24.76	30.34	21.93	10.57	15.10	21.21	13.09
3-yr (N)	0.65%	-218.49	-243.74	-281.20	-231.35	-102.24	-127.45	-166.48	-118.55
5-yr (N)	0.91%	757.66	810.83	820.84	729.94	334.49	385.80	410.69	317.84
7-yr (N)	1.17%	-788.08	-825.91	-779.61	-723.58	-339.46	-376.39	-350.25	-287.31
10-yr (N)	1.71%	236.62	243.95	219.23	212.55	100.15	107.57	89.40	79.37

Panel B: 5 Nominal and real bonds under the nominal SDF								
		$\gamma = 4$				$\gamma = 10$		
	Premia	T=0	T=1	T=5	T=10	T=0	T=1	T=5	T=10
1-yr (N)	0.20%	-1.47	-1.85	-5.81	-5.22	-0.81	-0.56	-3.69	-3.44
3-yr (R)	0.28%	17.32	17.81	15.63	15.19	7.27	7.74	6.09	5.42
5-yr (N)	0.91%	11.59	13.72	15.65	13.21	4.64	5.64	7.52	5.72
7-yr (R)	-1.39%	-16.57	-16.75	-13.67	-13.93	-6.79	-6.95	-4.46	-4.41
10-yr (N)	1.71%	-2.45	-3.23	-2.42	0.06	-0.97	-1.34	-1.01	1.09

Notes: This Table shows the optimal portfolio choice of an investor who has utility over real terminal wealth and access to five bonds. Panel A shows the portfolio choice among five nominal bonds as derived in (31) for the macroeconomic conditions prevailing in 1975:Q1 ($y = -5.89\%$, $\pi = 10.43\%$, $\rho = -5.21\%$, $\pi^* = 4.55\%$ and $\rho^* = 0.67\%$). Panel B shows the corresponding portfolio choice among 3 nominal and 2 real bonds. The dynamics for the returns of these 2 real bonds that have been employed in Panel B are given by the SDE in (41) and they are stated under the nominal SDF. The hedging demand for horizon T is given by the difference between the total demand for horizon T and the demand for $T = 0$ (myopic demand). The allocation to the nominal instantaneously riskless asset is equal to one minus the sum of the total bond demands in each case.

Table 6: Bond portfolio choice for an infinitely risk-averse investor

Panel A: 5 Nominal bonds						
	T=0	T=1	T=3	T=5	T=7	T=10
1-yr (N)	3.69	4.90	9.44	10.41	8.69	6.09
3-yr (N)	-24.74	-23.43	-50.18	-52.05	-40.38	-26.79
5-yr (N)	52.21	45.98	88.61	78.96	49.93	25.56
7-yr (N)	-40.38	-33.41	-55.75	-38.44	-12.52	3.57
10-yr (N)	9.17	6.99	8.97	2.23	-4.64	-7.38
Instantaneously risk-free asset	1.05	-0.03	-0.09	-0.10	-0.08	-0.05
Panel B: 5 Real bonds						
	T=0	T=1	T=3	T=5	T=7	T=10
1-yr (R)	0.00	1.00	0.00	0.00	0.00	0.00
3-yr (R)	0.00	0.00	1.00	0.00	0.00	0.00
5-yr (R)	0.00	0.00	0.00	1.00	0.00	0.00
7-yr (R)	0.00	0.00	0.00	0.00	1.00	0.00
10-yr (R)	0.00	0.00	0.00	0.00	0.00	1.00
Instantaneously risk-free asset	1.00	0.00	0.00	0.00	0.00	0.00

Notes: This Table shows the optimal portfolio choice of an infinitely risk averse investor ($\gamma = 100,000$), who has utility over real terminal wealth and access to five bonds. Panel A shows the portfolio choice among five nominal bonds for the macroeconomic conditions prevailing in 1975:Q1. Panel B shows the corresponding portfolio choice when the investor has access to five real bonds. The hedging demand for horizon T is given by the difference between the total demand for horizon T and the demand when $T = 0$ (myopic demand). The allocation to the corresponding instantaneously riskless asset is equal to one minus the sum of the total bond demands in each case, provided in the last line of each panel.

Table 7: Portfolio choice among two nominal bonds

		$\gamma = 4$				$\gamma = 10$			
Panel A: Benchmark case 1975:Q1									
	Premia	T=0	T=3	T=5	T=10	T=0	T=3	T=5	T=10
3-yr	0.65%	0.05	5.17	4.66	1.94	0.03	3.79	3.88	2.01
10-yr	1.71%	0.26	0.29	1.28	3.12	0.10	0.02	0.63	2.12
Panel B: Increase in the inflation central tendency									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=3	T=5	T=10	T=0	T=3	T=5	T=10
3-yr	0.88%	0.84	9.39	8.77	5.47	0.35	6.07	6.13	3.82
10-yr	1.96%	0.002	-0.24	0.94	3.15	-0.004	-0.23	0.52	2.32
Panel C: Decrease in the inflation central tendency									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=3	T=5	T=10	T=0	T=3	T=5	T=10
3-yr	0.42%	-0.74	0.94	0.56	-1.59	-0.28	1.52	1.64	0.19
10-yr	1.46%	0.52	0.82	1.62	3.09	0.20	0.26	0.74	1.91

Notes: This Table shows the optimal portfolio choice of an investor who has utility over real terminal wealth and access to two nominal bonds, given by (65). Panel A shows the portfolio choice for the macroeconomic conditions prevailing in 1975:Q1 ($y = -5.89\%$, $\pi = 10.43\%$, $\rho = -5.21\%$, $\pi^* = 4.55\%$ and $\rho^* = 0.67\%$). Panel B shows the corresponding portfolio choice when the inflation central tendency π^* is increased by one standard deviation ($\sigma_{\pi^*} = 0.81\%$), while Panel C shows the corresponding choice when the inflation central tendency π^* is decreased by one standard deviation. The hedging demand for horizon T is given by the difference between the total demand for horizon T and the demand when $T = 0$ (myopic demand). The allocation to the nominal instantaneously riskless asset is equal to one minus the sum of the total bond demands in each case.

Table 8: Portfolio choice among two real bonds

| | | $\gamma = 4$ | | | | $\gamma = 10$ | | | |
	Premia	T=0	T=3	T=5	T=10	T=0	T=3	T=5	T=10
Panel A: Benchmark case 1975:Q1									
3-yr	-0.25%	4.76	5.17	3.13	0.03	1.90	2.63	1.38	-0.82
10-yr	-1.59%	-3.01	-0.85	1.16	3.33	-1.20	-0.02	1.41	3.04
Panel B: Increase in the inflation central tendency									
3-yr	0.12%	5.65	6.32	3.89	0.19	2.26	3.12	1.59	-1.05
10-yr	-1.14%	-3.39	-0.02	2.37	4.96	-1.36	0.52	2.24	4.21
Panel C: Decrease in the inflation central tendency									
3-yr	-0.62%	3.87	4.01	2.37	-0.14	1.55	2.14	1.16	-0.59
10-yr	-2.06%	-2.63	-1.68	-0.05	1.69	-1.05	-0.56	0.58	1.87

Notes: This Table shows the optimal portfolio choice of an investor who has utility over real terminal wealth and access to real bonds, given by (77). Panel A shows the portfolio choice for the macroeconomic conditions prevailing in 1975:Q1 ($y = -5.89\%$, $\pi = 10.43\%$, $\rho = -5.21\%$, $\pi^* = 4.55\%$ and $\rho^* = 0.67\%$). Panel B shows the corresponding portfolio choice when the inflation central tendency π^* is increased by one standard deviation ($\sigma_{\pi^*} = 0.81\%$), while Panel C shows the corresponding choice when the inflation central tendency π^* is decreased by one standard deviation. The hedging demand for horizon T is given by the difference between the total demand for horizon T and the demand when $T = 0$ (myopic demand). The allocation to the real instantaneously riskless asset is equal to one minus the sum of the total bond demands in each case.

Table 9: Portfolio choice among three bonds

|		$\gamma = 4$				$\gamma = 10$			
Panel A: 3 Nominal bonds (nominal SDF)									
	Premia	T=0	T=1	T=5	T=10	T=0	T=1	T=5	T=10
1-yr (N)	0.20%	1.74	0.73	-5.96	-3.89	0.14	-0.25	-5.77	-4.31
5-yr (N)	0.91%	-1.61	1.27	8.66	4.09	-0.27	1.53	8.08	4.74
10-yr (N)	1.71%	0.99	-0.07	-1.07	2.13	0.25	-0.42	-1.66	0.91
Panel B: 3 Real bonds (real SDF)									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=1	T=5	T=10	T=0	T=1	T=5	T=10
1-yr (R)	0.34%	29.69	31.68	30.51	31.71	11.87	13.42	12.52	13.40
5-yr (R)	-1.37%	-12.96	-13.45	-15.90	-20.92	-5.18	-5.44	-6.64	-10.67
10-yr (R)	-1.59%	3.83	4.47	9.46	13.76	1.53	1.87	5.13	8.78
Panel C: 3 Nominal and real bonds under the nominal SDF									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=1	T=5	T=10	T=0	T=1	T=5	T=10
1-yr (N)	0.20%	2.98	4.71	4.81	2.28	0.87	2.00	1.99	1.17
5-yr (R)	-0.84%	-1.42	-0.99	0.83	-0.37	-0.46	-0.14	1.08	0.63
10-yr (N)	1.71%	0.60	0.63	3.15	3.74	0.23	0.23	1.30	2.43

Notes: This Table shows the optimal portfolio choice of an investor who has utility over real terminal wealth and access to three bonds for the macroeconomic conditions prevailing in 1975:Q1. Panel A shows the allocation among three nominal bonds, given by (65), while Panel B shows the allocation among three real bonds, given by (77). Panel C shows the corresponding allocation among two nominal and one real bond. The dynamics for the returns of the real bond are given by the SDE in (41) and they are stated under the nominal SDF. The hedging demand for horizon T is given by the difference between the total demand for horizon T and the demand when $T = 0$ (myopic demand). The allocation to the instantaneously riskless asset is equal to one minus the sum of the total bond demands in each case.

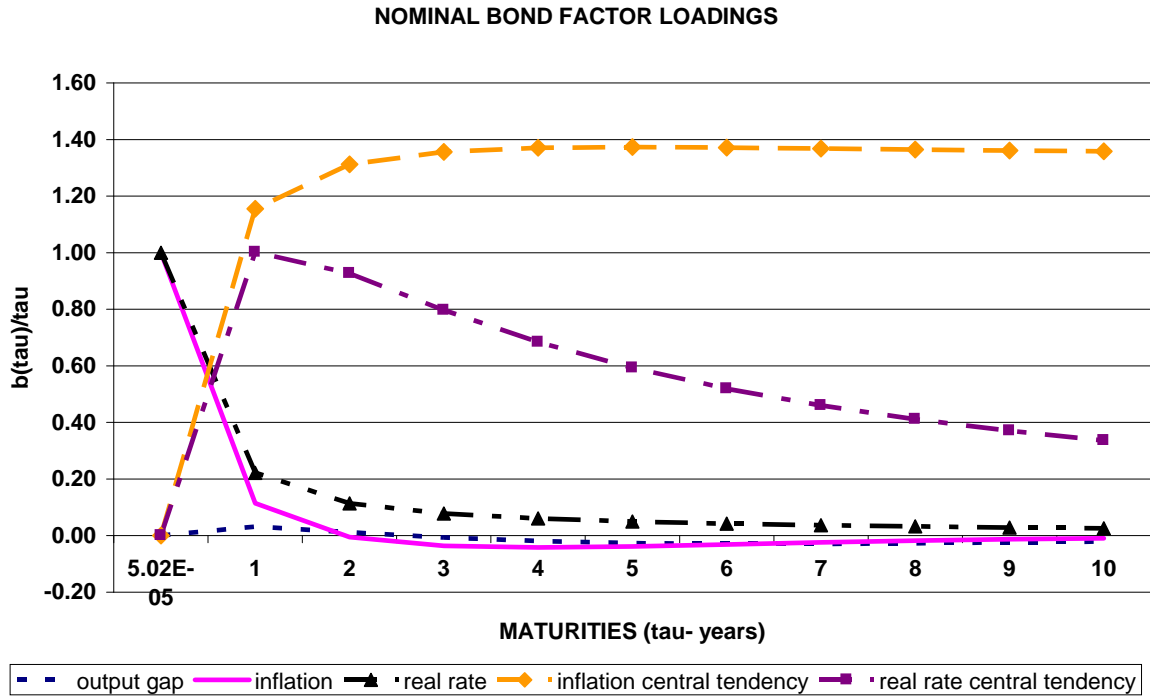Figure 1: Output gap, inflation and inflation central tendency
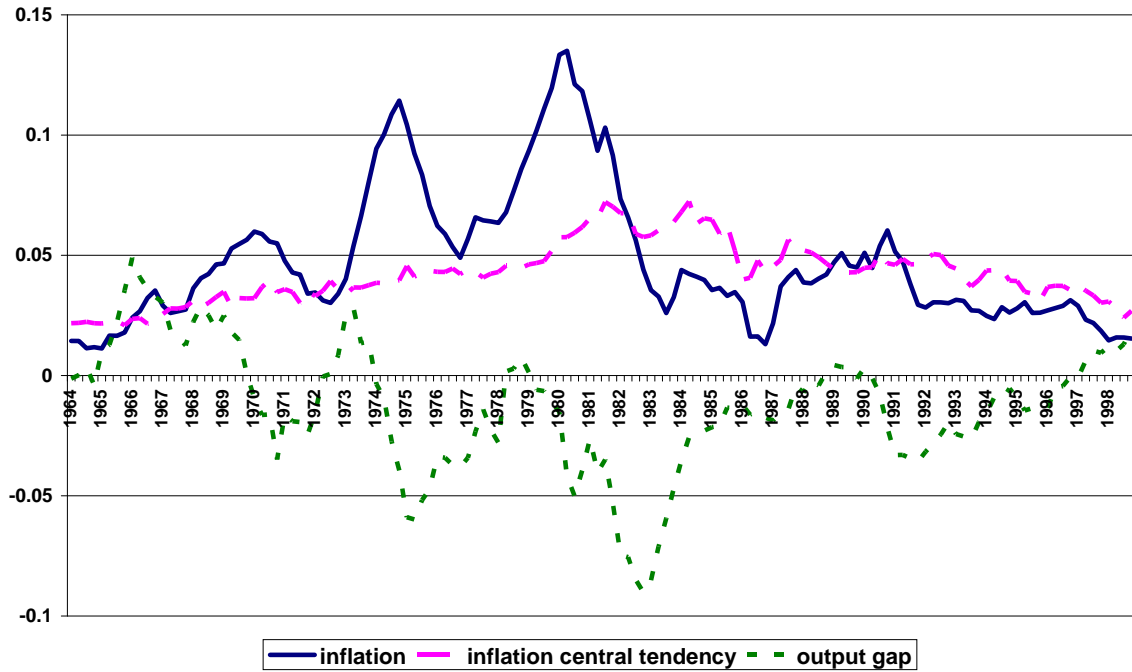
Notes: This figure plots the output gap y (dotted line) and the inflation series π (solid line), along with the filtered series for the central tendency of inflation π^* (dashed line) during the sample period 1964:Q1 to 1998:Q4.

Figure 2: Real interest rate and real interest rate central tendency
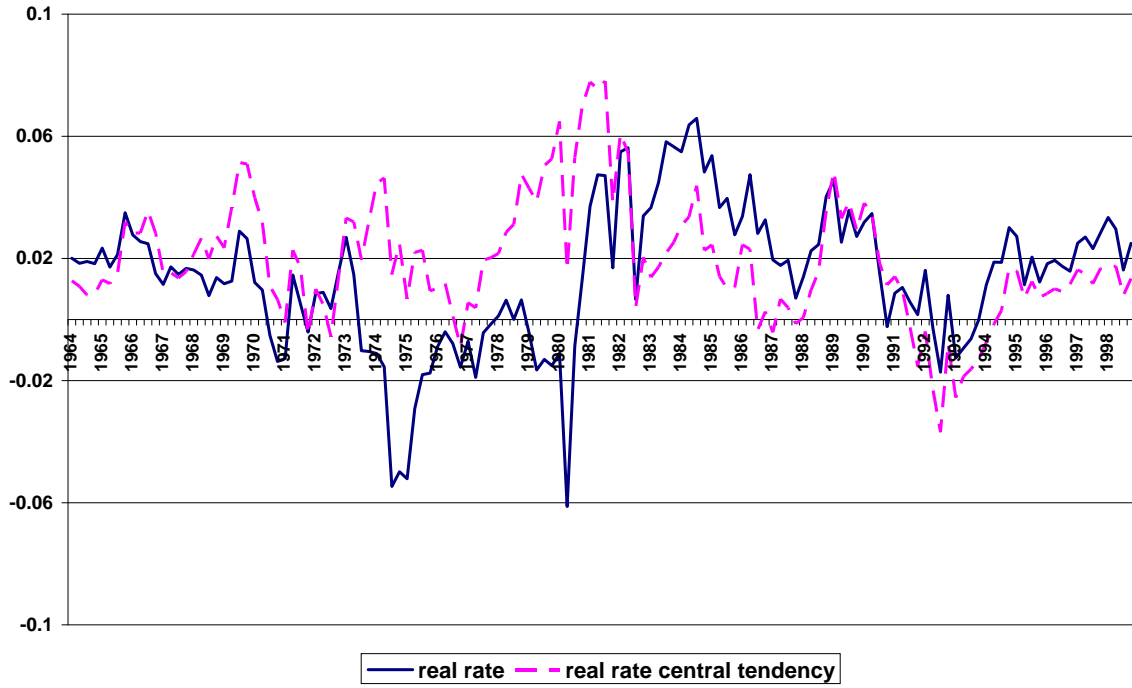
Notes: This figure plots the filtered series of the real interest rate ρ (solid line) and the central tendency of the real interest rate ρ^* (dashed line), during the sample period 1964:Q1 to 1998:Q4.

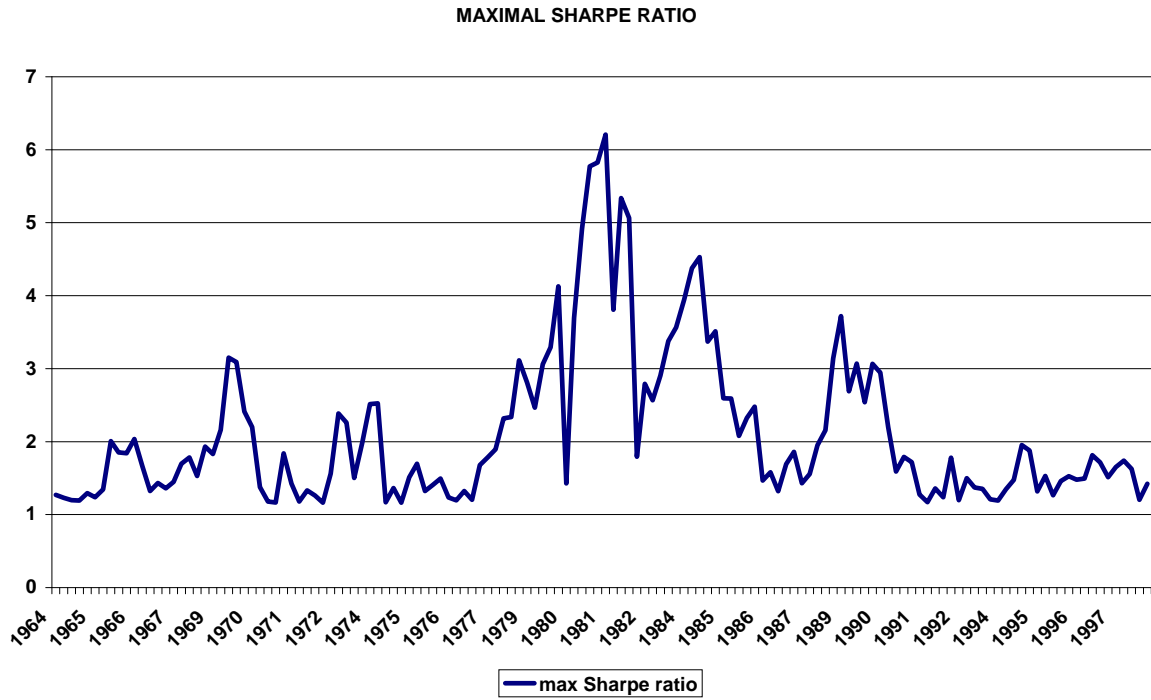Figure 3: Factor Loadings of nominal zero-coupon bonds

NOMINAL BOND FACTOR LOADINGS

Notes: This figure plots the loadings of the output gap y (dotted line), inflation π (solid line), real rate ρ (double dot-dashed line), inflation central tendency π^* (dashed line) and real rate central tendency ρ^* (dot-dashed line) on the nominal zero-coupon bonds, adjusted for the corresponding maturities, i.e. $b(\tau)/\tau$.

Figure 4: Expected excess returns of nominal zero-coupon bonds
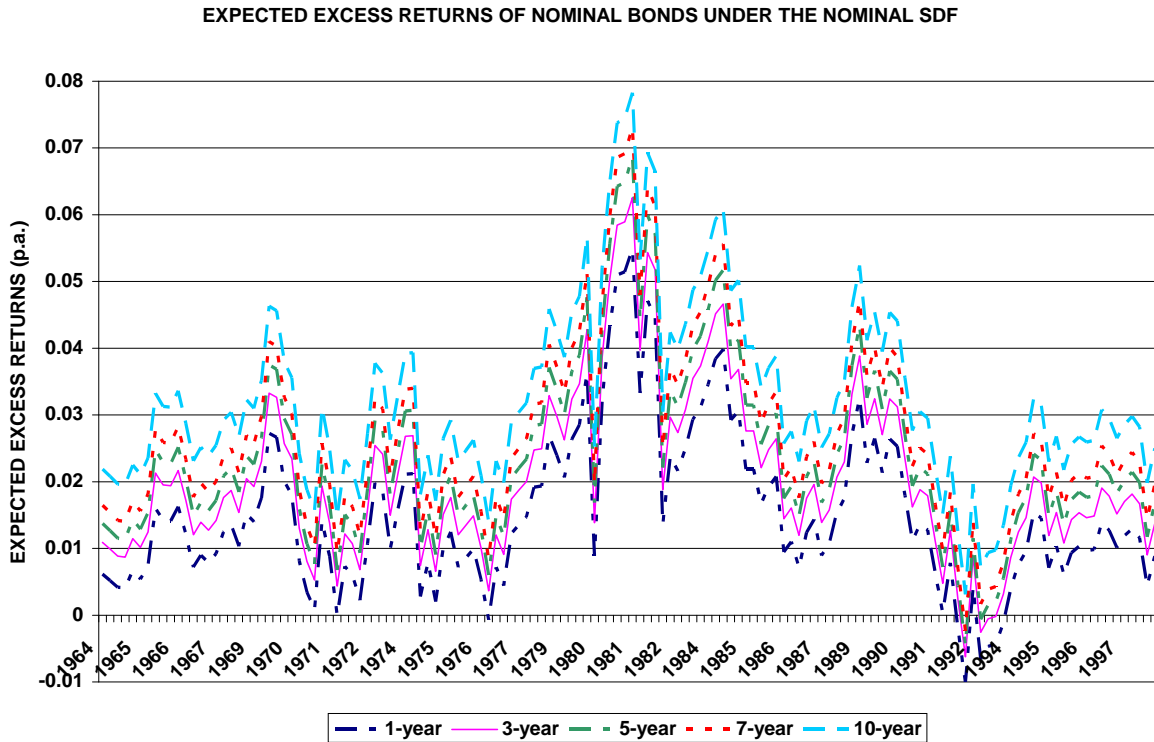
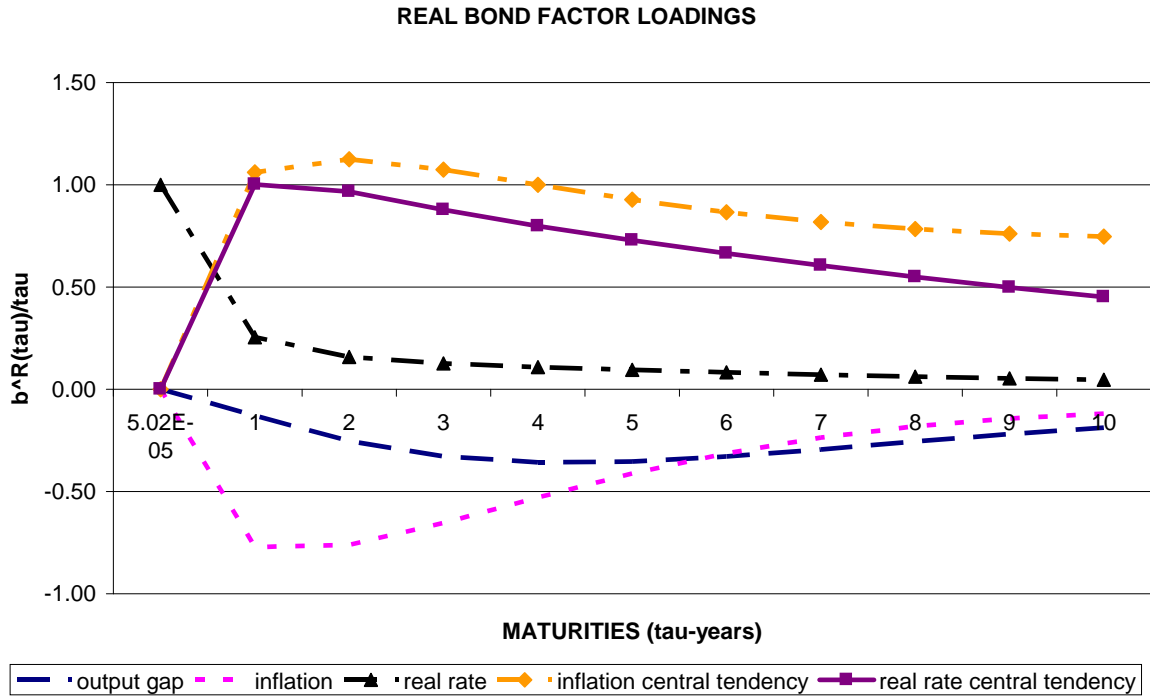**EXPECTED EXCESS RETURNS OF NOMINAL BONDS UNDER THE NOMINAL SDF**

Notes: This figure shows the expected excess returns of the nominal zero-coupon bonds with maturities of 1, 3, 5, 7 and 10 years over the sample period 1964:Q1 to 1998:Q4.

Figure 5: Maximal Sharpe Ratio

MAXIMAL SHARPE RATIO

Notes: This figure plots the Maximal Sharpe ratio, given by the norm $\sqrt{\xi^T \xi}$, over the sample period 1964:Q1 to 1998:Q4.

Figure 6: Factor Loadings of real zero-coupon bonds

REAL BOND FACTOR LOADINGS

Notes: This figure plots the loadings of the output gap y (dashed line), inflation π (dotted line), real rate ρ (dot-dashed line), inflation central tendency π^* (double dot-dashed line) and real rate central tendency ρ^* (solid line) on the real zero-coupon bonds, adjusted for the corresponding maturities, i.e. $b^R(\tau)/\tau$.

Figure 7: Expected excess returns of real zero-coupon bonds

EXPECTED EXCESS RETURNS OF REAL BONDS UNDER THE REAL SDF
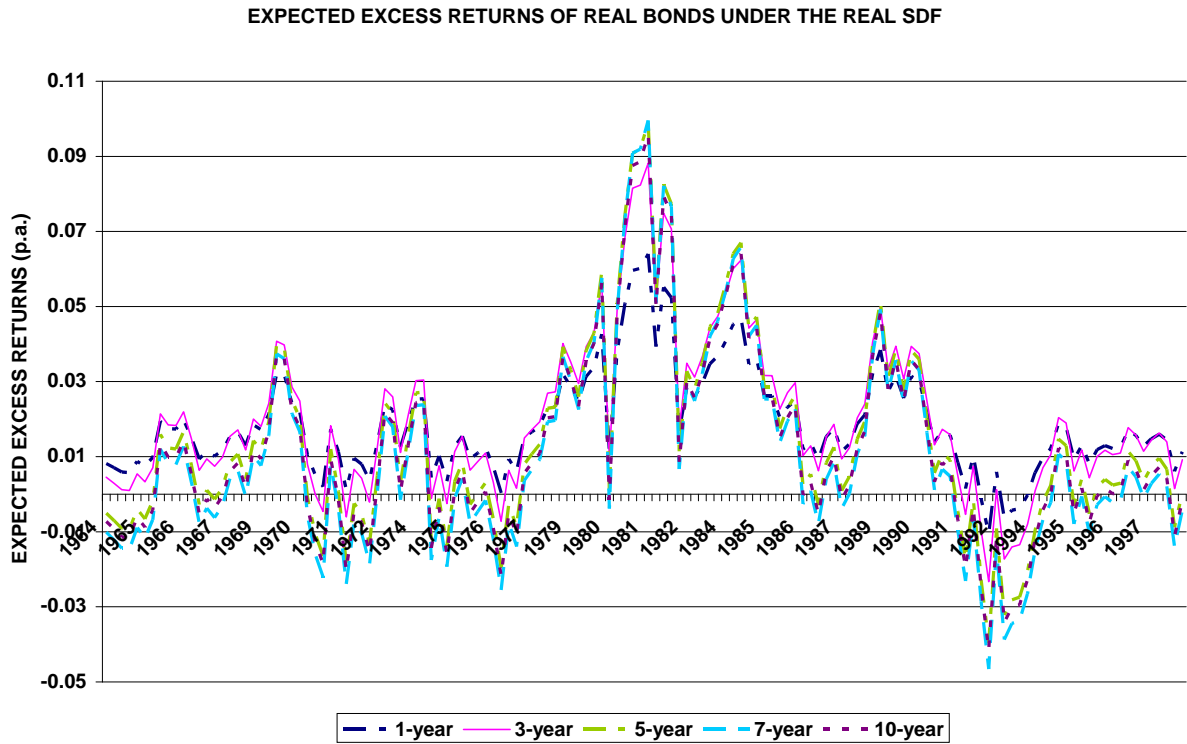

Notes: This figure shows the expected excess returns of the real zero-coupon bonds under the real SDF with maturities of 1, 3, 5, 7 and 10 years, over the sample period 1964:Q1 to 1998:Q4.

Figure 8: Wealth sensitivities with respect to the risk factors
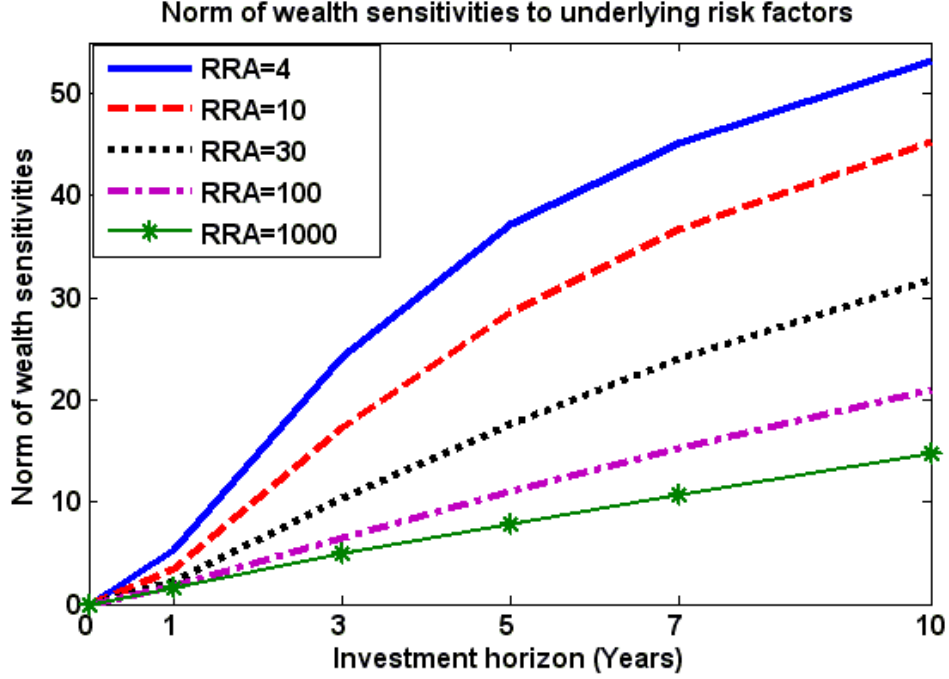
Notes: This figure shows the norm of the wealth sensitivities with respect to the underlying macroeconomic risk factors for various degrees of Relative Risk Aversion (RRA) as the investment horizon increases. The choice for the macroeconomic risk factors are those prevailing in 1975:Q1 ($y = -5.89\%$, $\pi = 10.43\%$, $\rho = -5.21\%$, $\pi^* = 4.55\%$ and $\rho^* = 0.67\%$). These norms correspond to the case of an investor with utility over real terminal wealth and they are given by $\sqrt{(\frac{\partial G}{\partial X}\frac{1}{G})^T(\frac{\partial G}{\partial X}\frac{1}{G})}$, where G is defined in (30).

Figure 9: Total myopic and hedging portfolio choices for 2 nominal bonds
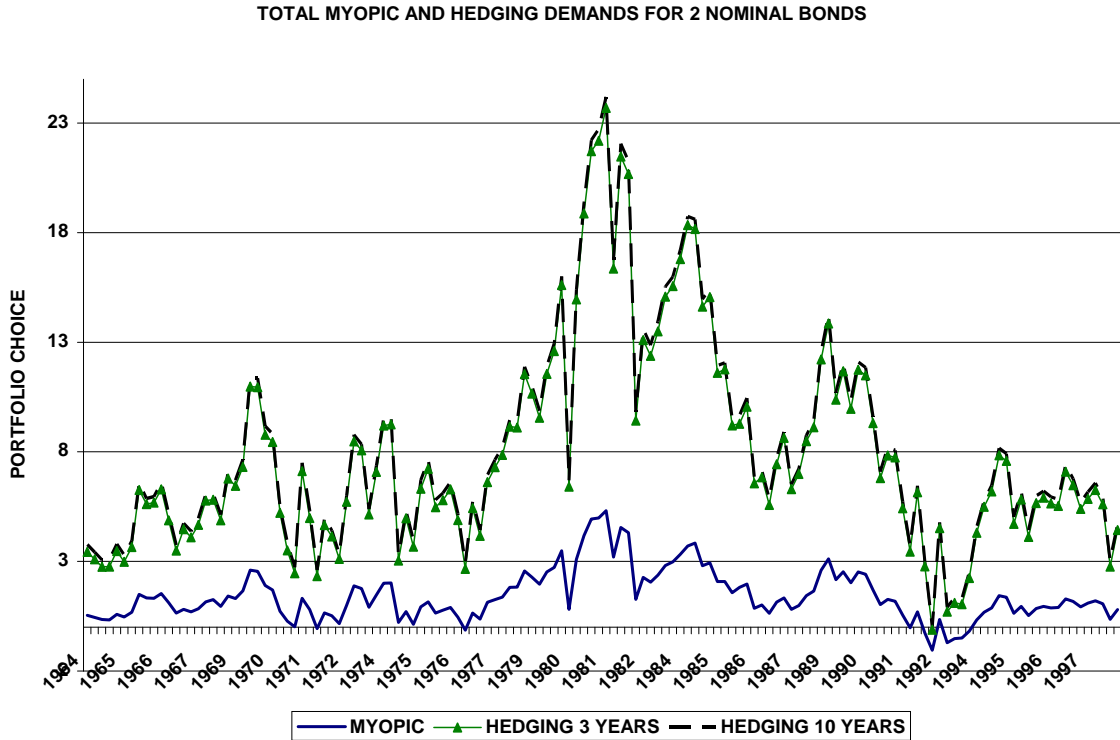
TOTAL MYOPIC AND HEDGING DEMANDS FOR 2 NOMINAL BONDS

MYOPIC — HEDGING 3 YEARS — HEDGING 10 YEARS

Notes: This figure shows the total myopic bond portfolio choice (solid line) of a power utility investor with coefficient of relative risk aversion $\gamma = 10$, who has access to a 3-year and a 10-year nominal zero-coupon bonds for the period 1964:Q1 to 1998:Q4. This sum is given by $i^T \phi^{myopic}$ where ϕ^{myopic} consists of the first two terms in equation (65). The triangle-marked line shows the total hedging bond demand for the power utility investor with $\gamma = 10$, who has a horizon of $T = 3$ years and access to the same nominal bonds. The dashed line shows the corresponding total hedging bond demand for the investor with a horizon of $T = 10$ years. For each case, this sum is given by $i^T \phi^{hedging}$, where $\phi^{hedging}$ is given by the third term in equation (65).

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Figure 10: Total myopic and hedging portfolio choices for 2 real bonds
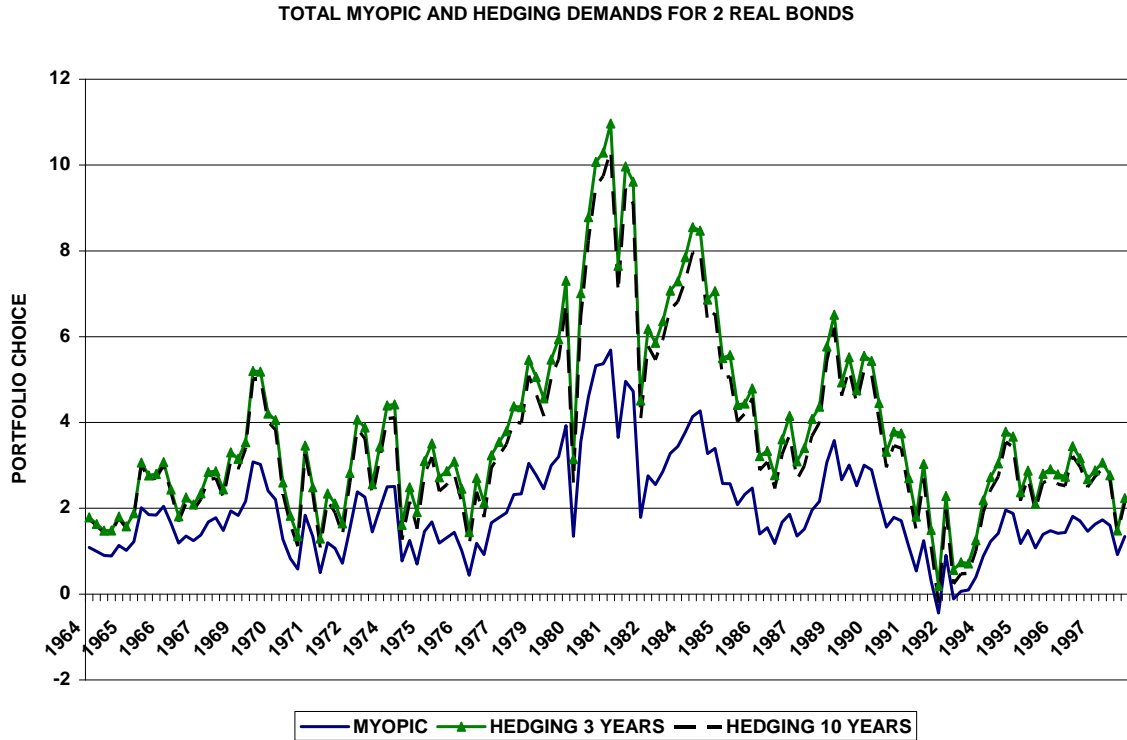
TOTAL MYOPIC AND HEDGING DEMANDS FOR 2 REAL BONDS

Notes: This figure shows the total myopic bond portfolio choice (solid line) of a power utility investor with coefficient of relative risk aversion $\gamma = 10$, who has access to a 3-year and a 10-year real zero-coupon bonds for the period 1964:Q1 to 1998:Q4. This sum is given by $i^T \phi^{myopic}$ where ϕ^{myopic} consists of the first term in equation (77). The triangle-marked line shows the total hedging bond demand for the power utility investor with $\gamma = 10$, who has a horizon of $T = 3$ years and access to the same real zero-coupon bonds. The dashed line shows the corresponding total hedging bond demand for the investor with a horizon of $T = 10$ years. For each case, this sum is given by $i^T \phi^{hedging}$, where $\phi^{hedging}$ is given by the second term in equation (77).

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