Multiple Trees Subject to Event Risk

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ABSTRACT

We study an equilibrium asset pricing model with several Lucas (1978) trees subject to event risk, that is, the possibility that trees experience unexpected disasters. We exploit the market clearing mechanism, in the presence of multiple positive net supply assets, to show that the implications of disasters for some cash-flows extend to the valuation of seemingly unrelated ones. Price-dividend ratios, risk premia, credit-spreads depend on the share of aggregate supply of each tree, but the endogeneity of risk neutral probabilities of disaster implies that the asset pricing implications of event risk go beyond the effects analyzed by the 'multiple tree' literature so far.

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I. Introduction

In an economy with multiple positive net supply assets, returns depends on the share of aggregate endowment that each asset supplies. This remains true also in case assets pay independent cash-flows. This issue has been investigated in Cochrane, Longstaff and Santa-Clara (2007). If assets can experience disasters and cease to supply output for a given time period, aggregate consumption decreases abruptly each time a disaster occurs. The likelihood and relative size of these jumps in consumption is time-varying, in that it depends on the extent to which consumption is diversified among supplying assets. This paper studies an equilibrium asset pricing model with several Lucas (1978) trees subject to event risk. We interpret 'event risk' as the possibility to experience a disaster, and to recover from this disaster state. Each tree's dividend stream is a constant multiple of unitary supply, described by a geometric Brownian motion, but upon 'disaster' trees cease to supply dividends. Once the trees recover from their disaster state, they resume their normal dividend supply. A continuous-time Markov chain governs the transition from 'regular supply' to disaster states, and conversely. The instantaneous probabilities of disaster of each tree can switch randomly between a 'high' and a 'low' state, according to the state of the economy, however, the representative agent faces incomplete information, because he doesn't observe the current state of these probabilities, but tries to infer them from past dividend observations.

The idea that the risk of rare disasters can help explain the equity premium, as well as stock market volatility, has been applied successfully in Barro (2006), Wachter (2009) and Gabaix (2009), among others. The influence of event risk in an economy with multiple supplying trees is unexplored though. Indeed, the asset pricing implications of our framework go beyond the event riskiness of each stand-alone tree, because risk premia depend on the fraction of aggregate endowment that each tree supplies and on the perceived likelihood of default of endowments. This highlights two mechanisms by which disaster or recovery events of some assets impacts the evaluation of seemingly uncorrelated endowments. The perceived probability of disaster increases as the agent observes more disasters, and decreases as recoveries occur. The share of a given endowment increases as different endowments undergo disasters. Our analysis emphasizes the importance of event correlations - an assessment of the likelihood to spend most of the time-horizon in the same disaster or 'normal supply' state - because the term premium is the relevant risk compensation measure for securities that do not expire instantaneously, and in our economy this measure is related to the expected future market share values. A disaster of a given tree generates a contagion effect, hence a negative return, for those securities that have high event correlation with the tree that has experienced the disaster, because their share of event risk has increased dramatically in a period where equilibrium state prices are also high, because of the aggregate consumption fall. Consistently with empirical findings, credit spreads experience upward jumps when different securities default. The higher the event correlation between the endowments, the higher the increase of the credit spread.

The article is organized as follows: Section II describes the economy. Section III analyzes the learning mechanism of the representative agent. In Section IV we solve in closed-form for security prices, risk premia and we study these quantities using parameters estimates from a simple calibration exercise. Section V discusses correlations. Section VI analyzes the term structure of credit spreads. Section VII concludes. All proofs are in the Appendices.

II. The Economy

On an infinite time horizon, we consider a standard pure exchange economy populated by a single, representative agent who maximizes isoelastic utility of intertemporal consumption, with Relative Risk Aversion coefficient γ and subjective discount rate δ:

$$U_0 = \mathbb{E}\left[\int_0^\infty e^{-\delta s}\frac{C_s^{1-\gamma}}{1-\gamma}ds\right] \tag{1}$$

The opportunity set of the investor comprises a locally risk-less security in zero net supply, with rate of return r_t (the interest rate), and N risky securities. These risky securities, or 'trees', are claim to a stochastic dividend process ϵ_t^i, $i = 1, \ldots, N$.

Trees can be identified with the industries of a single-country economy. Alternatively, they can be interpreted as single-country economies of a broader international framework. The distinctive feature of our model is that trees are subject to 'disasters', meaning that with some probability their dividend experiences an abrupt fall. To properly analyze the consequences of this feature, we assume a simple endowment structure. The $i-$th tree supplies a multiple x_t^i of an 'aggregate' state variable Y_t, which follows a geometric Brownian motion. This multiple 'normally' assumes a value $\overline{x}_h^i$, and with a small instantaneous probability λ_t^i can dramatically fall to zero. Empirical observations suggest that the dividend gradually reverts back to normality,, therefore we assume that the dividend reverts to normality with some probability η_t^i, the magnitude of which determines whether the 'disaster' state is more or less persistent. From an economic standpoint, it is intuitive that our results are going to be modulated by two opposing tensions - persistence of 'normal' against persistence of 'disaster' states. Individual endowments can be described as follows:

$$\begin{aligned} \epsilon_t^i &= x_t^i Y_t \quad i = 1, 2, \ldots N & (2)\\ dY_t &= Y_t\left(\mu_Y dt + \sigma_Y dZ_t\right) & (3)\end{aligned}$$

Technically speaking, endowments' disasters and recoveries are jump times of a continuous-time Markov chain. Both Watcher (2009) and Gabaix (2009) discuss the importance of time-varying disaster probabilities to match empirical regularities. We follow this direction and assume that these probabilities (or intensities) satisfy the following simple factor model:[1]

$$\lambda_t^i = f^i(z_t), \quad \eta_t^i = g^i(z_t) \quad i = 1, 2, \dots N \tag{5}$$

where $f^i(\cdot)$ and $g^i(\cdot)$ are positive functions of a factor process z_t, which in turns evolves as a two-state continuous-time Markov chain, with states $\overline{z}^h$ and $\overline{z}^l$ and transition matrix

$$I = \begin{bmatrix} -k_h(\mathbf{x}_t) & k_h(\mathbf{x}_t) \\ k_l(\mathbf{x}_t) & -k_l(\mathbf{x}_t) \end{bmatrix}$$

$k_i(\cdot)$, $i = h, l$ are positive function of the current vector of multiples $\mathbf{x}_t = (x_t^1, x_t^2, \dots x_t^N)'$. In other words, in the infinitesimal time interval $[t, t+\Delta]$, there is a probability $k_i(\mathbf{x}_t)\Delta$ that z_t jumps from $\overline{z}^i$ to $\overline{z}^j$, $i = h, l$, $j = h, l$, $i \neq j$. The role of the systematic variable z_t is to capture the state of the economy, therefore we assume that $f^i(\overline{z}^h) < f^i(\overline{z}^l)$ and $g^i(\overline{z}^h) > g^i(\overline{z}^l)$ for any tree i, in accordance with the intuition that the probability of a disaster (recovery from a disaster) should be higher (lower) in a 'bad' economic state. It is a well documented fact that the 'disaster' condition of a few sectors could propagate endemically economy-wide. This is essentially why we let the probability of a good (bad) state depend on disasters. The representative agent observes Y_t, disaster times of each tree and functional forms (f^i, g^i), but does not observe the state of the economy z_t, hence intensities of disaster and recovery. He infers them in a Bayesian fashion from observations of individual endowments, and an external vector of signals with dynamics:

$$\begin{aligned} dS_t^f &= \mathbf{f}(z_t)dt + \Omega_f^{-1}\, dB_t^f \qquad & \mathbf{f} = (f^1, f^2, \dots f^N)' \\ dS_t^g &= \mathbf{g}(z_t)dt + \Omega_g^{-1}\, dB_t^g \qquad & \mathbf{g} = (g^1, g^2, \dots g^N)' \end{aligned}$$

where B_t^j, $j = f, g$ is an $N-$dimensional standard Brownian motion and Ω_j is a constant diagonal matrix of signals precisions.

[1]In other words, the multiple x_t^i has a transition matrix of the form

$$\begin{bmatrix} -\lambda_t^i & \lambda_t^i \\ \eta_t^i & -\eta_t^i \end{bmatrix}$$

We could introduce an idiosyncratic component in the endowments' intensities. For instance, we may have:

$$\lambda_t^i = f^i(z_t)\varepsilon^i \qquad i = 1, 2, \dots N \tag{4}$$

where ε^i are i.i.d idiosyncratic shocks, uniformly distributed in the range $[1-u, 1+u]$, $u < 1$. This additional assumption would not alter significantly our conclusions.

This is an endowment economy, so prices adjust until aggregate consumption C_t coincides with the sum of the dividend processes:

$$C_t = Y_t \sum_{i=1}^{N} x_t^i \tag{6}$$

III. Learning

The information set of the representative agent is the sigma field generated by the dividend processes $\epsilon_t^i = Y_t x_t^i$ and the signal processes S_t^f and S_t^g, henceforth denoted as $\mathcal{F}_t^{x,S}$. Individual disaster and recovery probabilities are unobservable and inferred in a Bayesian fashion.

A. *Belief Dynamics*

Let $p_t^h = P(z_t = \overline{z}^h | \mathcal{F}_t^{x,S})$ denote the investor's belief that the economy is in a 'high' state. The expected instantaneous probability of disaster for the $i-$th endowment, conditional on available information, is then

$$\widehat{\lambda}_t^i = \mathbb{E}_t \left[\lambda_t^i \middle| \mathcal{F}_t^{H,S} \right] = p_t^h f^i(\overline{z}^h) + (1 - p_t^h) f^i(\overline{z}^l) \tag{7}$$

A similar expression holds for the expected probability of recovery, $\widehat{\eta}_t^i$. We have the following Proposition.

Proposition 1 *Let p_0^h denote investor's prior belief that the economy is in a 'high' state, and let also $H_t^i = \mathbf{1}(x_t^i = 0)$, so that dH_t^i is 1 (-1) if a disaster (recovery from disaster) occurs for tree i. Then*

$$\begin{aligned} p_t^h &= p_0^h + \int_0^t [k_l(\mathbf{x}_s) + k_h(\mathbf{x}_s)] \left[\frac{k_l(\mathbf{x}_s)}{k_l(\mathbf{x}_s) + k_h(\mathbf{x}_s)} - p_s^h \right] ds + \\ & \int_0^t p_s^h (1 - p_s^h) \left(\Lambda^f d\widetilde{B}_s^f + \Lambda^g d\widetilde{B}_s^g \right) \end{aligned} \tag{8}$$

$$\begin{aligned} + & \int_0^t p_s^h (1 - p_s^h) \sum_{i=1}^{N} \left[(1 - H_s^i) \frac{f^i(\overline{z}^h) - f^i(\overline{z}^l)}{\widehat{\lambda}_s^i} (dH_s^i - \widehat{\lambda}_s^i ds) - \right. \\ & \left. H_s^i \frac{g^i(\overline{z}^h) - g^i(\overline{z}^l)}{\widehat{\eta}_s^i} (dH_s^i + \widehat{\eta}_s^i ds) \right] \end{aligned} \tag{9}$$

where

$$d\widetilde{B}_t^f = \Omega^f [dS_t^f - (p_t^h \mathbf{f}(\overline{z}^h) + (1 - p_t^h) \mathbf{f}(\overline{z}^l))] \tag{10}$$

$$d\widetilde{B}_t^g = \Omega^g [dS_t^g - (p_t^h \mathbf{g}(\overline{z}^h) + (1 - p_t^h) \mathbf{g}(\overline{z}^l))] \tag{11}$$

is a standard Brownian motion with respect to $\mathcal{F}_t^{x,S}$*. Furthermore,*

$$\Lambda^f = [\mathbf{f}(\overline{z}^h) - \mathbf{f}(\overline{z}^l)]' \tag{12}$$

$$\Lambda^g = [\mathbf{g}(\overline{z}^h) - \mathbf{g}(\overline{z}^l)]'. \tag{13}$$

Expression (9) comprises a locally deterministic component and an innovation component. The former is a mean-reverting term that pulls p_s^h towards $k_l(\mathbf{x}_t)/(k_l(\mathbf{x}_t)+k_h(\mathbf{x}_t))$. If the current shares of aggregate consumption supplied by each tree did not change, this term would be the proportion of time that z_t spends on a 'high' state in the long run. The 'local' speed of mean reversion is the probability of occurrence of some regime switch, $k_l(\mathbf{x}_t)+k_h(\mathbf{x}_t)$, because convergence to the conditional long-run mean is faster if the frequency of regime transitions is higher. The innovation component comprises an updating rule for signal realizations and an updating rule for disaster and recovery observations. The former is proportional to the normalized innovation processes $\widetilde{B}^j$, $j = f, g$, where the normalization assigns smaller weights to less precise signals. The 'reaction' to these innovations, $p_t^h(1-p_t^h)\Lambda^j$, implies that a major update occurs when the difference of disaster or recovery probabilities in the two states is high, and when the uncertainty about the current state is maximal, $p_t^h = 0.5$. Intuitively, in this situation every innovation is interpreted as a univocal resolution of uncertainty. The updating rule for disaster (recovery) observations is also proportional to the normalized innovation process $(dH^i - \widehat{\lambda}_t^i)/\widehat{\lambda}_t^i$ $(-(dH^i - \widehat{\eta}_t^i)/\widehat{\eta}_t^i)$. The normalization assigns a smaller weight to innovations that pertain to a volatile disaster or recovery process. The 'reaction' these events is also proportional to $p_t^h(1-p_t^h)\Lambda$, because when uncertainty is high and states of the world imply a very heterogeneous instantaneous probability of disaster or recovery across states, the agent is more willing to interpret an event as evidence of an high intensity state, hence 'bad' economic state in case of disaster or 'good' state in case of recovery. Obviously, a disaster event leads to a lower posterior probability of 'good' state, because the difference of intensities, $f^i(\overline{z}^h) - f^i(\overline{z}^l)$ is negative. The opposite is true for a recovery event.

B. Characteristics of Disaster and Recovery Processes

Disaster and recovery events are the main constituent of the asset prices' behavior in our simple economy. It is important to understand the main characteristics of these events. What is the probability that some tree will not undergo a disaster before a given future date T and what is the expected time until the next disaster ? Provided that a disaster has occurred, how long does it take on average for the tree to recover ? Finally, which fraction of time will the economy spend free of disasters ? The next proposition addresses these questions.

Proposition 2 *Let* $D := \{d_1, d_2, \dots, d_L\}$ *and* τ_{d_i} *denote a collection of* $L \leq N$ *trees and the first disaster time for tree* d_i*, respectively. The posterior joint probability of no disaster for the elements of* D *until time* T*, conditional on information at time* $s < T$*, is:*

$$P(\tau_{d_1} > T, \tau_{d_2} > T, \dots \tau_{d_L} > T | \mathcal{F}_s^{x,S}) = \mathbf{1}(\tau_{d_1} > s, \tau_{d_2} > s, \dots \tau_{d_L} > s) \times \\ [p_t^h, 1 - p_t^h, 0_{\mathcal{N}-2}] \cdot \exp\left(-\mathbf{A}(T-s)\right) \overline{1}_{\mathcal{N}} \quad (14)$$

where $\exp(\,\cdot\,)$ *denotes the matrix exponential operator,* $\overline{1}_{\cdot}$ *(*$0_{\cdot}$*) denotes a vector of ones (zeros),* $\mathcal{N} = 2^{N-L}$ *and matrix* $\mathbf{A}$ *is reported in the Appendix. Furthermore the expected time until the disaster of any of the trees in group* D *is*

$$\mathbb{E}\left[\min_{i \in D} \tau^i - s \middle| \mathcal{F}_s^{H,S}\right] = [p_t^h, 1 - p_t^h, 0_{\mathcal{N}-2}] \cdot \mathbf{A}^{-1} \overline{1}_{\mathcal{N}}$$

Let τ_r^i *denote the first recovery time of a tree* i *that is currently in disaster state. The expected time until recovery is:*[2]

$$\mathbb{E}\left[\tau_r^i - s \middle| \mathcal{F}_s^{x,S}\right] = \mathbf{1}(\tau_r^i > s)[p_t^h, 1 - p_t^h, 0_{\mathcal{N}^i-2}] \cdot (\mathbf{A}^r)^{-1} \overline{1}_{\mathcal{N}^i}$$

The expected fraction of time on the horizon $[s, T]$ *with no disaster state for members of group* D *is:*

$$[p_t^h, 1 - p_t^h, 0_{\mathcal{N}-2}] \cdot \frac{(\mathbf{A}^H)^{-1}}{T - s} \left[I_d - \exp(-\mathbf{A}^H(T-s))\right] \overline{W}_{\mathcal{N}} \quad (15)$$

where expressions for $\mathbf{A}^r$*,* $\mathbf{A}^H$*, and* $\overline{W}_{\mathcal{N}}$ *are reported in the Appendix, while* $\mathcal{N}^i = 2^{N-1}$*.*

The elements of the vector $e^{-\mathbf{A}(T-s)} \overline{1}_{\mathcal{N}}$ are the full information probabilities of no disaster, conditional on any possible combination of disasters occurred. The probability of 'survival' of the L trees depends on the disaster history of the remaining trees, because the state of the economy z_t is influenced by the occurrence of disasters. To gauge more intuition about expression (14), assume that disaster intensities λ^i are constant, so that I reduces to a matrix of zeros. No-disaster probabilities then assume the familiar expression $e^{-(T-s)\sum_{i \in D} \lambda^i}$. But disaster intensities in our model react to different states of the economy, therefore these probabilities depend on the value of disaster intensities in 'good' or 'bad' states and on the probability of a regime switch. Figure 1 reports the term structure of full information survival probabilities for a given tree, assuming that the probability of an economic downturn is independent of disasters, that is, matrix I is constant.

Insert Figure 1 about here

[2] We have assumed without loss of generality that only the $i-$th tree is in disaster state.

Disaster intensities are very different across states of the economy, however in Panel 1 the 'good' state of the economy is very persistent, whereas the 'bad' state is scarcely persistent. No-disaster probabilities are close. In Panel 2 we have assumed instead that the low state is also very persistent, whereas intensity values $f(\overline{z}^h)$ and $f(\overline{z}^l)$ have been left unchanged. For every time horizon, the no disaster probability conditional on a current 'bad' state (dotted line) is sensibly lower than its high default state counterpart.

The no-disaster probability of a given tree depends in two ways on the disaster/recovery record of other trees. *i)* Perceived probabilities are weighted averages of full information conditional probabilities. If several trees have undergone disasters, then the posterior probability of a bad economic state is likely to be high, so that posterior probability of no-disaster are lower. However, each recovery leads to an increase in the probability of a good state, and to a higher posterior 'survival' probability. This channel acts ex-post and does not take into account the future perspective of disasters and recoveries. *ii)* Disasters and recoveries influence the likelihood of a good or bad economic state (entries of the matrix I), which in their turn influence the probability of disasters and recoveries. 'Survival' probabilities reported in (60) take this perspective feed-back into account. Figure 2 illustrates this point.

Insert Figure 2 about here

The dotted line is the posterior no-disaster probability of a given tree with probabilities of regime switches as in Panel 2 of Figure 1. The solid line reports the no-disaster probability of the same tree, when a disaster of a different tree causes a three-fold increase in the probability of an economic downturn. This assumption captures the idea that a turmoil occurring in a key economic sector can propagate economy-wise. In case of no feed-back the no disaster probability is sensibly higher, to the extent that the expected first disaster time for the tree drops from 31 years to 16 years, when feed-backs are taken into account. Parallely, the expected recovery time from a disaster for the same tree increases, from 25 to 35 years approximately. Figure 3 summarizes the general attitude to undergo disasters of a simple illustrative economy, by plotting, for different time horizons, the expected fraction of time that trees spend jointly free of disasters.

Insert Figure 3

As the horizon increases, the percentage of time with no disasters converges to zero. If the disaster of a given tree negatively affects the state of the economy, this fraction decreases much faster. This is especially apparent after a few years, when the 'crucial' tree has likely experienced a prolonged disaster. Besides the illustrative purpose of this stylized example, our framework can account for a number of realistic interactions.

IV. Model results

A. Aggregate Endowment and Endowment Share Dynamics

Aggregate consumption evolves continuously in time unless some tree experiences a disaster (or a recovery from disaster), in which case the relative consumption fall (increase) is proportional to the supply of the 'disastered' ('recovered') tree, relative to the aggregate supply before the disaster (recovery):[3]

$$\frac{dC_t}{C_t} = \mu dt + \sigma dZ_t - \sum_{i=1}^{N} \left(H_t^i \frac{\overline{x}_h^i}{\sum_{j\neq i} x_t^j} + (1 - H_t^i) \frac{\overline{x}_h^i}{\sum_{j=1}^{N} x_t^j} \right) dH_t^i \tag{16}$$

Expected consumption growth is the resultant of expected growth in the unitary supply Y_t, and of a component due to expected disaster and recovery events. This component is decreasing (increasing) in the instantaneous probability of disasters (recoveries) perceived by the representative agent:

$$\mathbb{E}\left[\frac{dC_t}{C_t} \middle| \mathcal{F}_t^{x,S} \right] = \mu - \sum_{i=1}^{N} \left[(1 - H_t^i) \frac{\overline{x}_h^i}{\sum_{j=1}^{N} x_t^j} \widehat{\lambda}_t^i - H_t^i \frac{\overline{x}_h^i}{\sum_{j\neq i} x_t^j} \widehat{\eta}_t^i \right] \tag{17}$$

Intuitively, if a high fraction of the aggregate product $Y_t \sum_i x_t^i$ is provided by a few sectors perceived as highly prone to disasters and with high expected recovery times then expected consumption growth is low. Consumption growth volatility is also time-varying:

$$\begin{aligned} \text{Var}\left[\frac{dC_t}{C_t} \middle| \mathcal{F}_t^{x,S} \right] = \sigma^2 + \sum_{i=1}^{N} \Bigg[& H_t^i \left(\frac{\overline{x}_h^i}{\sum_{j\neq i} x_t^j} \right)^2 \left[\widehat{\eta}_t^i + p_t^h (1 - p_t^h) (g^i(\overline{z}^h) - g^i(\overline{z}^l))^2 \right] + \\ & (1 - H_t^i) \left(\frac{\overline{x}_h^i}{\sum_{j=1}^{N} x_t^j} \right)^2 \left[\widehat{\lambda}_t^i + p_t^h (1 - p_t^h) (f^i(\overline{z}^h) - f^i(\overline{z}^l))^2 \right] \Bigg] \end{aligned} \tag{18}$$

When trees are seen as scarcely likely to experience or recover from disasters, consumption volatility is low. As a result of incomplete information: *i)* Volatility is high if there is high uncertainty about the current state of the economy, hence likelihood of an event ($p_t^h = 0.5$). *ii)* Volatility is high if by mistakenly assessing as likely a given state of the economy, the estimation error for event probabilities is large. This happens when $f^i(\overline{z}^h) - f^i(\overline{z}^l)$ and $g^i(\overline{z}^h) - g^i(\overline{z}^l)$ are large. When consumption is diversified between many supplying endowments consumption volatility is intuitively low, because disaster or recovery events lead to minor changes in consumption growth. More in general, it is clear that the relative

[3] Remind that since $H_t^i = \mathbf{1}(x_t^i = 0)$ is the indicator of a disaster for tree i, $dH_t^i = -1$ if $H_t^i = 1$.

size of a tree with respect to the market is an important quantity in this economy, and that it deserves attention. The $i-$th endowment share process,

$$s_t^i = \frac{x_t^i}{\sum_{j=1}^N x_t^j} \tag{19}$$

evolves as:

$$ds_t^i = \sum_{j \neq i} \left[(1 - H_t^i) \left(\frac{x_t^i}{\sum_{z \neq j} x_t^z} - \frac{x_t^i}{\sum_{z=1}^N x_t^z} \right) - H_t^i \left(\frac{x_t^i}{\sum_{z=1}^N x_t^z} - \frac{x_t^i}{\sum_{z \neq j} x_t^z} \right) \right] dH_t^j$$
$$- \left[(1 - H_t^i) s_t^i + H_t^i \frac{\overline{x}_h^i}{\sum_{z=1}^N x_t^z} \right] dH_t^i \tag{20}$$

It is a pure jump process. If some tree j different from the $i-$th undergoes a disaster, the $i-$th share gain (loss) is low if the 'normal' supply of j, $\overline{x}_h^j$, is low relative to the aggregate. If the $i-$th endowment undergoes a disaster, its share declines to zero. Since endowments jump simultaneously in states of high or low disaster or recovery probability, the properties of the $i-$th share process depend on how the $i-$th endowment disaster and recovery intensity compares to intensities of the remaining trees in the same state. Intuitively, the expected share increment is an increasing function of the estimated cumulative probability of disaster of the remaining endowments, and a decreasing function of their expected recovery time. It is also intuitively decreasing in the probability of disaster of the $i-$th endowment. In the Appendix, we report for completeness the posterior conditional distribution of the $i-$th endowment share.

B. *Equilibrium interest rate and risk premia*

According to the optimality conditions for the representative agent, the equilibrium state price density ξ_t coincides with his marginal utility evaluated at aggregate consumption:

$$\xi_t = e^{-\delta t} Y_t^{-\gamma} \left(\sum_{i=1}^N x_t^i \right)^{-\gamma} \tag{21}$$

The following proposition provides details of the equilibrium interest rate and market prices of risk.

Proposition 3 *Let κ_t denote the market price of diffusive risk and θ_t^i the market price of event risk for the $i-th$ dividend process.*[4] *The equilibrium expressions for these prices of risk*

[4]An event is a disaster if the tree is not in disaster state, i.e. $H_t^i = 0$, a recovery otherwise.

and the interest rate are:

$$r_t = \delta + \gamma\mu_Y - \frac{1}{2}\gamma(\gamma+1)\sigma_Y^2 + \sum_{i=1}^{N}\left\{H_t^i\left[1-\left(\frac{\overline{x}_h^i + \sum_{j\neq i} x_t^j}{\sum_{j\neq i} x_t^j}\right)^{-\gamma}\right]\widehat{\eta}_t^i + \right. \tag{22}$$

$$\left. (1-H_t^i)\left[1-\left(\frac{\sum_{j\neq i} x_t^j}{\overline{x}_h^i + \sum_{j\neq i} x_t^j}\right)^{-\gamma}\right]\widehat{\lambda}_t^i\right\} \tag{23}$$

$$\kappa_t = \gamma\sigma_Y \tag{24}$$

$$\theta_t^i = H_t^i\left(\frac{\overline{x}_h^i + \sum_{j\neq i} x_t^j}{\sum_{j\neq i} x_t^j}\right)^{-\gamma} + (1-H_t^i)\left(\frac{\sum_{j\neq i} x_t^j}{\overline{x}_h^i + \sum_{j\neq i} x_t^j}\right)^{-\gamma} \quad i = 1, 2, \ldots N \tag{25}$$

In the expression for the interest rate, we can identify the usual consumption growth and precautionary savings effects. The term in curly brackets in equation (22) summarizes the impact of event risk on the interest rate. Disaster risk decreases the interest rate, while the recovery possibility of a currently 'disastered' tree increases it. Intuitively, a high perceived probability of disaster (recovery) for a tree with high output share substantially decreases (increases) expected consumption growth, thereby decreasing (increasing) the interest rate. Expression (24) is the usual risk compensation for aggregate diffusive risk, which is proportional to the relative risk aversion coefficient and to the diffusive risk volatility. The market price of event risk for the $i-th$ tree is clearly a market price of recovery (disaster) risk if the three is (not) in a disaster state. To interpret expression (25), we should note that in a risk neutral world with complete information the disaster (recovery) instantaneous probability is $\theta_t^i\lambda_t^i$ $(\theta_t^i\eta_t^i)$.[5] The agent demands a compensation for event risk by fictitiously considering a different instantaneous probability for this event. This compensation depends on how the event would affect his desired consumption plan, which is mainly a function of the market share of the tree that experiences the event. Figure 4 shows this market premium for a given tree as a function of the risk aversion coefficient, for two different pre-event share values of the tree ($s_t^i = 1/6$, dashed line, and $s_t^i = 1/10$, solid line).

Insert Figure 4 about here

Panel 1 reports a price of recovery risk, because the tree is in disaster state, while Panel 2 plots the prices of disaster risk, because the tree is not in disaster state. A disaster always entails a decrease in aggregate consumption, therefore the premium increases monotonically with risk aversion, and this decrease is proportionally more severe the less aggregate dividend is diversified across contributing trees, that is, the higher the share of the tree. The price of disaster risk is always higher than one, because the compensation required by the agent mandates that the risk adjusted probability of disaster, $\theta_t^i\lambda_t^i$ is higher than the objective

[5] The risk neutral posterior disaster intensity is then $\widehat{\lambda}_t^i\theta_t^i$, while the recovery intensity is $\widehat{\eta}_t^i\theta_t^i$.

probability λ_t^i. Conversely, a recovery implies an increase in aggregate consumption, therefore the price of recovery risk is smaller than one, decreasing in risk aversion, and it is small when the share value of the tree is high. In a risk neutral world, the (instantaneous) likelihood of each endowment's default depends also on the disaster condition of different trees. For asset pricing purposes, this is an important channel of contagion.

C. Price-Dividend Ratios, Risk Premia and Returns Volatility

The market portfolio is the security that pays the aggregate dividend process, $C_t = Y_t \sum_{i=1}^N x_t^i$, while individual securities are claim to individual dividend process rates $\epsilon_t^i = Y_t x_t^i$. Prices and the risk premia for these securities are reported in the next proposition:

Proposition 4 *The equilibrium price of the market portfolio (V_t^M) and the price of the claim to the i−th dividend process (V_t^i) are given by the following expressions:*

$$V_t^M = C_t \left(\sum_{i=1}^N x_t^i \right)^{\gamma-1} [p_t^h, 1-p_t^h, 0_{\mathcal{N}-2}] \cdot (\mathbf{a} + \mathbf{A}^H)^{-1} \mathbf{C} \tag{26}$$

$$V_t^i = \frac{\epsilon_t^i}{s_t^i \left(\sum_{i=1}^N x_t^i \right)^{1-\gamma}} [p_t^h, 1-p_t^h, 0_{\mathcal{N}-2}] \cdot (\mathbf{a} + \mathbf{A}^H)^{-1} \mathbf{C}^i \tag{27}$$

where $0_.$ *denotes a column vector of zeros,* $\mathcal{N} = 2^N$*, matrices* $\mathbf{A}^H$ *and* $\mathbf{a}$*, vectors* $\mathbf{C}$ *and* $\mathbf{C}^i$ *are reported in the Appendix.*

The equilibrium risk premia of the market portfolio (μ_t^M) and of the i-th individual secu-

rities (μ_t^i) read:

$$\mu_t^M = \gamma\sigma_Y^2 \tag{28}$$

$$-\sum_{i=1}^{N}\left\{(1-H_t^i)\left[\left(\frac{\sum_{j\neq i}x_t^j}{\overline{x}_h^i+\sum_{j\neq i}x_t^j}\right)^{-\gamma}-1\right]\left[\left(\frac{\sum_{j\neq i}x_t^j}{\overline{x}_h^i+\sum_{j\neq i}x_t^j}\right)^{\gamma}\times\right.\right. \tag{29}$$

$$\left.\times\left(\frac{\left[p_t^h\frac{f^i(\overline{z}^h)}{\widehat{\lambda}_t^i},(1-p_t^h)\frac{f^i(\overline{z}^l)}{\widehat{\lambda}_t^i}\right]\cdot\overline{V}^M(H-i)}{[p_t^h,1-p_t^h]\cdot\overline{V}^M(H)}\right)-1\right]\widehat{\lambda}_t^i$$

$$+H_t^i\left[\left(\frac{\overline{x}_h^i+\sum_{j\neq i}x_t^j}{\sum_{j\neq i}x_t^j}\right)^{-\gamma}-1\right]\left[\left(\frac{\overline{x}_h^i+\sum_{j\neq i}x_t^j}{\sum_{j\neq i}x_t^j}\right)^{\gamma}\times\right. \tag{30}$$

$$\left.\left.\times\left(\frac{\left[p_t^h\frac{g^i(\overline{z}^h)}{\widehat{\eta}_t^i},(1-p_t^h)\frac{g^i(\overline{z}^l)}{\widehat{\eta}_t^i}\right]\cdot\overline{V}^M(H+i)}{[p_t^h,1-p_t^h]\cdot\overline{V}^M(H)}\right)-1\right]\widehat{\eta}_t^i\right\}$$

$$\mu_t^i = \gamma\sigma_Y^2 \tag{31}$$

$$-\sum_{j=1}^{N}\left\{(1-H_t^j)\left[\left(\frac{\sum_{z\neq j}x_t^z}{\overline{x}_h^j+\sum_{z\neq j}x_t^z}\right)^{-\gamma}-1\right]\left[\left(\frac{\sum_{z\neq j}x_t^z}{\overline{x}_h^j+\sum_{z\neq j}x_t^z}\right)^{\gamma}\times\right.\right. \tag{32}$$

$$\left.\times\left(\frac{\left[p_t^h\frac{f^j(\overline{z}^h)}{\widehat{\lambda}_t^j},(1-p_t^h)\frac{f^j(\overline{z}^l)}{\widehat{\lambda}_t^j}\right]\cdot\overline{V}^i(H-j)}{[p_t^h,1-p_t^h]\cdot\overline{V}^i(H)}\right)-1\right]\widehat{\lambda}_t^j$$

$$+H_t^j\left[\left(\frac{\overline{x}_h^j+\sum_{z\neq j}x_t^z}{\sum_{z\neq j}x_t^z}\right)^{-\gamma}-1\right]\left[\left(\frac{\overline{x}_h^j+\sum_{z\neq j}x_t^z}{\sum_{z\neq j}x_t^z}\right)^{\gamma}\times\right. \tag{33}$$

$$\left.\left.\times\left(\frac{\left[p_t^h\frac{g^j(\overline{z}^h)}{\widehat{\eta}_t^j},(1-p_t^h)\frac{g^j(\overline{z}^l)}{\widehat{\eta}_t^j}\right]\cdot\overline{V}^i(H+j)}{[p_t^h,1-p_t^h]\cdot\overline{V}^i(H)}\right)-1\right]\widehat{\eta}_t^j\right\}$$

where vectors $\overline{V}^M(H)$ ($\overline{V}^i(H)$), $\overline{V}^M(H-i)$ ($\overline{V}^i(H-j)$), $\overline{V}^M(H+i)$ ($\overline{V}^i(H+j)$) are reported in the Appendix.

The components of the vector $(\sum_{z\neq j}x_t^z)^{\gamma-1}\overline{V}^M(H-j)$ are the full information price-dividend ratios for the market, after tree j has experienced a disaster, conditional on the state of the economy z_t. In particular, bivariate functions $\overline{V}^M(H)$ and $\overline{V}^i(H)$ are expected discounted cash-flow streams from the market portfolio and individual securities. The empirical illustrations that follow assume a simple economy with 6 supplying trees. Without loss of generality, none of the trees is assumed to have yet undergone a disaster, unless otherwise noticed. Parameters have been calibrated using the procedure outlined in Appendix B.

C.1. Market price-dividend ratio and risk premium

Figure 5 reports prices and price-dividend ratios for the market portfolio corresponding to different level of risk aversions and different number of disasters occurred among trees.

Insert Figure 5 about here

As deemed reasonable from an empirical standpoint, trees with the highest disaster intensities λ_t are also the trees with lowest recovery intensity η_t. Dividends of the market portfolio are perfectly negatively correlated with equilibrium state prices. When the agent is sufficiently risk averse, as in Panel 2, the increase in marginal utility following a disaster is higher than the corresponding consumption fall, because the agent is averse to extreme losses but does not benefit from extreme gains. If a tree with high risk of disaster undergoes a default, trees less prone to disaster risk are left to supply the output until the (unlikely) recovery of the tree, and this forecasts a lower future dividend payments in states where the marginal utility is high - because the likely disasters of riskier trees will cumulate. If, paradoxically, 'higher rated' trees default sooner, a more steady consumption path is going to prevail, because disasters will occur sooner, when the marginal utility is lower and 'disastered' trees might have recovered. In other words, Panel 2 shows that, if the risk aversion is sufficiently high, a disaster of a very risky tree generates a negative market return, while a positive return may follow the disaster of a scarcely risky tree. The latter behavior is observed especially when the number of remaining trees is small, because expected consumption growth is very high, thanks to the likely recoveries, is a context where equilibrium state prices are also high. In Panel 1 the risk aversion coefficient is too low, and post-disaster returns are always negative. Note that this effect is scarcely dependent of the market share of each tree and largely due to the perfect negative correlation between consumption and state-prices. Furthermore, the effect mostly relies on the impact of disasters and recoveries on equilibrium state prices, therefore it is not surprising that price-dividend ratios display this effect even when the risk aversion is low, as in Panel 3.

Prices in Figure 5 are plotted assuming that a disaster for a given tree enhances dramatically the probability that the economy switches to a 'bad' state, because this tree is crucial for the correct functioning of the rest of the economy. Figure 6 shows the effect of this assumption on market prices.

Insert Figure 6 about here

When the assumption is removed and the state of the economy is purely exogenous the economy is more likely to persist in a 'good' state, whereby disaster risk is lower. When the most likely history of disasters occurs (solid line), the influence on the price of the market portfolio is very limited, because riskier trees default first and are scarcely likely to recover in both states of the economy, as much as those left are scarcely likely to undergo disasters. However, in the unlikely scenario where riskier trees default last, the price of the market portfolio is higher with no feed-back, because 'disastered' trees are more likely to recover soon and to generate high consumption growth, in a situation where equilibrium state prices are high. As a matter of fact, in Figure 6 the price increases at each disaster when only a few trees are left, despite the fact that the risk aversion is low.

We should emphasize that in an empirically relevant scenario disaster occur first for riskier trees. In this respect, the predictions of this model are intuitively correct, with post-disaster decreasing market prices and price dividend-ratios which decrease less.

We now turn to analyze the risk premium of the market portfolio. Equation (28) is the compensation for the systematic diffusive risk Y_t. Aggregate output shares supplied by individual trees are subject to abrupt changes because of disasters and recoveries, and this risk component demands two layers of reward. In the absence of incomplete information, expression (32) becomes

$$\begin{aligned}
-\sum_{i=1}^{N}\Bigg\{&(1-H_t^i)\left[\left(\frac{\sum_{j\neq i}x_t^j}{\overline{x}_h^i+\sum_{j\neq i}x_t^j}\right)^{-\gamma}-1\right]\left[\left(\frac{\sum_{j\neq i}x_t^j}{\overline{x}_h^i+\sum_{j\neq i}x_t^j}\right)^{\gamma}\left(\frac{\overline{V}^M(H-i)_u}{\overline{V}^M(H)_u}\right)-1\right]\lambda_t^i\\
&+H_t^i\left[\left(\frac{\overline{x}_h^i+\sum_{j\neq i}x_t^j}{\sum_{j\neq i}x_t^j}\right)^{-\gamma}-1\right]\left[\left(\frac{\overline{x}_h^i+\sum_{j\neq i}x_t^j}{\sum_{j\neq i}x_t^j}\right)^{\gamma}\left(\frac{\overline{V}^M(H+i)_u}{\overline{V}^M(H)_u}\right)-1\right]\eta_t^i\Bigg\}\quad u=h,l
\end{aligned} \tag{34}$$

This term describes the compensation required for the direct impact of disaster and recoveries on the aggregate dividend. For every tree j that could undergo a disaster (recovery), the premium is proportional to the risk-adjusted probability of disaster (recovery) $\theta_t^j\lambda_t^j$ $(\theta_t^j\eta_t^j)$, weighted by the market return response to disaster (recovery) shocks. This compensation is increasing in cumulative disaster likelihood $\sum_j(1-H_t^j)\lambda_t^j$ and decreasing in the cumulative recovery likelihood $(\sum_j H_t^j\lambda_t^j)$. Intuitively, covariance between returns and aggregate consumption is increasing in the former and decreasing in the latter. Leaving market diversification, hence the number trees and supplied multiples $\mathbf{x}_t$ unaltered, an increase in the likelihood of some disaster (recovery) requires additional (lesser) reward. For a given tree i, the sign of this full information event premium is driven by

$$\begin{aligned}
(1-H_t^i)&\left(\frac{\sum_{j\neq i}x_t^j}{\overline{x}_h^i+\sum_{j\neq i}x_t^j}\right)^{\gamma}\left(\frac{\overline{V}^M(H-i)_u}{\overline{V}^M(H)_u}\right)\\
&+H_t^i\left(\frac{\overline{x}_h^i+\sum_{j\neq i}x_t^j}{\sum_{j\neq i}x_t^j}\right)^{\gamma}\left(\frac{\overline{V}^M(H+i)_u}{\overline{V}^M(H)_u}\right)\quad u=h,l
\end{aligned} \tag{35}$$

which is the ratio of the prices of the market portfolio after and before a potential disaster (if $H_t^i = 0$) or recovery (if $H_t^i = 1$) of the i-th tree. If the tree is not in disaster state, and this ratio is smaller than one, because the price of the market portfolio decreases after the disaster, then disaster risk always demands a positive premium. On the other hand, recovery risk demands a negative premium if the ratio in (35) is higher than one.

The investor realizes that his belief fluctuates randomly in response to disaster and recovery shocks, and the premium he charges is adjusted for this risk component. We discuss the case of disasters, the intuition being similar for recovery events.

The effect of learning is captured by the partial information ratio of prices before and after a disaster event:

$$\left(\frac{\sum_{j\neq i} x_t^j}{\overline{x}_h^i + \sum_{j\neq i} x_t^j}\right)^\gamma \left(\frac{\left[p_t^h \frac{f^i(\overline{z}^h)}{\widehat{\lambda}_t^i}, (1-p_t^h)\frac{f^i(\overline{z}^l)}{\widehat{\lambda}_t^i}\right] \cdot \overline{V}^M(H-i)}{[p_t^h, 1-p_t^h] \cdot \overline{V}^M(H)}\right) \tag{36}$$

This expression is similar to the ratio of partial information market prices, with the difference that after a disaster the investors regards the 'bad' economic state as more likely. The more so, the more a disaster was regarded unlikely according to the ex-ante belief. If $\frac{f^i(\overline{z}^l)}{\widehat{\lambda}_t^i}$ is large, a disaster event is interpreted as evidence that the disaster intensity is underestimated, so that the economy is more likely to be in a 'bad' state. The agent takes this into account by overweighting the post-default market price conditional on a 'bad' state. This price is higher than its 'good' state counterpart when the risk aversion is greater than one, because the utility function of the agent is bounded above. Since, for the market portfolio, dividends are perfectly negatively correlated with marginal utility, a decrease in the dividend paid due to a disaster is compensated by a higher percentage increase in marginal utility. However both conditional prices can decrease or increase after a disaster, therefore the partial information ratio (36) is not granted a priori to be smaller or greater than the full information ratio (35). It follows that learning may increase or decrease the risk premium.

According to the familiar consumption-CAPM representation, the market instantaneous risk premium is proportional to the instantaneous covariance of the market return with the state-price deflator:

$$\begin{aligned}
\mu_t^M &= -\mathbb{E}\left[\frac{dR_t^M}{R_t^M}\frac{d\xi_t}{\xi_t}|\mathcal{F}_t^{H,S}\right] \\
&= -\left(\mathbb{E}\left[\frac{dC_t}{C_t}\frac{d\xi_t}{\xi_t}|\mathcal{F}_t^{H,S}\right] + \mathbb{E}\left[\frac{d(V_t^M/C_t)}{(V_t^M/C_t)}\frac{d\xi_t}{\xi_t}|\mathcal{F}_t^{H,S}\right] + \mathbb{E}\left[\frac{d(V_t^M/C_t)dC_t}{(V_t^M/C_t)C_t}\frac{d\xi_t}{\xi_t}|\mathcal{F}_t^{H,S}\right]\right)
\end{aligned} \tag{37}$$

where R_t^M denotes the market cum-dividend return.[6] Expression (37) decomposes the risk premium into the covariance between endowment growth and marginal utility of consumption, a 'cash-flow beta', the covariance between price-dividend ratio and marginal utility of consumption, a 'valuation beta', and the covariance between marginal utility and the product of jumps in the price-dividend ratio and dividend growth. We call this last covariance the 'jump' beta.[7] The 'cash-flow beta' bears compensation for the systematic fluctuations of the dividend paid by the market portfolio, due both to the volatility of the unitary output Y_t and to disaster or recovery occurrence. The latter component consists of the percentage loss (gain) in consumption in case of a disaster (recovery) weighted by the risk adjusted probability that a disaster (recovery) occurs next instant. Since the market portfolio pays-off aggregate consumption, its cash-flow is low when equilibrium state-prices are high, because it is perfectly negatively correlated with marginal utility of consumption. The 'cash-flow' beta is then intuitively positive. The price-dividend ratio of the market portfolio is also time-varying, because the fluctuations of marginal utility of consumption due to disasters and recoveries cannot be hedged by holding the market portfolio alone. The 'valuation beta' captures the layer of compensation required by this fluctuations in the evaluation of states of the world. It also comprises the compensation for learning risk. The agent knows that his assessment about the likelihood of being in a 'good' state will react to random default events, hence will covary with aggregate consumption and, consequently, marginal utility. Since aggregate consumption is observable, learning risk acts entirely on the price-dividend ratio, hence its premium is part of the 'valuation beta'. The price-dividend ratio and the aggregate consumption jump simultaneously as a disaster or recovery occurs. The market return has an (instantaneously) unpredictable component proportional to this co-jump, and this component Co-varies with marginal utility of consumption. This is the essence of the 'jump beta'.

Figure 7 plots the market risk premium from Equations (28)-(32) as a function of the

[6]Equation (37) follows from the fact that returns can always be decomposed as the sum of the dividend yield, dividend growth, the change in valuation or growth in the price-dividend ratio, and of an instantaneous covariation term. Namely:

$$dR_t^i = \frac{C_t}{V_t^M}dt + \frac{dC_t}{C_t} + \frac{d(V_t^M/C_t)}{(V_t^M/C_t)} + \frac{d(V_t^M/C_t)dC_t}{(V_t^M/C_t)C_t} \tag{38}$$

Note that by Ito's lemma the covariation terms has a jump component, which will display a nonzero covariance with the stochastic discount factor:

$$\frac{d(V_t^M/C_t)dC_t}{(V_t^M/C_t)C_t} = (...)dt + \left(\frac{[(V_t^M/C_t) - (V_{t-}^M/C_{t-})][C_t - C_{t-}]}{(V_{t-}^M/C_{t-})C_{t-}}\right)dN_t \tag{39}$$

[7]Expression for these 'betas' are easily computed and they are not reported.

number of disasters occurred. For this simple illustration, we have used the canonical case reported in Figure 5, when riskier trees default first. The 'betas' are also reported.

Insert Figure 7 about here

As anticipated above, the 'cash-flow beta' is always positive. This component is relatively more prominent when lesser disasters have occurred and there are more trees supplying the endowment. In this situation, when a disaster (recovery) occurs, the relative share of dividend payed by the market that goes lost (recovered) is close to the corresponding increase (decrease) of equilibrium state prices. Contrarily, in a scarcely diversified economy with only a few suppliers, each disaster or recovery event leads to a dividend variation that is far outweighed by the equilibrium valuation of this dividend. It is therefore intuitive that most part of the correlation of market returns with marginal utility is due to price-dividend ratio changes, when only a few trees are not in disaster state. The learning component of the 'valuation beta', however, is relatively less important after more disasters, because uncertainty has been resolved towards a high disaster probability - hence 'bad' economic state. With this uncertainty almost fully resolved, there is less reward for the risk that the belief fluctuates.

If a given tree is not currently in disaster state, its market price of disaster risk is increasing in its market share, because a potential disaster for this tree would lead to high relative fall of consumption. If the tree is in disaster state, its market price of recovery is decreasing in its share, because a potential recovery would lead to a high relative consumption gain, thus requiring less compensation for risk. However, risk premia also depend on the posterior instantaneous probability of disaster or recovery occurrence for each tree. As more riskier trees undergo disasters, the likelihood of further disasters decreases at high rates, which more than compensate the higher premium required 'for unit of disaster risk'. At the same time, the likelihood that some tree recovers is increasing, and the latter requires a negative premium. This combined effect leads to a market risk premium that is decreasing in the number of disasters. Note that, in the same setting, the market price and price-dividend ratio also decrease when a disaster occurs. This apparently contradictory behavior of prices and excess returns can be clarified by noting that *instantaneous* risk premia are not the appropriate measure of reward to analyze the valuation ratio. The long-run excess return should be considered instead.

As noted by Dumas, Kurshev and Uppal (2008), we need to consider the long-run response of the stochastic discount factor to shocks occurring today, $\mathcal{D}_t^{H^i,Z}\xi_T$, where $\mathcal{D}_t$ denotes the Malliavin derivative operator with respect to the systematic diffusive shock dZ and the disaster/shocks event dH^i:

$$\frac{\mathcal{D}_t^{H^i,Z}\xi_T}{\xi_T} = -\gamma\sigma_Y + (1-H_t^i)\left[\left(\frac{\sum_{j=1}^N x_T^j - \overline{x}_h^i}{\sum_{j=1}^N x_T^j}\right)^{-\gamma} - 1\right] + H_t^i\left[\left(\frac{\overline{x}_h^i + \sum_{j=1}^N x_T^j}{\sum_{j=1}^N x_T^j}\right)^{-\gamma} - 1\right] \tag{40}$$

The long-term response to systematic diffusive shocks coincides with the instantaneous response, because of the IID nature of systematic consumption growth Y_t. The response to disaster or recover shocks, albeit also formally similar to the instantaneous response, depends on future output supplied by remaining trees. The riskier[8] the endowment i that undergoes a disaster or recovery, the higher the expected number of trees that will not be in disaster state at time T. If more 'healthy' trees are expected, and the expected residual output $\sum_{j\neq i} x_T^j$ is consequently higher, the response of state prices to a current disaster or recovery is lower. To summarize, the term response is lower the higher the riskiness of the tree that experiences the event. Thus, a disaster or recovery event of a risky tree forecasts higher consumption in high marginal utility states. The long-term event risk premium on a security that pays one unit of consumption at time T is proportional to the expected event covariance of returns and marginal utility along the holding period return of the security. This can be written as follows in terms of the long term response:

$$\mu_{t,T}^M = \sum_{i=1}^N \mathbb{E}\left[\mathbb{E}\left[\mathcal{D}_s^{H^i,Z}(\xi_T C_T)\middle|\mathcal{F}_s^{H^i,S}\right]\frac{\mathcal{D}_s^{H^i,Z}\xi_T}{\xi_T}\middle|\mathcal{F}_t^{H^i,S}\right] \tag{41}$$

$$= \sum_{i=1}^N \mathbb{E}\left[(1-H_t^i)\left[\left(\sum_{j=1}^N x_T^j - \overline{x}_h^i\right)^{1-\gamma} - \left(\sum_{i=1}^N x_T^j\right)^{1-\gamma}\right]\times\right.$$

$$\left[\left(\frac{\sum_{j=1}^N x_T^j - \overline{x}_h^i}{\sum_{j=1}^N x_T^j}\right)^{-\gamma} - 1\right] + H_t^i\left[\left(\overline{x}_h^i + \sum_{j=1}^N x_T^j\right)^{1-\gamma} - \left(\sum_{i=1}^N x_T^j\right)^{1-\gamma}\right]\times \tag{42}$$

$$\left.\left[\left(\frac{\sum_{j=1}^N x_T^j + \overline{x}_h^i}{\sum_{j=1}^N x_T^j}\right)^{-\gamma} - 1\right]\middle|\mathcal{F}_t^{H^i,S}\right] \qquad s\in[t,T] \tag{43}$$

Each expectation is the contribution of a single tree's event risk to the expected covariance between returns and state prices. In light of the reasoning above, when a risky tree undergoes a disaster, the reduction of cumulative event risk faced is small. However, the output is now supplied by fewer trees, so that each remaining expected covariance term increases, because aggregate consumption changes due to disaster or recovery shocks will be more correlated with changes in marginal utility. It follows that the term premium increases if a tree experiences a disaster, in situations where the price-dividend ratio decreases.

[8] We use the term 'risky' as synonymous of prone to disaster risk, therefore characterized by high disaster and low recovery intensity.

C.2. Prices and Risk Premia of Individual Dividend Claims

Figure 8 shows returns on individual securities prices immediately following the disaster of a different tree.

Insert Figure 8 about here

When a disaster occurs, individual securities become more similar to the market portfolio, because the covariance between dividends and state-prices increases. If the event risk of a given tree is small, this tree is likely to be still supplying at full regime ($\overline{x}_h$) while the remaining trees have already undergone disasters. If the risk aversion of the agent is high (Panel 1), the security has a high pay-off in circumstances where equilibrium state prices are high, because the output fall has risen marginal utility (Panel 1). Therefore a security is positively evaluated by the market when its event risk is small relative to the aggregate. When the riskiest tree experiences a disaster, the proportion of disaster risk beared by the security increases dramatically, and the market determines a negative return for the security. A contagion effect. If less risky trees default, the security benefits from a scarce increase in (the absolute value of) its covariance with state-prices, and its return is higher, the smaller the risk of the defaulted security. When the market share of the tree is high, its volatility due to event risk is small, because disasters and recoveries will only marginally alter its current share. Panel 3 shows disaster-induced returns on the claim to the tree with the higherst event risk. In principle, the mechanism at work is the same, the higher the risk of the defaulted tree the higher the return on the security. However, the risk share of this tree is only marginally affected by any disaster, so returns are smaller in magnitude. When the risk aversion of the agent is low, as in Panel 2 and 4, the market mechanism described is weaker, because of the less pronounced response of equilibrium state prices to disaster events. In Panel 2, for instance, the return on the safest security following a disaster is always negative, because the increase in state prices is too limited to compensate the increase of the event risk share carried by the dividend paying tree.

The *term premium* of individual security prices has the following representation:

$$\mu^i_{t,T} = \sum_{j=1}^N \mathbb{E}\left[\mathbb{E}\left[\mathcal{D}_s^{H^j,Z}(\xi_T C_T) \middle| \mathcal{F}_s^{H^j,S}\right] \frac{\mathcal{D}_s^{H^j,Z}\xi_T}{\xi_T} \middle| \mathcal{F}_t^{H^j,S} \right] \tag{44}$$

$$\begin{aligned} &= \sum_{j=1}^N \mathbb{E}\left[(1-H_t^j)\left[\left(\sum_{z=1}^N x_T^z - \overline{x}_h^j\right)^{-\gamma} \mathbf{1}(j\neq i)x_T^i - \left(\sum_{i=1}^N x_T^j\right)^{-\gamma} x_T^i \right] \times \right. \\ &\left[\left(\frac{\sum_{z=1}^N x_T^z - \overline{x}_h^j}{\sum_{z=1}^N x_T^z}\right)^{-\gamma} - 1\right] + H_t^j \left[\left(\overline{x}_h^j + \sum_{z=1}^N x_T^z \right)^{-\gamma} (\mathbf{1}(i\neq j)x_t^i + \mathbf{1}(i=j)\overline{x}_h^i) \right. \\ &\left. \left. - \left(\sum_{z=1}^N x_T^z\right)^{-\gamma} x_T^i \right] \left[\left(\frac{\sum_{j=1}^N x_T^j + \overline{x}_h^i}{\sum_{j=1}^N x_T^j} \right)^{-\gamma} - 1 \right] \middle| \mathcal{F}_t^{H^i,S} \right] \quad s \in [t,T] \end{aligned} \tag{45}$$

These expected covariances are increasing in the expected market share of the endowment, because lower share values imply less covariation between aggregate consumption and endowment. If the endowment has the lowest propensity to experience a disaster, it is likely to supply a high multiple of output when most of the remaining trees might be in disaster state. However, if the most risky of the endowments experiences a disaster, and the disaster risk share of the evaluated tree increases, the security is likely to provide a high dividend when many of the trees are also supplying full output, aggregate consumption is high and equilibrium state-prices are low. Hence, the disaster of the most risky tree forecasts a negative future covariance between the dividend of the security and state-prices, therefore it leads to an increase of the term premium of the security. Conversely, disasters of less risky trees imply a high and positive expected covariance between the dividend and state-prices, hence lead to a decrease of the security's term premium. When the risk aversion is low, however, as in Panel 2 of Figure 8, the expected increase of equilibrium state prices is modest, and the effect induced by the increased market share of the endowment dominates, leading to higher term premia, and lower security prices.

C.3. Returns volatility

The variance of returns on the i-th endowment claim reads:

$$\mathrm{Var}\left[\frac{dV_t^i}{V_t^i}\middle|\mathcal{F}_t^{H,S}\right] = \sigma_Y^2 + \sum_{u=1}^{N}\left\{(1-H_t^u)\left[\left(\frac{\sum_{j\neq u} x_t^j}{\sum_{j=1}^N x_t^j}\right)^{\gamma} \times\right.\right.$$

$$\times\left(\frac{\left[p_t^h\frac{f^u(\overline{z}^h)}{\widehat{\lambda}_t^u},(1-p_t^h)\frac{f^u(\overline{z}^l)}{\widehat{\lambda}_t^u}\right]\cdot\overline{V}^i(H-u)}{[p_t^h,1-p_t^h]\cdot\overline{V}^i(H)}\right)-1\Bigg]^2(\widehat{\lambda}_t^u+p_t^h(1-p_t^h)(f^u(\overline{z}^h)-f^u(\overline{z}^l))^2)$$

$$+H_t^u\left[\left(\frac{\sum_{j=1}^N x_t^j}{\sum_{j\neq u} x_t^j}\right)^{\gamma}\left(\frac{\left[p_t^h\frac{g^u(\overline{z}^h)}{\widehat{\eta}_t^u},(1-p_t^h)\frac{g^u(\overline{z}^l)}{\widehat{\eta}_t^u}\right]\cdot\overline{V}^i(H+u)}{[p_t^h,1-p_t^h]\cdot\overline{V}^i(H)}\right)-1\right]^2\times$$

$$\left.(\widehat{\eta}_t^u+p_t^h(1-p_t^h)(g^u(\overline{z}^h)-g^u(\overline{z}^l))^2)\right)\Big\}$$

$$+\sum_{j=1}^{N}\left(\frac{(e_1-e_2)\cdot\overline{V}^i}{[p_t^h,1-p_t^h]\cdot\overline{V}^i}\right)^2[p_t^h(1-p_t^h)]^2[(f^j(\overline{z}^h)-f^j(\overline{z}^l))^2+(g^j(\overline{z}^h)-g^j(\overline{z}^l))^2]$$

The terms

$$\widehat{\lambda}_t^j+p_t^h(1-p_t^h)(f^j(\overline{z}^h)-f^j(\overline{z}^l))^2$$
$$\widehat{\eta}_t^j+p_t^h(1-p_t^h)(g^j(\overline{z}^h)-g^j(\overline{z}^l))^2$$

denote, respectively, the posterior instantaneous variance of the $j-$th disaster event and of the $j-$th recovery event. With full information, these terms would reduce to the instantaneous probability of disaster and recovery, $\lambda_t^j = f^j(z_t)$ and $\eta_t^j = g^j(z_t)$. Partial information determines an additional Jensen inequality adjustment, which is maximal in situations of higher uncertainty about the current state of the economy, that is, when the agent regards very different disaster or recovery probabilities as equally likely ($p_t^h = 0.5$).

Periods of 'excess volatility' arise when the volatility of returns exceeds the volatility of underlying dividend growth. When expected dividend growth is time-varying, this phenomenon depends on the volatility of the stochastic discount factor, which impacts the volatility of the valuation (price-dividend) ratio. If the risk aversion is high, the influence of disaster risk on state-prices is higher than its influence on cash-flows. The full information dividend variance reads

$$\mathrm{Var}\left[d\epsilon_t^i\middle|\mathcal{F}_t\right] = \sigma_Y^2+(\overline{x}_h^i)^2[(1-H_t^i)\lambda_t^i+H_t^i\eta_t^i], \tag{46}$$

According to our discussion on the behavior of security prices, the reaction of price-dividend ratios to disaster or recovery events is maximal when the market share of the evaluated tree is small and the risk aversion is high. Comparing volatility of returns to volatility of dividend, we conclude that in this situation disaster risk is most likely to generate excess volatility. If the risk aversion is moderate and the underlying tree provides a small output share, the instantaneous volatility of returns is small, because the price-dividend ratio of the security is scarcely responsive to event risk and dividend volatility is also small, unless the instantaneous probability of default or recovery of the tree is disproportionally high.

Learning adds one more degree of uncertainty. A lower $i-$th market share due to a high number of suppliers raises the number of signal to learn from, hence their cumulative contribution to volatility of returns. But a lower $i-$th market share also raises the probability of an update in the posterior price-dividend ratio due to event risk, hence raises the perceived volatility of the price-dividend ratio.

Figure 9 reports instantaneous volatilities of returns for claims to the endowments bearing the highest and lowest event risk, respectively. In Panel 1 and Panel 2, these volatilities are plotted as functions of the number of disasters, under the canonical assumption that disasters occur first for riskiest trees . Panel 3 and Panel 4 plot post-disaster volatilities for different disaster events.

Insert Figure 9 about here

In Panel 1, the volatility of the least risky security increases as disasters of remaining trees occur. To understanding this behavior, we should take into account two opposit effects. While riskier trees default first and the post-disaster cumulative event risk decreases, the post-disaster share of event risk characterizing the tree increases. The latter effect means that the price-dividend ratio of the security is expected to be highly volatile, because the likely disasters to occur will lead to high state prices when the security is still paying off a conspicuous dividend. If the initial event risk share of the tree is small enough, so that the tree is likely to be in 'normal' supply state while remaining trees are in disaster state, then this effect dominates. This is the situation depicted in Panel 1. In Panel 2, the secutity has the highest event risk, therefore its price dividend ratio is scarcely influenced by disasters of remaining trees, so that the returns volatility decreases after a disaster. The only exception is the disaster of the second to riskiest tree. The post-disaster variation of volatility is less pronounced when the tree holds a high share of the market, because state-prices, hence the price-dividend ratio of the security, are then scarcely influenced by a disaster of remaining trees. Panels 3 and 4 show that a disaster of a scarcely risky tree leads to a higher volatility increase. In light of the discussion above this is not surprising. The less risky the tree that experiences a disaster, the smaller the increase in the event risk share of the tree, while equilibrium state prices hace increased due to the consumption fall, hence the higher the volatility of the price-dividend ratio. The learning component mentioned above has marginal influence and does not alter the main intuition.

‘Volatility leakage’ has been extensively investigated in the literature. Empirical evidence acknowledges that shocks to volatilities of returns of some asset classes also affect the volatilities of different asset classes returns. Our model copes well with this empirical regularity. The analytical expression for the correlation between volatilities of returns on assets i and j is cumbersome and difficult to interpret. Direct intuition is easier to convey. Common shocks to volatilities are those that affect price-dividend ratios and posterior probabilities. If assets i and j have scarce ‘excess volatility’, their volatilities will covary mainly because of posterior probability updates. If the variance of the posterior probability of an ‘high’ economic state is low, then the co-volatility will also be low and scarcely affected by market shares of the endowments. This is the situation depicted in Panel 1 of Figure 10. This panel reports the instantaneous correlation between retruns volatilities of the securities with the highest and second to highest event risk. These assets have high likelihood of instantaneous disaster, and scarce recovery chances, therefore display weak ‘excess volatility’, as noted above.

Insert Figure 10 about here

Panel 2 reports the same graph for the securities bearing the highest and second to highest event risk. These securities have significant ‘excess volatility’, therefore shocks to the volatility of the stochastic discount factor propagate simultaneously to the volatility of their price-dividend ratio. In Panel 2 the ‘volatility leakage’ is then higher and more sensitive to endowments’ market share than in Panel 1.

V. Correlations

Returns of securities are correlated even if their dividends are scarcely correlated or independent. This is a feature of any equilibrium model where individual endowments are correlated with aggregate output, albeit uncorrelated among them. In our set-up however this phenomenon is very relevant.

Partial information implies that perceived (posterior) correlations between endowments differ from correlations arising in a full information framework.

We analyze these aspects from the point of view of individual endowments and asset returns.

A. Trees event correlations

The probability that two trees experience a disaster at the same time is zero. Therefore the instantaneous covariance between endowments growth is entirely captured by the variance of unitary output, σ_Y^2. On a finite time horizon $[t, T]$, the covariance is driven by the cumulative variance of output $\sigma_Y^2(T-t)$ and by the event covariance, which provides an assessment about the probability that trees will spend most of the time horizon in the same disaster of 'normal' supply state. The latter is by far the most important component. Given the observed information set $\mathcal{F}_t^{x,S}$, the conditional event correlation between tree i and tree j is reported in Proposition 8 of the Appendix. Figure 11, Panel 1, shows the behavior of the posterior 6-month conditional event correlation between two endowments. The posterior correlation is compared to the behavior of the full information conditional correlations.

Insert Figure 11 about here

These endowments are both scarcely prone to event risk, hence their full information correlation is negligible when the economy is in a 'high' state, and moderate in a 'bad' state. In general, security prices are averages of conditional full information security prices, with weights given by posterior probabilities. However, the impact of incomplete information on correlations is nonlinear. The posterior correlation is sensibly higher than full information correlation when the uncertainty about the current state of the world is high, that is, for p_t^h around 0.5. In Panel 1, as in Panel 2, a starting value of $p_t^h = 0.5$ is progressively updated downwards after each disaster, and the partial information correlation slowly converges to the full information counterpart. In this illustration, we have assumed that a given tree (the 6-th) is crucial for the rest of the economy, and that a disaster event for this tree enhances the probability of a 'bad' economic state. Since the 'bad' state becomes extremely persistent when a disaster occurs for the 6-th tree, the trees' event risk experiences an upward jump, and trees are more likely to persist in the same supply or 'normal' disaster state. Since the 'high' state of the economy is extremely transitory in this situation, the full information correlation in this state is only remotely affected.

Figure 12, is the analog of Figure 11 for the risk neutral event correlation between the same trees.

Insert Figure 12 around here

The market price of event risk for a given tree depends on the relative risk aversion and on the output share that the tree is currently supplying (or would supply if it recovered). With the degree of market concentration used in our exercise, where at most 6 trees supply the aggregate endowment, the correlation premium is dramatic. When all 6 trees are yet to default, the impact of each disaster on state prices is such that to price a security whose payoff depends on two objectively scarcely uncorrelated endowments, the relevant correlation that the agent would use is 0.55 (as compared to an objective 0.26). It should be noted that these figures are obtained with a relatively moderate risk aversion coefficient ($\gamma = 5$). Not surprisingly, Panel 1 shows that the risk adjusted correlation increases at each successive disaster, because the market share of the trees is progressively increased. Without risk adjustments, the only tree which affects correlations is the three whose state affects the economy (the 6-th). Intuitively, in the risk adjusted world, every tree's disaster has an impact on correlation. Panel 2 shows that this influence is highest for disasters of trees with the lower event risk (4th and 3rd), because the risk adjusted event risk share of the trees increases dramatically in this case. Informally, we may say that the evaluated trees are 'next' to default.

B. Assets return correlations

The expression for the instantaneous covariance between returns is reported in the Appendix. Event risk does not influence (instantaneous) endowments' covariance, but it does influence returns covariance by means of its impact on price-dividend ratios. Let us consider the contribution of the $z-$th disaster event to the full information covariance between the $i-$th and the $j-$th securities' returns:

$$\left[\left(\frac{\sum_{u\neq z} x_t^u}{\sum_{u=1}^N x_t^u}\right)^\gamma \frac{\overline{V}^i(H-z)_v}{\overline{V}^i(H)_v} - 1\right]\left[\left(\frac{\sum_{u\neq z} x_t^u}{\sum_{u=1}^N x_t^u}\right)^\gamma \frac{\overline{V}^j(H-z)_v}{\overline{V}^j(H)_v} - 1\right]\lambda_t^z \qquad v = h, l. \tag{47}$$

Terms in square brackets are full information returns on each security following the disaster of tree z. We have pointed out that a security responds positively to disasters if the post-disaster share of event risk has increased less than the output supply share of the underlying tree. If tree z has very high (low) event risk, its disaster will homogeneously impact prices of most securities, therefore contributing positively to correlation. We can conclude that: *i)* After the disaster (recovery) of a risky or scarcely risky security correlation between any two securities decreases (increases). *ii)* Correlation is high and positive for those securities sharing similar event risk, that is, disaster and recovery probabilities, because their response to shocks is homogeneous. This intuition is substantiated by Figure 13. Panel 1 reports instantaneous correlation of returns between the security with smallest event risk and the remaining, while Panel 2 reports instantaneous correlation of returns between the most event risky security and the remaining. Market shares are homogeneous among securities.

Insert Figure 13 about here

C. Disaster Contagion and Security Price Contagion

Disaster contagion is related to the concept of event correlation that we have already analyzed. In our model, a given disaster event cannot directly cause an additional disaster, but it can modify the probability of disaster or recovery of remaining trees. If we consider instantaneous probabilities, learning is fully in charge of this effect. Over a finite time-horizon instead, the possibility to foster a 'good' or 'bad' economic state provides an additional channel of contagion . Disaster events have also a direct impact on returns. It seems appropriate then to distinguish a 'cash-flow', or simply 'disaster' contagion, from the ''valuation' contagion displayed by security prices.

C.1. Disaster contagion

Following Frey, Schmidt and Gabith (2007), we consider the following measure of disaster contagion:

$$\widehat{\lambda}^i_{\tau^j} - \widehat{\lambda}^i_{\tau^j-} = p^h_{\tau^j-}(1-p^h_{\tau^j-})\frac{[f^i(\overline{z}^h)-f^i(\overline{z}^l)][f^j(\overline{z}^h)-f^j(\overline{z}^l)]}{\widehat{\lambda}^j_{\tau^j-}} \tag{48}$$

where τ^j is a disaster time for tree j. According to the LHS of (48), disaster contagion is the variation of the instantaneous posterior probability of disaster for tree i following a disaster of tree j. This is given by the posterior covariance between $\lambda^j_{\tau^j-}$ and $\lambda^i_{\tau^j-}$ (the numerator of (48)), divided by the posterior disaster intensity of tree j. Intuitively, contagion increases when the disaster of tree j leads to a major update in the posterior probability of being in a 'bad' economic - and high disaster probability - state. This happens when there is high uncertainty about the current state ($p^h_t = 0.5$) and when the difference between disaster intensities in the two states is large. In this case the default event is unambiguously interpreted as evidence of an high disaster likelihood state. Tthe more the disaster is unexpected according to the current belief, because the posterior intensity $\widehat{\lambda}^j_{\tau^j-}$ is low, the larger the update. If we consider finite horizon disaster probabilities instead of instantaneous probabilities, then contagion effects are reacher, because a disaster event propagates to the state of the economy, and this in turn influences the likelihood of disaster of remaining trees. In Panel 1 of Figure 14 we have plotted the survival probability of a given tree after the disaster of a 'sector' that is crucial for the economy, so that the likelihood of transition to a 'bad' state has increased dramatically.[9]

Insert Figure 14 about here

The probability of disaster for the given tree has increased substantially after the collapse of the key sector. The pronounced difference with the pre-default situation is due to the fact

[9]The reported quantity,

$$\frac{P\left(\tau^i > T \middle| \mathcal{F}^{x,S}_{\tau^j}\right) - P\left(\tau^i > T \middle| \mathcal{F}^{x,S}_{\tau^j-}\right)}{P\left(\tau^i > T \middle| \mathcal{F}^{x,S}_{\tau^j-}\right)}, \tag{49}$$

can be written in closed-form, following expression (14).

that the 'crucial' tree had scarce ex-ante probability of disaster.

C.2. Security price contagion

We have already discussed the response of asset prices to disaster events. It is instructive to analyze the risk-neutral counterpart of the contagion measure reported in (48)

$$\begin{aligned}\theta_{\tau^j}\widehat{\lambda}^i_{\tau^j} - \theta_{\tau^j-}\widehat{\lambda}^i_{\tau^j-} \quad &= \\ \left(\frac{\sum_{u\neq i,j} x^u_t}{\overline{x}^i_h + \sum_{u\neq i,j} x^u_t}\right)^{-\gamma} &\frac{\mathrm{Cov}[\lambda^i_{\tau^j-}\lambda^j_{\tau^j-}|\mathcal{F}^{x,S}_{\tau^j-}]}{\widehat{\lambda}^j_{\tau^j-}} \\ &+\left[\left(\frac{\sum_{u\neq i,j} x^u_t}{\overline{x}^i_h + \sum_{u\neq i,j} x^u_t}\right)^{-\gamma} - \left(\frac{\overline{x}^j_h + \sum_{u\neq i,j} x^u_t}{\overline{x}^i_h + \overline{x}^j_h + \sum_{u\neq i,j} x^u_t}\right)^{-\gamma}\right]\widehat{\lambda}^i_{\tau^j}\end{aligned} \tag{50}$$

The risk neutral disaster contagion measure for tree i is given by the objective contagion scaled by the market price of disaster risk after the disaster of tree j, plus the jump in market price of disaster risk due to the event, scaled by the posterior intensity of default of the $i-$th asset. The market price of disaster risk is always above 1 and it increases when the market share of the $i-$th tree increases. Therefore the jump of the market price of risk is positive and it is increasing in the market share of the $i-$th tree. It follows that the risk-neutral contagion is higher than the objective contagion, and the difference is larger the more the aggregate output supply is concentrated. The behavior of the risk neutral probability of default is reflected on security prices, which, as extensively commented above, depend on the risk neutral survival probability of trees. Panel 2 of Figure 14 mimicks the exercise of Panel 1, but shows that the increase in disaster probability after the disaster of the 'crucial' tree is much more pronounced in the risk-adjusted case.

VI. Term structure of credit-spreads

In our framework the $i-$th defaultable zero coupon bond with maturity T is the security that pays one unit of consumption in T if the $i-$th tree has not undergone a disaster before maturity. The $i-$th credit spread is defined as the yield of this bond in excess of the yield of a default-free bond.

Proposition 5 *The equilibrium price of a zero coupon bond with time to maturity $T-t$ is:*

$$P(t,T) = e^{b(T-t)}\left(\sum_{i=1}^{N} x^i_t\right)^{\gamma}\left[p^h_t, 1-p^h_t, 0_{\mathcal{N}-2}\right]\exp\left(-(T-t)\mathbf{A}^H\right)\mathbf{E} \tag{51}$$

The equilibrium price of a defaultable zero coupon bond, linked to the default event of the $i-$*th tree, with zero recovery value and time to maturity* $T-t$ *is:*

$$P^i(t,T) = \mathbf{1}\left(\tau^i > t\right) e^{b(T-t)} \left(\sum_{j=1}^{N} x_t^j\right)^{\gamma} \left[p_t^h, 1-p_t^h, 0_{\mathcal{N}^i-2}\right] \exp\left(-(T-t)\mathbf{A}^{ZD}\right) \mathbf{E}^D \qquad (52)$$

where $\exp(\cdot)$ *denotes the matrix exponential operator,*

$$b = -\delta - \mu_Y \gamma + \sigma_Y^2 \frac{\gamma(\gamma+1)}{2},$$

$\mathcal{N} = 2^N$, $\mathcal{N}^i = 2^{N-1}$ *and expressions for matrices* $\mathbf{A}^H$, $\mathbf{A}^{ZD}$, *vectors* $\mathbf{E}^D$ *and* $\mathbf{E}$ *are reported in the Appendix. The equilibrium* $i-$*th credit spread at time* t *for maturity* T *is:*

$$cs_t^i = -\frac{\log P^i(t,T)}{T-t} + \frac{\log P(t,T)}{T-t} \qquad (53)$$

Figure 15 plots the term structure of credit spreads for the endowment with the highest event risk and for the endowment with the lowest event risk.

Insert Figure 15 about here

To understand the behavior of these spreads, we should take two opposite effects into account. A deferral of the pay-off of the bond implies a higher likelihood of default until maturity. The spread is increasing with time to maturity because of this effect. Higher time to maturity, however, implies also higher state prices at which the pay-off will be evaluated if default does not occur, because some tree will likely have undergone disasters and aggregate consumption will be lower. In Panel 1 the endowment is almost default-risk free, therefore its spread size is small. The probability of no-disaster before 30 years is still high, to the extent that the first effect dominates only until the 12-year maturity, approximately, thereby determining an increasing pattern for the spread. After this maturity, the likelihood of disasters is high for the worst rated trees, and the increasing state prices determine a decreasing pattern for the spread. On the other hand, the no-disaster probability of the tree with the highest event risk reaches quickly small values, so that a further deferral of the pay-off time beyond a critical date does not add significant risk. The perspective of higher state prices in case of payments dominates in fact, determining a decreasing pattern for the spread, with the exception of matirities between 8 and 12 years, when the probability of no-disaster converges to zero.

Figure 16 shows the behavior of credit spreads for the same trees in relation to the number of disasters occurred, assuming that more risky trees undergo a disaster first.

Insert Figure 16 about here

Credit spreads are increasing in the endowment's market share. The intuition is simple if we keep in mind our discussion on disaster risk premia. The credit spread depends on the risk neutral no-disaster probability of the endowment. The risk neutral intensity of disaster $\theta_t\widehat{\lambda}_t$ is increasing in the tree's market share, so that a current higher share forecasts a steadily higher future pattern for the risk neutral intensity and probability of default. In our simple calibration exercise, indeed the dynamics of credit spreads can experience abrupt jumps. Figure 17 shows a simulated path of a constant-time-to-maturity 1-year credit spread for a given endowment.

Insert Figure 17 about here

VII. Conclusions

We have applied the classic Lucas-tree pure exchange framework to investigate the consequences of having multiple supplying trees that are subject to event risk. As already pointed out by Cochrane, Longstaff and Santa-Clara (2007), in such an economy results are driven by random fluctuations of the share of aggregate consumption that each tree supplies. In our paper, disaster and recovery events are solely responsible for market shares fluctuations. Security prices react to default events according to the magnitude of their event correlation with the tree that has experienced the disaster. The higher the event correlation, the more pronounced the contagion effect, that is, the lower the return shock. Incomplete information provides a source of 'perceived' contagion, because the agent updates upwards the probability of an high disaster intensity state when some disaster event occur. Consistently with empirical findings, credit spreads experience upward jumps when different securities default. The higher the event correlation between the endowments, the higher the increase of the credit spread.

We conclude with a brief, heuristic discussion about the robustness of our results to the presence of an arbitrary high number of trees, infinite in fact. Assume that output multiples x^i is such that aggregate output $Y\sum_{i=1}^{\infty} x^i$ is finite. Since each output share is then arbitrary small, the full information response of the $i-$th security return to a disaster for the $j-$th security,

$$\left(\frac{\sum_{z\neq j} x_t^z}{\sum_{z=1}^{\infty} x_t^z}\right)^{\gamma}\left(\frac{\overline{V}^{P^i}(H-j)}{\overline{V}^{P^i}(H)}-1\right) \tag{54}$$

is also arbitrary small. The partial information response must obey the same rule. However, we should take into account that the overall disaster and recovery 'activity' also increases unboundedly, unless event probabilities depend of the number of trees, because

$$\lim_{N\to\infty}\sum_{i=1}^{N}(1-H_t^i)\widehat{\lambda}_t^i=\infty \tag{55}$$

Asset returns will then be subject to high frequency jumps of smaller magnitude. Indeed, the market price of default risk θ_t will vanish with an infinite number of trees, but the event risk premium on the $i-$th security, being the sum of an infinite number of vanishing sources of risk, will converge to a positive limit or diverge to infinity, depending on trees posterior disaster and recovery intensities.

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Appendix A: Proofs

Proof of Proposition 1

Let $H_t^i = \mathbf{1}(x_t^i = 0)$. Note that the continuous-time Markov chain x_t^i can be equivalently written as the pure jump process:

$$x_t^i = \overline{x}_h^i - \int_0^t \overline{x}_h^i dH_s^i$$

where the $\mathcal{F}_t-$intensity of the compound Poisson process dH_t^i is

$$-H_t^i \eta_t^i + (1 - H_t^i)\lambda_t^i$$

Then let $\psi_t = [H_t^1, H_t^2, \ldots, H_t^N, S_t^{f'}, S_t^{g'}]'$ denote the vector of observation processes. Let also $\mathcal{F}_t^{\psi}$ denote the sigma field that the observation process generates. This is equivalent to the sigma-field $\mathcal{F}_t^{\psi} = \mathcal{F}_t^{x,S}$ used in the text. It follows from Theorem 18.3 and Theorem 7.17 in Lipster and Shyriaev (2001) that

$$\widetilde{B}_t^j = \Omega_j \left(S_t^j - \left[P(z_t = \overline{z}^h | \mathcal{F}_t^{\psi}) \odot \mathbf{j}(\overline{z}^h) + P(z_t = \overline{z}^l | \mathcal{F}_t^{\psi}) \odot \mathbf{j}(\overline{z}^l) \right] \right) \quad j = f, g \tag{56}$$

is an $\mathcal{F}_t^{\psi}$-Brownian motion and H_t^i is an $\mathcal{F}_t^{\psi}-$point process with compensator

$$\widehat{\lambda}_t^{H^i} = -H_t^i [g^i(\overline{z}^h) P(z_t = \overline{z}^h | \mathcal{F}_t^{\psi}) + g^i(\overline{z}^l) P(z_t = \overline{z}^l | \mathcal{F}_t^{\psi})] + (1 - H_t^i)[f^i(\overline{z}^h) P(z_t = \overline{z}^h | \mathcal{F}_t^{\psi}) + f^i(\overline{z}^l) P(z_t = \overline{z}^l | \mathcal{F}_t^{\psi})] \tag{57}$$

The representation of the signal vector process S_t^j, $j = f, g$, with respect to the observation filtration becomes

$$dS_t^j \quad = \quad \left[P(z_t = \overline{z}^h | \mathcal{F}_t^{\psi}) \odot \mathbf{j}(\overline{z}^h) + P(z_t = \overline{z}^l | \mathcal{F}_t^{\psi}) \odot \mathbf{j}(\overline{z}^l) \right] + \Omega_j^{-1} d\widetilde{B}_t^j$$

The following Proposition is a straightforward adaptation of Theorem 19.1 and Theorem 5.17 in Lipster and Shyriaev (2001).

Proposition 6 *Any $\mathcal{F}_t^{\psi}-$ martingale Y_t admits the representation:*

$$Y_t = Y_0 + \int_0^t h_s^f \cdot d\widetilde{B}_s^f + \int_0^t h_s^g \cdot d\widetilde{B}_s^g + \int_0^t \sum_{i=1}^N f_s^{H^i} (dH_s^i - \widehat{\lambda}_s^{H^i} ds)$$

where adapted processes h_t^f and h_t^g satisfy the integrability conditions in Theorem 5.17 and Theorem 19.1, respectively, of Lipster and Shyriaev (2001).

The following Proposition is Lemma 9.2 in Lipster and Shyriaev (2001)

Proposition 7 *For $j = h, l$, the random process*

$$y_t^j = \mathbf{1}(z_t = \overline{z}^j) - \mathbf{1}(z_0 = \overline{z}^j) - \int_0^t [-\mathbf{1}(z_s = \overline{z}^j) k_j(\mathbf{x}_s) + \mathbf{1}(z_s = \overline{z}^{j^c}) k_{j^c}(\mathbf{x}_s)] ds$$

is an $\mathcal{F}_t-$martingale, where j^c denotes the complement of j.

Taking conditional expectations with respect to $\mathcal{F}_t^{\psi}$ in the definition of x_t^j, we obtain:

$$P(z_t = \overline{z}^j | \mathcal{F}_t^{\psi}) = P(z_0 = \overline{z}^j | \mathcal{F}_t^{\psi}) + \int_0^t [-P(z_s = \overline{z}^j | \mathcal{F}_t^{\psi}) k_j(\mathbf{x}_s) + P(z_s = \overline{z}^{j^c} | \mathcal{F}_t^{\psi}) k_{j^c}(\mathbf{x}_s)] ds + \mathbb{E}[y_t^j | \mathcal{F}_t^{\psi}] \tag{58}$$

We can now apply the martingale representation theorem above to the martingale $\mathbb{E}[y_t^j | \mathcal{F}_t^{\psi}]$ and by conditional independence of the vector processes H_t^i and S_t^j identify stochastic integrands as in Lipster and Shyriaev (2001), Theorem 19.5 and Theorem 9.2. We end up with the representation given in the Proposition. This ends the proof.

Proof of Proposition 2

Let $H_t = [H_t^1, H_t^2, \ldots, H_t^N]'$ and assume, without loss of generality, that none of the N trees is in a disaster state at time s, so that $H_s = 0'_N$, an $N-$dimensional vector of zeros. Note that 'survival' probabilities for individual trees are obtained as a special case of this methodology, when set D is a singleton. We can write:

$$P(\tau_{d_1} > T, \tau_{d_2} > T, \ldots \tau_{d_L} > T | \mathcal{F}_s^{x,S}) = \mathbb{E}\left[P(\tau_{d_1} > T, \tau_{d_2} > T, \ldots \tau_{d_L} > T | \mathcal{F}_s) | \mathcal{F}_s^{x,S}\right] \tag{59}$$

Assume $z_s = \overline{z}_h$. By the law of iterated expectations the inner expectation is an $\mathcal{F}_s-$martingale, therefore the 'drift' component of its Ito representation must vanish. By the Markov property we must have

$$P(\tau_{d_1} > T, \tau_{d_2} > T, \ldots \tau_{d_L} > T | \mathcal{F}_s) = \mathbf{1}(\tau_{d_1} > s, \tau_{d_2} > s, \ldots \tau_{d_L} > s) V^h(s, H_t) \tag{60}$$

Let $\mathcal{S}^D(N-L, K)$ denote the set of combinations of the $N-L$ available trees excluding those in D, into groups of K, and let $\mathcal{S}^D(N-L, K)_h$ denote the $h-$th element of this set. The set $\mathcal{S}^D$ will be used to denote the trees that are not in a disaster state. We use the notation $\overline{\mathcal{S}}^D(N-L, K)_h$ to denote the complement of the $h-$th element, that is, the trees that are in a disaster state. We apply Ito's lemma to the RHS of (60), take conditional expectations and impose the martingale property, according to which the conditional mean of the RHS of (60) must vanish. Applying this argument also to the probability conditional on the low state of the economy, $\overline{z}^l$, we obtain the following system of ordinary differential equations:

$$\frac{\partial}{\partial s}\begin{bmatrix} V^h(s, H_s) \\ V^l(s, H_s) \end{bmatrix} = \left(\begin{bmatrix} \sum_{j=1}^N f^j(\overline{z}^h) & 0 \\ 0 & \sum_{j=1}^N f^j(\overline{z}^l) \end{bmatrix} - I\right)\begin{bmatrix} V^h(s, H_s) \\ V^l(s, H_s) \end{bmatrix} - \begin{bmatrix} \sum_{j=1}^{\#(\mathcal{S}^D(N-L,N-L-1))} f^j(\overline{z}^h) V^h(s, \mathcal{S}^D(N-L, N-L-1)_j) \\ \sum_{j=1}^{\#(\mathcal{S}^D(N-L,N-L-1))} f^j(\overline{z}^l) V^l(s, \mathcal{S}^D(N-L, N-L-1)_j) \end{bmatrix} \tag{61}$$

The system involves all functions V conditional on any combination of supply x_t^i for the N trees excluding those L for which we want to compute the probability of no-disaster. This system of equations can be written compactly in vectorial notation.

$$\frac{d}{ds}\begin{bmatrix} \mathbf{V}(s, H_s) \\ \mathbf{V}(s, \mathcal{S}^D(N-L, N-L-1)_1) \\ \vdots \\ \mathbf{V}(s, \mathcal{S}^D(N-L, N-L-1)_{N-L}) \\ \mathbf{V}(s, \mathcal{S}^D(N-L, N-L-2)_1) \\ \vdots \\ \mathbf{V}(s, \mathcal{S}^D(N-L, 1)) \\ \mathbf{V}(s, \mathcal{S}^D) \end{bmatrix} = \mathbf{A}\begin{bmatrix} \mathbf{V}(s, H_s) \\ \mathbf{V}(s, \mathcal{S}^D(N-L, N-L-1)_1) \\ \vdots \\ \mathbf{V}(s, \mathcal{S}^D(N-L, N-L-1)_{N-L}) \\ \mathbf{V}(s, \mathcal{S}^D(N-L, N-L-2)_1) \\ \vdots \\ \mathbf{V}(s, \mathcal{S}^D(N-L, 1)) \\ \mathbf{V}(s, \mathcal{S}^D) \end{bmatrix} \tag{62}$$

where $\mathbf{V}(s, \cdot) = [V^h(s, \cdot), V^l(s, \cdot)]'$, and $\mathbf{V}(s, \mathcal{S}^D)$ denotes the function $\mathbf{V}$ conditional on all trees excluding those in D being in a disaster state.

$$\begin{aligned}\mathbf{A} = \text{diag}[&\boldsymbol{\Upsilon}^{N-L}, \boldsymbol{\Upsilon}^{\mathcal{S}^D(N-L,N-L-1)_1}, \ldots, \boldsymbol{\Upsilon}^{\mathcal{S}^D(N-L,N-L-1)_{N-L}}, \boldsymbol{\Upsilon}^{\mathcal{S}^D(N-L,N-L-2)_1}, \\ &\ldots, \boldsymbol{\Upsilon}^{\mathcal{S}^D(N-L,1)_{N-L}}, \boldsymbol{\Upsilon}^{\mathcal{S}^D}] - \begin{bmatrix} \mathbf{0} & \mathbf{F}^1 & \mathbf{F}^2 & \ldots & \mathbf{F}^{N-L} & \mathbf{0} & \ldots & \mathbf{0} & \ldots \\ \mathbf{G}^1 & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{F}^2 & \mathbf{F}^3 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}\end{aligned} \tag{63}$$

with

$$\begin{aligned}
\mathbf{F}^i &= \text{diag}[f^i(\overline{z}^h), f^i(\overline{z}^l)] \\
\mathbf{G}^i &= \text{diag}[g^i(\overline{z}^h), g^i(\overline{z}^l)] \\
\boldsymbol{\Upsilon}^{\mathcal{S}^D(K,K-1)_j} &= \text{diag}[\sum_{u\in D} f^u(\overline{z}^h), \sum_{u\in D} f^u(\overline{z}^l)] + \text{diag}[\sum_{u\in\mathcal{S}^D(K,K-1)_j} f^u(\overline{z}^h), \sum_{u\in\mathcal{S}^D(K,K-1)_j} f^u(\overline{z}^l)] \\
&\quad + \text{diag}[\sum_{u\in\overline{\mathcal{S}}^D(K,K-1)_j} g^u(\overline{z}^h), \sum_{u\in\overline{\mathcal{S}}^D(K,K-1)_j} g^u(\overline{z}^l)] - I \\
\boldsymbol{\Upsilon}^{N-L} &= \text{diag}[\sum_{u=1}^N f^u(\overline{z}^h), \sum_{u\in D} f^u(\overline{z}^l)] - I \\
\boldsymbol{\Upsilon}^{\mathcal{S}^D} &= \text{diag}[\sum_{u=1}^{N-L} g^u(\overline{z}^h), \sum_{u=1}^{N-L} g^u(\overline{z}^l)] + \text{diag}[\sum_{u\in D} f^u(\overline{z}^h), \sum_{u\in D} f^u(\overline{z}^l)] - I,
\end{aligned}$$

and $\mathbf{0} = \text{diag}[0,0]$. The terminal condition is $V(T) = 1$. The solution of this system is immediately characterized in terms of matrix exponential operator, so that:

$$P(\tau_{d_1} > T, \tau_{d_2} > T, \dots \tau_{d_L} > T | \mathcal{F}_s^{x,S}) = \mathbf{1}(\tau_{d_1} > s, \tau_{d_2} > s, \dots \tau_{d_L} > s)[p_t^h, 1 - p_t^h, 0_{\mathcal{N}-2}] \cdot \exp\left(-\mathbf{A}(T-s)\right) \overline{1}_{\mathcal{N}} \tag{64}$$

where $\mathcal{N} = 2^{N-L}$

The expected time until the next disaster of any of the trees in group D is given by

$$\begin{aligned}
\mathbb{E}\left[\min_{i \in D} \tau^i - s \middle| \mathcal{F}_s^{x,S}\right] &= \int_s^\infty u \frac{\partial}{\partial u}\left[1 - P(\tau_{d_1} > u, \tau_{d_2} > u, \dots \tau_{d_L} > u | \mathcal{F}_s^{x,S})\right] du \\
&= \int_s^\infty -u \frac{\partial}{\partial u} P(\tau_{d_1} > u, \tau_{d_2} > u, \dots \tau_{d_L} > u | \mathcal{F}_s^{x,S}) du - s \\
&= \mathbf{1}(\tau_{d_1} > s, \tau_{d_2} > s, \dots \tau_{d_L} > s) \times \\
&\quad [p_t^h, 1 - p_t^h, 0_{\mathcal{N}-2}] \cdot \left[\int_s^\infty u \mathbf{A} \exp\left(-\mathbf{A}(u-s)\right) du\right] \overline{1}_{\mathcal{N}} - s \\
&= [p_t^h, 1 - p_t^h, 0_{\mathcal{N}-2}] \cdot \mathbf{A}^{-1} \overline{1}_{\mathcal{N}}
\end{aligned} \tag{65}$$

Given a tree i which is currently in disaster state, i.e. $H_t^i = 1$, its expected recovery time is computed with a similar methodology. Without loss of generality we assume that all remaining trees are not in a disaster state. Any different combination of state can be accomodated with obvious modifications. Let τ_r^i be the first recovery time of tree i. We have

$$P(\tau_r^i > T | \mathcal{F}_s^{H,S}) = \mathbf{1}(\tau_r^i > s)[p_t^h, 1 - p_t^h, 0_{\mathcal{N}-2}] \cdot \exp\left(-\mathbf{A}^r(T-s)\right) \overline{1}_{\mathcal{N}^i} \tag{66}$$

where $\mathcal{N}^i = 2^{N-1}$

$$\begin{aligned}
\mathbf{A}^r = \text{diag}[&\mathbf{\Upsilon}_r^{N-1}, \mathbf{\Upsilon}_r^{\mathcal{S}^i(N-1,N-1)_1}, \dots, \mathbf{\Upsilon}_r^{\mathcal{S}^i(N-1,N-1)_{N-1}}, \mathbf{\Upsilon}_r^{\mathcal{S}^i(N-1,N-2)_1}, \dots, \mathbf{\Upsilon}_r^{\mathcal{S}^i(N-1,1)_{N-1}}, \mathbf{\Upsilon}_r^{\mathcal{S}^i}] - \\
&\begin{bmatrix} \mathbf{0} & \mathbf{F}^1 & \mathbf{F}^2 & \dots & \mathbf{F}^{N-1} & \mathbf{0} & \dots & \mathbf{0} & \dots \\ \mathbf{G}^1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{F}^2 & \mathbf{F}^3 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}
\end{aligned} \tag{67}$$

with

$$\begin{aligned}
\mathbf{F}^i &= \text{diag}[f^i(\overline{z}^h), f^i(\overline{z}^l)] \\
\mathbf{G}^i &= \text{diag}[g^i(\overline{z}^h), g^i(\overline{z}^l)] \\
\mathbf{\Upsilon}_r^{\mathcal{S}^i(N-1,K)_j} &= \text{diag}[g^i(\overline{z}^h), g^i(\overline{z}^l)] + \text{diag}[\sum_{u \in \mathcal{S}^i(N-1,K)_j} f^u(\overline{z}^h), \sum_{u \in \mathcal{S}^i(N-1,K)_j} f^u(\overline{z}^l)] \\
&\quad + \text{diag}[\sum_{u \in \overline{\mathcal{S}}^i(N-1,K)_j} g^u(\overline{z}^h), \sum_{u \in \overline{\mathcal{S}}^i(N-1,K)_j} g^u(\overline{z}^l)] - I \\
\mathbf{\Upsilon}_r^{N-1} &= \text{diag}[g^i(\overline{z}^h), g^i(\overline{z}^l)] + \sum_{j \neq i} \text{diag}[f^j(\overline{z}^h), f^j(\overline{z}^l)] - I \\
\mathbf{\Upsilon}_r^{\mathcal{S}^i} &= \sum_{i=1}^N \text{diag}[g^i(\overline{z}^h), g^i(\overline{z}^l)] - I,
\end{aligned}$$

and $\mathbf{0} = \text{diag}[0,0]$. Notation is the same used for no-disaster probabilities, with the exception that $\mathcal{S}^i(N-1,K)$ now denotes the set of combinations of the $N-1$ trees that have not yet undergone a disaster - excluding the ith - in groups of K. The expected time until recovery for tree i is then:

$$\begin{aligned}
\mathbb{E}\left[\tau_r^i - s \middle| \mathcal{F}_s^{x,S}\right] &= \int_s^\infty u \frac{\partial}{\partial u} P(\tau_r^i \le u | \mathcal{F}_s^{x,S}) du - s \\
&= \int_s^\infty -u \frac{\partial}{\partial u} P(\tau_r^i > u | \mathcal{F}_s^{x,S}) du - s \\
&= \mathbf{1}(\tau_r^i > s)[p_t^h, 1 - p_t^h, 0_{\mathcal{N}^i-2}] \cdot \left[\int_s^\infty \mathbf{A}^r \exp\left(-\mathbf{A}^r(u-s)\right) du\right] \overline{1}_{\mathcal{N}^i} - s \\
&= \mathbf{1}(\tau_r^i > s)[p_t^h, 1 - p_t^h, 0_{\mathcal{N}^i-2}] \cdot (\mathbf{A}^r)^{-1} \overline{1}_{\mathcal{N}^i}
\end{aligned} \tag{68}$$

We apply once again a similar methodology to compute the expected fraction of time spent in a disaster state. Assume without loss of generality that none of the N trees is currently in a disaster state. We remind that $H_t^i = \mathbf{1}(x_t^i = 0)$. For a collection $D = \{d_1, d_2, \dots d_L\}$ of trees, we have

$$P(H_T^{d_1} = 0, H_T^{d_2} = 0, \dots H_T^{d_L} = 0 | \mathcal{F}_s^{x,S}) = [p_t^h, 1 - p_t^h, 0_{\mathcal{N}-2}] \cdot \exp\left(-\mathbf{A}^H (T-s)\right) \overline{W}_{\mathcal{N}} \tag{69}$$

where $\overline{W}_{\mathcal{N}}$ is the column vector with $j-$th element $\mathbf{1}(D \in \mathcal{S}(N,K)_j)$ and dimension given by $\mathcal{N} = 2^N$

$$\mathbf{A}^H = \text{diag}[\mathbf{\Upsilon}_H^N, \mathbf{\Upsilon}_H^{\mathcal{S}(N,N-1)_1}, \dots, \mathbf{\Upsilon}_H^{\mathcal{S}(N,N-1)_{N-1}}, \mathbf{\Upsilon}_H^{\mathcal{S}(N,N-2)_1}, \dots, \mathbf{\Upsilon}_H^{\mathcal{S}(N,1)}, \mathbf{\Upsilon}_H^{\mathcal{S}}] - \begin{bmatrix} \mathbf{0} & \mathbf{F}^1 & \mathbf{F}^2 & \dots & \mathbf{F}^N & \mathbf{0} & \dots & \mathbf{0} & \dots \\ \mathbf{G}^1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{F}^2 & \mathbf{F}^3 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{70}$$

with

$$\begin{aligned}
\mathbf{F}^i &= \text{diag}[f^i(\overline{z}^h), f^i(\overline{z}^l)] \\
\mathbf{G}^i &= \text{diag}[g^i(\overline{z}^h), g^i(\overline{z}^l)] \\
\mathbf{\Upsilon}_H^{\mathcal{S}(N,K)_j} &= \text{diag}[\sum_{u \in \mathcal{S}(N,K)_j} f^u(\overline{z}^h), \sum_{u \in \mathcal{S}(N,K)_j} f^u(\overline{z}^l)] + \text{diag}[\sum_{u \in \overline{\mathcal{S}}(N,K)_j} g^u(\overline{z}^h), \sum_{u \in \overline{\mathcal{S}}(N,K)_j} g^u(\overline{z}^l)] - I \\
\mathbf{\Upsilon}_H^N &= \sum_{i=1}^N \text{diag}[f^i(\overline{z}^h), f^i(\overline{z}^l)] - I \\
\mathbf{\Upsilon}_H^{\mathcal{S}} &= \sum_{i=1}^N \text{diag}[g^i(\overline{z}^h), g^i(\overline{z}^l)] - I,
\end{aligned}$$

and $\mathbf{0} = \text{diag}[0,0]$. $\mathcal{S}(N,K)$ now denotes the set of combinations of all the N trees in groups of K. The expected fraction of time on the horizon $[s,T]$ with no disaster state for members of group D is then:

$$\begin{aligned}
&\frac{1}{T-s}\mathbb{E}\left[\int_s^T \mathbf{1}(H_u^{d_1} = 0, H_u^{d_2} = 0, \dots H_u^{d_2} = 0) \Big| \mathcal{F}_s^{x,H} du\right] = \frac{1}{T-s}\int_s^T P(H_u^{d_1} = 0, H_u^{d_2} = 0, \dots H_u^{d_2} = 0 | \mathcal{F}_s^{x,S}) du \\
&= \frac{[p_t^h, 1-p_t^h, 0_{\mathcal{N}-2}]}{T-s} \cdot \left[\int_0^{T-s} \exp\left(-\mathbf{A}^H \tau\right) d\tau\right] \overline{W}_{\mathcal{N}} \quad = [p_t^h, 1-p_t^h, 0_{\mathcal{N}-2}] \cdot \frac{(\mathbf{A}^H)^{-1}}{T-s}\left[I_d - \exp(-\mathbf{A}^H(T-s))\right] \overline{W}_{\mathcal{N}}
\end{aligned} \tag{71}$$

This ends the proof of the proposition.

Conditional probabilities of i-th share

Thanks to the proof of Proposition 1, it is immediate to identify the conditional distribution for the market share of the $i-th$ tree. As before, for $T > v$, assume without loss of generality that none of the N threes has yet (at time v) undergone a disaster. We have, for $K = 1, \dots, N-1$ and $j = 1, \dots, \#(\mathcal{S}^i(N-1,K))$:

$$\begin{aligned}
P\left(s_T^i = 0 \middle| \mathcal{F}_v^{x,S}\right) &= 1 - P\left(H_T^i = 0 \middle| \mathcal{F}_v^{x,S}\right) \\
P\left(s_T^i = \frac{\overline{x}_h^i}{\sum_{u \in \mathcal{S}^i(N-1,K)_j} \overline{x}_h^u} \middle| \mathcal{F}_v^{x,S}\right) &= P\left(H_T^i = 0, H_T^u = 0, H_T^z = 1 \middle| \mathcal{F}_v^{x,S}\right), \\
& \quad u \in \mathcal{S}^i(N-1,K)_j, z \in \overline{\mathcal{S}}^i(N-1,K)_j
\end{aligned}$$

where

$$P\left(H_T^i = 0, H_T^u = 0, H_T^z = 1 \middle| \mathcal{F}_v^{x,S}\right) = [p_t^h, 1-p_t^h, 0_{\mathcal{N}-2}] \cdot \exp\left(-\mathbf{A}^H(T-v)\right) \overline{W}_{\overline{\mathcal{N}}} \tag{72}$$

where $\overline{W}_{\overline{\mathcal{N}}}$ is the column vector with $k-$th element $\mathbf{1}(i \cup \mathcal{S}^i(N-1,K)_j \in \mathcal{S}(N,K)_k)$ and dimension given by $\overline{\mathcal{N}} = 2^N$

Proof of Proposition 3

We remind that $H_t^i = \mathbf{1}(x_t^i = 0)$, therefore $dH_t^i = 1$ if a disaster occurs, i.e. $H_t^i = 0$, and $dH_t^i = -1$ if a recovery occurs, i.e. $H_t^i = 1$.

The state-price density is:

$$\xi_t = e^{-\delta t} Y_t^{-\gamma} \left(\sum_{i=1}^{N} x_t^i \right)^{-\gamma} \tag{73}$$

On the other hand, the state-price density must also obey:

$$\xi_t = \exp\left(-\int_0^t (r_s + \frac{\kappa_s^2}{2}) ds - \int_0^t \kappa_s dZ_s + \int_0^t \sum_{i=1}^{N} \widehat{\lambda}_s^{H^i} (1-\theta_s^i) ds + \int_0^t \sum_{i=1}^{N} -\text{sgn}(H_t^i) \log(\theta_s^i) dH_s^i \right) \tag{74}$$

where $\text{sgn}(H_t^i) = -1$ if $H_t^i \leq 0$ and $\text{sgn}(H_t^i) = 1$ if $H_t^i > 0$. Furthermore

$$\widehat{\lambda}_t^{H^i} = -H_t^i \widehat{\eta}^i + (1 - H_t^i) \widehat{\lambda}_t^i$$

By applying Ito's lemma to (74) we obtain:

$$d\xi_t = -\xi_t r_t dt - \xi_t \kappa dZ_t + \xi_t \left[\sum_{i=1}^{N} -\text{sgn}(H_t^i)(\theta_s^i - 1)(dH_t^i - \widehat{\lambda}_t^{H^i}) \right] \tag{75}$$

By Ito's lemma applied to (73) we obtain the alternative representation:

$$\begin{aligned} d\xi_t = &-\delta\xi_t - \gamma\mu_Y \xi_t dt + \frac{1}{2}\gamma(\gamma+1)\sigma_Y^2 \xi_t dt + \xi_t \sum_{i=1}^{N} \left[(1-H_t) \frac{\left[(\sum_{j\neq i} x_t^j)^{-\gamma} - (\sum_{j=1}^{N} x_t^j)^{-\gamma} \right]}{(\sum_{j=1}^{N} x_t^j)^{-\gamma}} \widehat{\lambda}_t^i \right. \\ &\left. + H_t \frac{\left[(\sum_{j=1}^{N} x_t^j)^{-\gamma} - (\sum_{j\neq i} x_t^j)^{-\gamma} \right]}{(\sum_{j\neq i} x_t^j)^{-\gamma}} \widehat{\eta}_t^i \right] - \gamma\xi_t \sigma_Y dZ_t + \xi_t \sum_{i=1}^{N} \left[(1-H_t) \frac{\left[(\sum_{j\neq i} x_t^j)^{-\gamma} - (\sum_{j=1}^{N} x_t^j)^{-\gamma} \right]}{(\sum_{j=1}^{N} x_t^j)^{-\gamma}} (dH_t^i - \widehat{\lambda}_t^i) \right. \\ &\left. - H_t \frac{\left[(\sum_{j=1}^{N} x_t^j)^{-\gamma} - (\sum_{j\neq i} x_t^j)^{-\gamma} \right]}{(\sum_{j\neq i} x_t^j)^{-\gamma}} (dH_t^i + \widehat{\eta}_t^i) \right] \end{aligned}$$

which, compared with (74) yields the expressions reported in the text. This ends the proof.

Proof of Proposition 4

We assume without loss of generality that none of the N trees has yet (at time t) undergone a disaster. The price of the market portfolio is:

$$\begin{aligned} V_t^M &= \frac{1}{\xi_t} \mathbb{E}\left[\int_t^\infty \xi_s C_s ds \middle| \mathcal{F}_t^{H,S} \right] && (76) \\ &= \frac{Y_t}{(\sum_{i=1}^{N} x_t^i)^{-\gamma}} \mathbb{E}\left[\int_t^\infty e^{-\delta(s-t)} \left(\frac{Y_s}{Y_t} \right)^{1-\gamma} \left(\sum_{i=1}^{N} x_s^i \right)^{1-\gamma} ds \middle| \mathcal{F}_t^{H,S} \right] && (77) \\ &= \frac{Y_t}{(\sum_{i=1}^{N} x_t^i)^{-\gamma}} \mathbb{E}\left[\mathbb{E}\left[\int_t^\infty e^{-a(s-t)} \left(\sum_{i=1}^{N} x_s^i \right)^{1-\gamma} ds \middle| \mathcal{F}_t \right] \middle| \mathcal{F}_t^{H,S} \right] && (78) \end{aligned}$$

where

$$a = \delta - \mu_Y(1-\gamma) + \frac{\sigma_Y^2}{2}(1-\gamma)\gamma$$

The last step follows from the independence of Y_t and from the law of iterated expectations. Similarly, the price of the claim to the $i-$th endowment process is:

$$\begin{aligned} V_t^i &= \frac{1}{\xi_t} \mathbb{E}\left[\int_t^\infty \xi_s Y_s x_s^i ds \middle| \mathcal{F}_t^{H,S} \right] && (79) \\ &= \frac{Y_t}{(\sum_{i=1}^{N} x_t^i)^{-\gamma}} \mathbb{E}\left[\mathbb{E}\left[\int_t^\infty e^{-a(s-t)} s_s^i \left(\sum_{i=1}^{N} x_s^i \right)^{1-\gamma} ds \middle| \mathcal{F}_t \right] \middle| \mathcal{F}_t^{H,S} \right] && (80) \end{aligned}$$

where s_t^i is the $i-$th tree market share at time t.

Assume $z_t = \overline{z}_h$, and let $V^h(H_t)$ denote the inner (full information) conditional expectation in (78). The inner expectation in (80) is computed similarly after the obvious modifications. By the law of iterated expectations

$$\int_0^t e^{-\delta s}\left(\sum_{i=1}^N x_s^i\right)^{1-\gamma} ds + e^{-\delta t} V^h(H_t) \tag{81}$$

is an $\mathcal{F}_t-$martingale, therefore the 'drift' component of its Ito representation must vanish. We use the same notation of the proof of Proposition 2 to identify the collection of trees that are in disaster state and those that are not. We apply Ito's lemma to (81), take conditional expectations and impose the martingale property. Applying this argument also to the full information price of the market portfolio conditional on the low state of the economy, $\overline{z}^l$, we obtain the following system of equations:

$$\begin{bmatrix}0\\0\end{bmatrix} = \left(\begin{bmatrix}-a-\sum_{j=1}^N f^j(\overline{z}^h) & 0\\ 0 & -a-\sum_{j=1}^N f^j(\overline{z}^l)\end{bmatrix} + I\right)\begin{bmatrix}V^h(H_t)\\V^l(H_t)\end{bmatrix} + \begin{bmatrix}\sum_{j=1}^{\#(\mathcal{S}(N,N-1))} f^j(\overline{z}^h)V^h(\mathcal{S}(N,N-1)_j)\\ \sum_{j=1}^{\#(\mathcal{S}(N,N-1))} f^j(\overline{z}^l)V^l(\mathcal{S}(N,N-1)_j)\end{bmatrix} + \begin{bmatrix}\left(\sum_{i=1}^N \overline{x}_h^i\right)^{1-\gamma}\\ \left(\sum_{i=1}^N \overline{x}_h^i\right)^{1-\gamma}\end{bmatrix} \tag{82}$$

Using notation the notation of the proof of Proposition 2, this system of equations can be written compactly in vectorial notation.

$$\begin{bmatrix}0\\0\\ \vdots\\0\\0\\ \vdots\\0\end{bmatrix} = -(\mathbf{a}+\mathbf{A}^H)\begin{bmatrix}\mathbf{V}(H_t)\\ \mathbf{V}(\mathcal{S}(N,N-1)_1)\\ \vdots\\ \mathbf{V}(\mathcal{S}(N,N-1)_N)\\ \mathbf{V}(\mathcal{S}(N,N-2)_1)\\ \vdots\\ \mathbf{V}(\mathcal{S}(N,1))\\ \mathbf{V}(\mathcal{S})\end{bmatrix} + \mathbf{C} \tag{83}$$

where $\mathbf{V}(\cdot) = [V^h(\cdot), V^l(\cdot)]'$, $\mathbf{V}(\mathcal{S})$ denotes the function $\mathbf{V}$ conditional on all trees being in disaster state, and

$$\mathbf{C} = \begin{bmatrix}\left(\sum_{i=1}^N \overline{x}_h^i\right)^{1-\gamma}\overline{1}_2\\ \left(\sum_{i=2}^N \overline{x}_h^i\right)^{1-\gamma}\overline{1}_2\\ \vdots\\ \left(\sum_{i=1}^{N-1} \overline{x}_h^i\right)^{1-\gamma}\overline{1}_2\\ \left(\sum_{i=3}^N \overline{x}_h^i\right)^{1-\gamma}\overline{1}_2\\ \vdots\\ (0)^{1-\gamma}\overline{1}_2\end{bmatrix}. \tag{84}$$

where $\overline{1}_2$ is a 2-dimensional column vector of ones. Finally:

$$V_t^M = Y_t\left(\sum_{i=1}^N x_t^i\right)^{\gamma}[p_t^h, 1-p_t^h, 0_{\mathcal{N}-2}]\cdot(\mathbf{a}+\mathbf{A}^H)^{-1}\mathbf{C} \tag{85}$$

$$V_t^i = Y_t\left(\sum_{i=1}^N x_t^i\right)^{\gamma}[p_t^h, 1-p_t^h, 0_{\mathcal{N}-2}]\cdot(\mathbf{a}+\mathbf{A}^H)^{-1}\mathbf{C}^i \tag{86}$$

where $\mathcal{N} = 2^N$ and

$$\mathbf{C}^i = \begin{bmatrix} \frac{\overline{x}_h^i}{\sum_{j=1}^N \overline{x}_h^j} \left(\sum_{j=1}^N \overline{x}_h^j\right)^{1-\gamma} \overline{1}_2 \\ \frac{\overline{x}_h^i}{\sum_{j=2}^N \overline{x}_h^j} \left(\sum_{j=2}^N \overline{x}_h^j\right)^{1-\gamma} \overline{1}_2 \\ \vdots \\ \frac{0}{\sum_{j\neq i} \overline{x}_h^j} \left(\sum_{j\neq i} \overline{x}_h^j\right)^{1-\gamma} \overline{1}_2 \\ \vdots \\ \frac{\overline{x}_h^i}{\sum_{j=1}^{N-1} \overline{x}_h^j} \left(\sum_{i=1}^{N-1} \overline{x}_h^i\right)^{1-\gamma} \overline{1}_2 \\ \vdots \\ (0)^{1-\gamma}\overline{1}_2 \end{bmatrix}. \tag{87}$$

and $\mathbf{a}$ is a $\mathcal{N}$-dimensional diagonal matrix with a on the main diagonal. In the Proposition, we have denoted by $\overline{V}^M(H)$ the expected full-information discounted cash-flow of the market portfolio, i.e

$$\overline{V}^M(H) = [V^h(H_t), V^l(H_t)] = [[1, 0_{\mathcal{N}-1}] \cdot (\mathbf{a} + \mathbf{A}^H)^{-1}\mathbf{C}, [0, 1, 0_{\mathcal{N}-2}] \cdot (\mathbf{a} + \mathbf{A}^H)^{-1}\mathbf{C}]]$$

We have also denoted by $\overline{V}^M(N-j)$ ($\overline{V}^M(N+j)$) the same quantity conditional on tree j having undergone a disaster (a recovery from disaster). This is the entry of the vector $(\mathbf{a} + \mathbf{A}^H)^{-1}\mathbf{C}$ corresponding to the specific combination of trees in disaster and 'normal' state. A similar definition holds for $\overline{V}^i(H)$ and $\overline{V}^i(H-j)$ ($\overline{V}^i(H+j)$).

The risk premium of the market portfolio is:

$$\mu_t^M = \mathbb{E}\left[\left.\frac{dV_t^M}{V_t^M}\right| \mathcal{F}_t^{H,S}\right] + \frac{C_t}{V_t^M} - r_t \tag{88}$$

The expression reported is obtained by applying Ito's lemma to the formula for the price process, taking expectations and taking into account the expression for the equilibrium interest rate. The risk premium of claims to individual endowment processes is computed similarly.

This concludes the proof.

Proposition 8 and its proof.

Proposition 8 *The conditional event correlation between trees i and j on the horizon $[t,T]$ is defined as:*

$$\rho_{t,T}^{ij} = \frac{P\left(H_T^i = 0, H_T^j = 0\middle| \mathcal{F}_t^{x,S}\right) - P\left(H_T^i = 0\middle| \mathcal{F}_t^{x,S}\right) P\left(H_T^j = 0\middle| \mathcal{F}_t^{x,S}\right)}{\sqrt{P\left(H_T^i = 0\middle| \mathcal{F}_t^{x,S}\right)\left(1 - P\left(H_T^i = 0\middle| \mathcal{F}_t^{x,S}\right)\right) P\left(H_T^j = 0\middle| \mathcal{F}_t^{x,S}\right)\left(1 - P\left(H_T^j = 0\middle| \mathcal{F}_t^{x,S}\right)\right)}} \tag{89}$$

Let $P^Q\left(\cdot \middle| \mathcal{F}_t^{x,S}\right)$ denote the conditional probability operator under the risk neutral measure. The risk neutral conditional default correlation between tree i and tree j on the horizon $[t,T]$ is given by equation (89) with risk neutral probabilities instead of objective probabilities. The risk neutral joint probability of 'high' economic state for trees of an L-dimensional group D reads explicitly as follows:

$$P^Q(H_T^{d_1} = 0, H_T^{d_2} = 0, \ldots H_T^{d_L} = 0 | \mathcal{F}_s^{x,S}) = [p_t^h, 1 - p_t^h, 0_{\mathcal{N}-2}] \cdot \exp\left(-\mathbf{A}^Q(T-s)\right) \overline{W}_{\mathcal{N}} \tag{90}$$

where $\overline{W}_{\mathcal{N}}$ is the column vector with $j-$th element $\mathbf{1}(D \in \mathcal{S}(N,K)_j)$ and dimension given by $\mathcal{N} = 2^N$

$$\mathbf{A}^Q = \text{diag}[\mathbf{\Upsilon}_H^N, \mathbf{\Upsilon}_H^{\mathcal{S}(N,N-1)_1}, \ldots, \mathbf{\Upsilon}_H^{\mathcal{S}(N,N-1)_{N-1}}, \mathbf{\Upsilon}_H^{\mathcal{S}(N,N-2)_1}, \ldots, \mathbf{\Upsilon}_H^{\mathcal{S}(N,1)}, \mathbf{\Upsilon}_H^{\mathcal{S}}] - \begin{bmatrix} \mathbf{0} & \mathbf{F}^1 & \mathbf{F}^2 & \ldots & \mathbf{F}^N & \mathbf{0} & \ldots & \mathbf{0} & \ldots \\ \mathbf{G}^1 & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{F}^2 & \mathbf{F}^3 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \tag{91}$$

with

$$\begin{aligned}
\mathbf{F}^i &= \mathrm{diag}\left[f^i(\overline{z}^h)\left(\frac{\sum_{j\neq i} x_t^j}{\sum_{j=1}^N x_t^j}\right)^{-\gamma}, f^i(\overline{z}^l)\left(\frac{\sum_{j\neq i} x_t^j}{\sum_{j=1}^N x_t^j}\right)^{-\gamma}\right] \\
\mathbf{G}^i &= \mathrm{diag}\left[g^i(\overline{z}^h)\left(\frac{\sum_{j=1}^N x_t^j}{\sum_{j\neq i} x_t^j}\right)^{-\gamma}, g^i(\overline{z}^l)\left(\frac{\sum_{j=1}^N x_t^j}{\sum_{j\neq i} x_t^j}\right)^{-\gamma}\right] \\
\boldsymbol{\Upsilon}_H^{\mathcal{S}(N,K)_j} &= \mathrm{diag}[\sum_{u\in\mathcal{S}(N,K)_j} f^u(\overline{z}^h)\left(\frac{\sum_{j\neq u} x_t^j}{\sum_{j=1}^N x_t^j}\right)^{-\gamma}, \sum_{u\in\mathcal{S}(N,K)_j} f^u(\overline{z}^l)\left(\frac{\sum_{j\neq u} x_t^j}{\sum_{j=1}^N x_t^j}\right)^{-\gamma}] \\
&\quad +\mathrm{diag}[\sum_{u\in\mathcal{S}(N,K)_j} g^u(\overline{z}^h)\left(\frac{\sum_{j=1}^N x_t^j}{\sum_{j\neq u} x_t^j}\right)^{-\gamma}, \sum_{u\in\overline{\mathcal{S}}(N,K)_j} g^u(\overline{z}^l)\left(\frac{\sum_{j=1}^N x_t^j}{\sum_{j\neq u} x_t^j}\right)^{-\gamma}] - I \\
\boldsymbol{\Upsilon}_H^N &= \sum_{i=1}^N \mathrm{diag}[f^i(\overline{z}^h), f^i(\overline{z}^l)\left(\frac{\sum_{j=1}^N x_t^j}{\sum_{j\neq u} x_t^j}\right)^{-\gamma}] - I \\
\boldsymbol{\Upsilon}_H^{\mathcal{S}} &= \sum_{i=1}^N \mathrm{diag}[g^i(\overline{z}^h)\left(\frac{\sum_{j\neq i} x_t^j}{\sum_{j=1}^N x_t^j}\right)^{-\gamma}, g^i(\overline{z}^l)\left(\frac{\sum_{j=1}^N x_t^j}{\sum_{j\neq u} x_t^j}\right)^{-\gamma}] - I,
\end{aligned}$$

and $\mathbf{0} = \mathrm{diag}[0,0]$. $\mathcal{S}(N,K)$ *now denotes the set of combinations of all the* N *trees in groups of* K.

Expressions for probabilities in (89) are reported in (71). The corresponding expressions for the risk neutral probabilities are computed along the same lines, once we take into account that the risk-neutral full information 'event' intensity of each tree is

$$(1-H_t^i)\lambda_t^i\left(\frac{\sum_{j\neq i} x_t^j}{\sum_{j=1}^N x_t^j}\right)^{-\gamma} - H_t^i\eta_t^i\left(\frac{\sum_{j=1}^N x_t^j}{\sum_{j\neq i} x_t^j}\right)^{-\gamma} \tag{92}$$

The covariance between returns on endowment claims

The covariance between returns on claims to the $i-$th and the $j-$th endowment reads.

$$\begin{aligned}
\mathbf{E}\left[\frac{dV_t^i}{V_t^i}\frac{dV_t^j}{V_t^j}\middle|\mathcal{F}_t^{H,S}\right] &= \sigma_Y^2 + \sum_{u=1}^N\left(\frac{(e_1-e_2)\cdot\overline{V}^i}{[p_t^h,1-p_t^h]\cdot\overline{V}^i}\right)\left(\frac{(e_1-e_2)\cdot\overline{V}^j}{[p_t^h,1-p_t^h]\cdot\overline{V}^j}\right)[p_t^h(1-p_t^h)]^2\times \\
&\times[(f^u(\overline{z}^h)-f^u(\overline{z}^l))^2+(g^u(\overline{z}^h)-g^u(\overline{z}^l))^2] + \sum_{u=1}^N\Bigg\{(1-H_t^u)\left[\left(\frac{\sum_{j\neq u} x_t^j}{\sum_{j=1}^N x_t^j}\right)^{\gamma}\times\right. \\
&\times\left(\frac{\left[p_t^h\frac{f^u(\overline{z}^h)}{\widehat{\lambda}_t^u},(1-p_t^h)\frac{f^u(\overline{z}^l)}{\widehat{\lambda}_t^u}\right]\cdot\overline{V}^i(H-u)}{[p_t^h,1-p_t^h]\cdot\overline{V}^i(H)}\right)-1\Bigg]\left[\left(\frac{\sum_{j\neq u} x_t^j}{\sum_{j=1}^N x_t^j}\right)^{\gamma}\times\right. \\
&\times\left(\frac{\left[p_t^h\frac{f^u(\overline{z}^h)}{\widehat{\lambda}_t^u},(1-p_t^h)\frac{f^u(\overline{z}^l)}{\widehat{\lambda}_t^u}\right]\cdot\overline{V}^j(H-u)}{[p_t^h,1-p_t^h]\cdot\overline{V}^j(H)}\right)-1\Bigg](\widehat{\lambda}_t^u+p_t^h(1-p_t^h)(f^u(\overline{z}^h)-f^u(\overline{z}^l))^2)+H_t^u\left[\left(\frac{\sum_{j=1}^N x_t^j}{\sum_{j\neq u} x_t^j}\right)^{\gamma}\times\right. \\
&\times\left(\frac{\left[p_t^h\frac{g^u(\overline{z}^h)}{\widehat{\eta}_t^u},(1-p_t^h)\frac{g^u(\overline{z}^l)}{\widehat{\eta}_t^u}\right]\cdot\overline{V}^i(H+u)}{[p_t^h,1-p_t^h]\cdot\overline{V}^i(H)}\right)-1\Bigg]\left[\left(\frac{\sum_{j=1}^N x_t^j}{\sum_{j\neq u} x_t^j}\right)^{\gamma}\times\right. \\
&\times\left(\frac{\left[p_t^h\frac{g^u(\overline{z}^h)}{\widehat{\eta}_t^u},(1-p_t^h)\frac{g^u(\overline{z}^l)}{\widehat{\eta}_t^u}\right]\cdot\overline{V}^j(H+u)}{[p_t^h,1-p_t^h]\cdot\overline{V}^j(H)}\right)-1\Bigg](\widehat{\eta}_t^u+p_t^h(1-p_t^h)(g^u(\overline{z}^h)-g^u(\overline{z}^l))^2)\Bigg\}
\end{aligned}$$

Proof of Proposition 5

We assume without loss of generality that none of the N trees has yet undergone a disaster. The price of a zero coupon bond that expires at time T is:

$$\begin{aligned} P(t,T) &= \mathbb{E}\left[\left.\frac{\xi_T}{\xi_t}\right|\mathcal{F}_t^{H,S}\right] \\ &= e^{b(T-t)}\frac{1}{\left(\sum_{i=1}^N x_t^i\right)^{-\gamma}}\mathbb{E}\left[\left.\left(\sum_{i=1}^N x_T^i\right)^{-\gamma}\right|\mathcal{F}_t^{H,S}\right] \end{aligned}$$

where

$$b = -\delta - \mu_Y\gamma + \sigma_Y^2\frac{\gamma(\gamma+1)}{2}$$

But $\mathbb{E}[(\sum_{i=1}^N x_T^i)^{-\gamma}|\mathcal{F}_t^{H,S}]$ is immediately computed with the same methodology used to compute security prices. We finally obtain:

$$\begin{aligned} P(t,T) &= e^{b(T-t)}\left(\sum_{i=1}^N x_t^i\right)^{\gamma}[p_t, 1-p_t^h, 0_{\mathcal{N}-2}]\exp\left(-(T-t)\mathbf{A}^H\right)\mathbf{E} \\ \mathbf{E} &= \begin{bmatrix} \left(\sum_{i=1}^N \overline{x}_h^i\right)^{-\gamma}\overline{1}_2 \\ \left(\sum_{i=2}^N \overline{x}_h^i\right)^{-\gamma}\overline{1}_2 \\ \vdots \\ \left(\sum_{i=1}^{N-1} \overline{x}_h^i\right)^{-\gamma}\overline{1}_2 \\ \left(\sum_{i=3}^N \overline{x}_h^i\right)^{-\gamma}\overline{1}_2 \\ \vdots \\ (0)^{-\gamma}\overline{1}_2 \end{bmatrix} \end{aligned}$$

where $\exp(\cdot)$ denotes the matrix exponential operator and $\mathbf{A}^H$ is reported in (70)

A similar line of reasoning yields to the following expression for the price of a defaultable zero coupon bond, linked to the $i-$tree default event, with zero recovery value:

$$\begin{aligned} P^i(t,T) &= \frac{1}{\xi_t}\mathbb{E}\left[\xi_T\mathbf{1}(\tau^i > T)\right] \\ &= \mathbf{1}\left(\tau^i > t\right)e^{b(T-t)}\left(\sum_{i=1}^N x_t^i\right)^{\gamma}\left[p_t^h, 1-p_t^h, 0_{\mathcal{N}_D-2}\right]\exp\left(-(T-t)\mathbf{A}^{ZD}\right)\mathbf{E}^D \\ \mathbf{E}^D &= \begin{bmatrix} \left(\sum_{u=1}^N \overline{x}_h^u\right)^{-\gamma}\overline{1}_2 \\ \left(\overline{x}_h^i + \sum_{u\in\mathcal{S}^i(N-1,N-2)_1}\overline{x}_h^u\right)^{-\gamma}\overline{1}_2 \\ \vdots \\ \left(\overline{x}_h^i + \sum_{u\in\mathcal{S}^i(N-1,N-2)_{N-1}}\overline{x}_h^u\right)^{-\gamma}\overline{1}_2 \\ \left(\overline{x}_h^i + \sum_{u\in\mathcal{S}^i(N-1,N-3)_1}\overline{x}_h^u\right)^{-\gamma}\overline{1}_2 \\ \vdots \\ \left(\overline{x}_h^i\right)^{-\gamma}\overline{1}_2 \end{bmatrix}. \end{aligned}$$

where $\mathbf{A}^{ZD}$ corresponds to matrix $\mathbf{A}$ in (63) when $D := \{i\}$.

This concludes the proof.

Appendix B: A Simple Illustrative Calibration

In the examples discussed in the text, we use a stylized economy populated by 6 trees. In this Appendix, we discuss the heuristic procedure by which we identify parameter values for these trees. The transition between states of high and low intensity of disaster or recovery is driven by the business cycle. In particular the intensity of disaster is low (high) in 'good' states of the economy. In the absence of feed-backs between disasters and business cycle, Ribeiro and Veronesi (2002) estimate a quarterly probability of 0.0501 of switching from "Peak" to "Trough", and a probability of 0.2716 for the opposite transition. It follows that:

$$\exp\left(\begin{bmatrix} -k_h & k_h \\ k_l & -k_l \end{bmatrix} \frac{1}{4}\right) = \begin{bmatrix} 1-0.0501 & 0.0501 \\ 0.2716 & 1-0.2716 \end{bmatrix} \tag{93}$$

that is

$$\begin{bmatrix} -k_h & k_h \\ k_l & -k_l \end{bmatrix} = \begin{bmatrix} -0.2418 & 0.2418 \\ 1.3109 & -1.3109 \end{bmatrix} \tag{94}$$

To estimate disaster intensities in the two states of the world, we assume that the likelihood of disaster is assigned by an 'agency', similarly to credit-worthiness criteria. The 'ratings' for the trees are AA, A BBB, BB, B and CCC, respectively. Moody's historical 1-year average default probabilities (ADP) for these classes are 0.0001, 0.0004, 0.0029, 0.012, 0.0571 and 0.28 respectively. The historical standard deviations (SDP) for these probabilities are 0.00003, 0.00013, 0.0008, 0.0035, 0.016, 0.1. We calibrate parameters $f^i(\overline{z}^h)$ and $f^i(\overline{z}^l)$, $i = 1, \ldots, 6$ by matching the unconditional mean and variance of full information default rates implied by our model, namely:

$$\begin{aligned} 1 - \mathbb{E}\left[e^{-\int_0^1 \lambda_t^i dt}\right] &= \mathrm{ADP}^i \\ \mathbb{E}\left[e^{-\int_0^1 \lambda_t^i dt}\right](1 - \mathbb{E}\left[e^{-\int_0^1 \lambda_t^i dt}\right]) &= (\mathrm{SDP}^i)^2 \qquad i = 1, \ldots, 6 \end{aligned}$$

where

$$\mathbb{E}\left[e^{-\int_0^1 \lambda_t^i dt}\right] = [\pi^h, 1-\pi^h] \cdot \exp(I - \mathrm{diag}[f^i(\overline{z}^h), f^i(\overline{z}^l)]) \cdot \overline{1}_2 \tag{95}$$

and π^h denotes the stationary probability of a "Peak" state, i.e.

$$\pi^h = \frac{k_l}{k_h + k_l} \tag{96}$$

The following table reports calibrated intensities of default for each rating class:

	$f^i(\overline{z}^h)$	$f^i(\overline{z}^l)$
AA	0.00012	0.000065
A	0.00048	0.000020
BBB	0.0035	0.00029
BB	0.014	0.0041
B	0.0670	0.015
CCC	0.3670	0.150

We assume that trees with the lowest disaster intensities have the highest instantaneous chance to recover, once in disaster state. For simplicity, we don't estimate recovery intensities, but we simply assume: $g^i(\overline{z}^j) = 0.5 * f^{6-i+1}(\overline{z}^j)$, $i = 1, \ldots, 6$, $j = h, l$

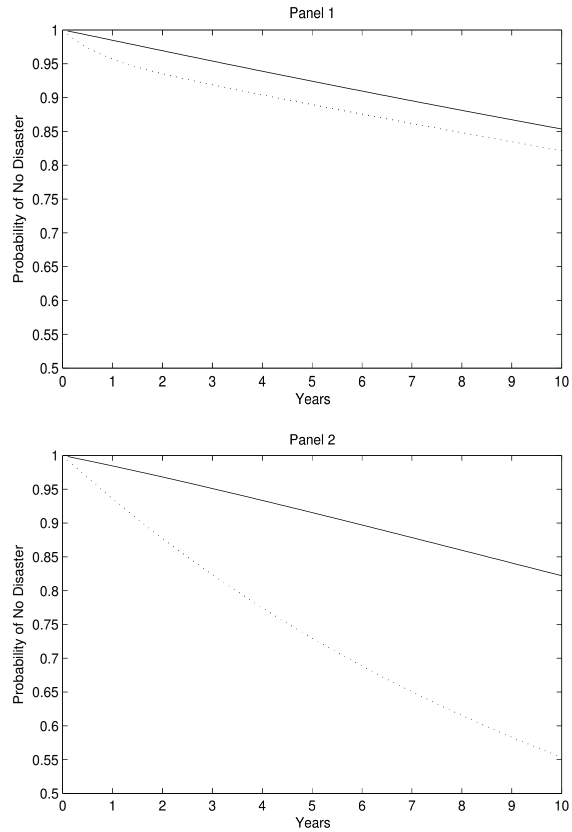

Figure 1. Full Information Survival Probabilities.
Term structure of survival probabilities for the tree with the highest disaster risk, when probabilities of regime switches of the state of the economy are assumed constant. *Panel 1* shows the survival probabilities conditional on a 'bad' (dotted line) and 'good' (solid line) state of the economy. In *Panel 2* the same quantities are displayed with a regime transition intensity 5 times smaller than in Panel 1, everything else being unchanged.

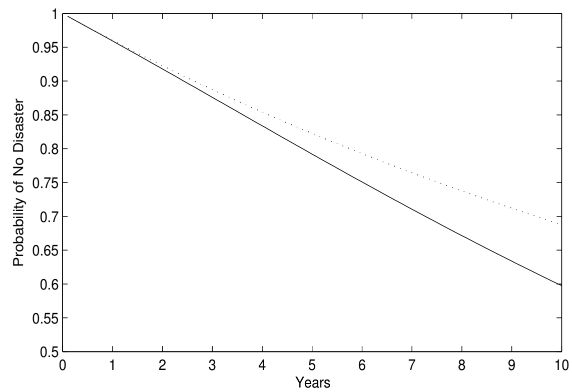

Figure 2. No-disaster probabilities.
Term structure of survival probabilities for the same tree of Figure 1, when a disaster in a different tree enhances the probability of an economic downturn (solid line). These survival probabilities are compared with those arising when the state of the economy is exogenous and it is not influenced by disasters of the economic sectors (dotted line).

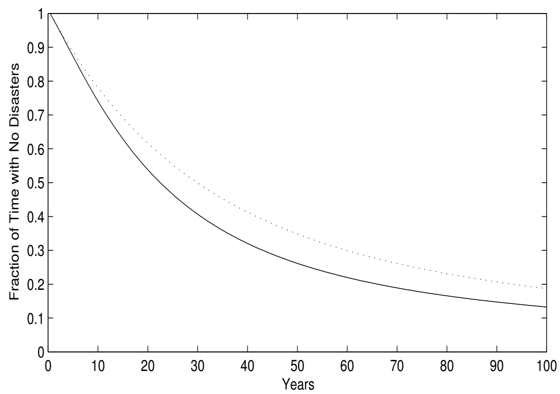

Figure 3. Fraction of time spent with no disaster.
Fractions of time that a simple economy composed of 5 trees spends with no disasters for any tree, plotted for different time horizons, when a disaster of one of the trees enhances the probability of an economic downturn (solid line). These fractions are compared with those arising when the state of the economy is exogenous and it is not influenced by disasters of the economic sectors (dotted line).

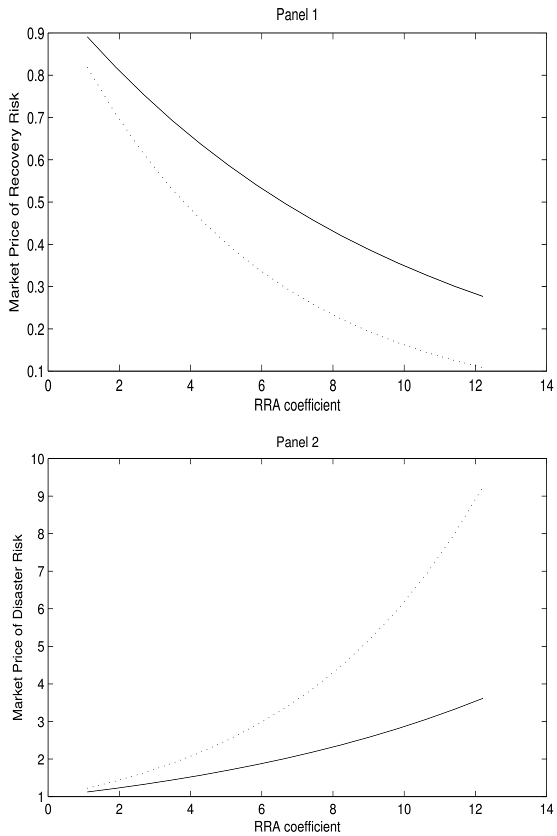

Figure 4. Market Price of Default Risk.
Panel 1: Equilibrium market price of recovery risk for a given tree plotted as a function of the Relative Risk Aversion coefficient, for two share values: $s_t^i = 1/6$ (dashed line) and $s_t^i = 1/10$ (solid line). There are 6 trees supplying the aggregate endowment and all remaining trees are not experiencing a disaster. *Panel 2*: Equilibrium market price of disaster risk for the same tree plotted as a function of the Relative Risk Aversion coefficient, for two share values: $s_t^i = 1/6$ (dashed line) and $s_t^i = 1/10$ (solid line). There are 6 trees supplying the aggregate endowment and all trees are not experiencing a disaster.

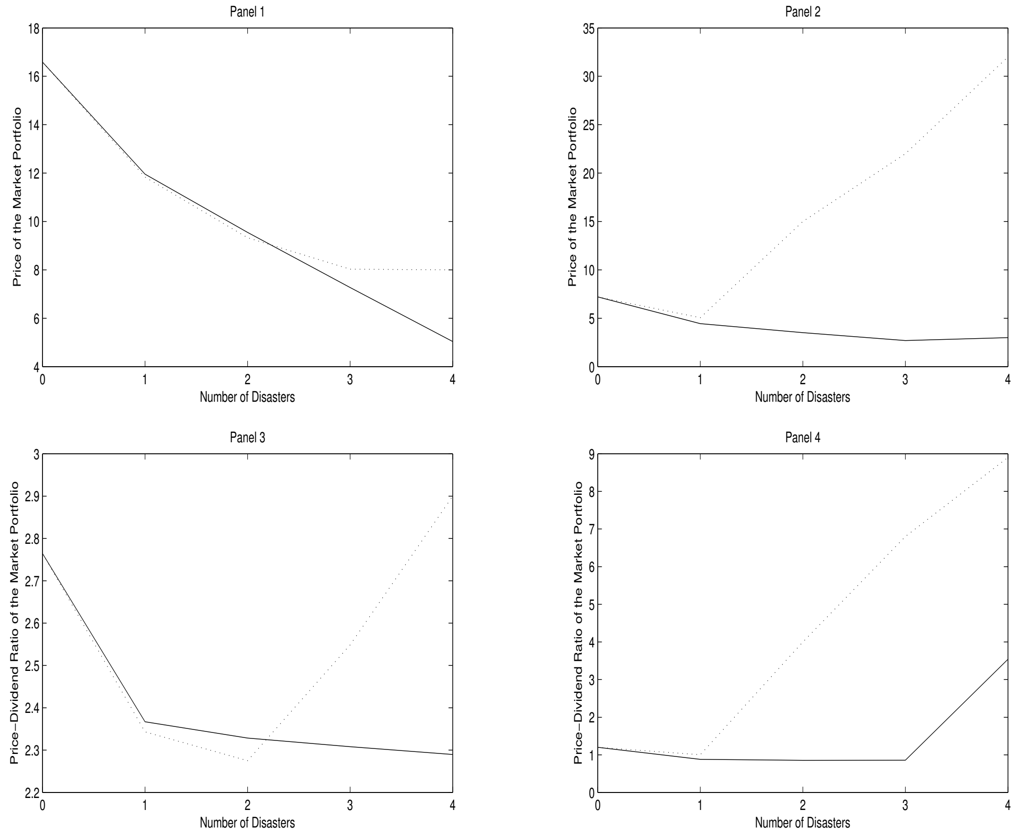

Figure 5. Price of the Market Portfolio.
Price of the market portfolio for a different number of occurred disasters when 6 trees supply the endowment and model parameters are calibrated as outlined in Appendix B. In Panel 1 we have assumed a RRA coefficient of 3, the solid line is plotted assuming that trees with highest disaster risk experience a disaster first, while the dashed line is plotted inverting the order of disasters. Panel 2 is as Panel 1, but the RRA coefficient is 7. Panel 3 plots the price-dividend ratio of the market portfolio with a RRA coefficient of 3, while Panel 4 plots the same quantity for a RRA coefficient of 7. Solid and dashed lines have the same interpretation as above.

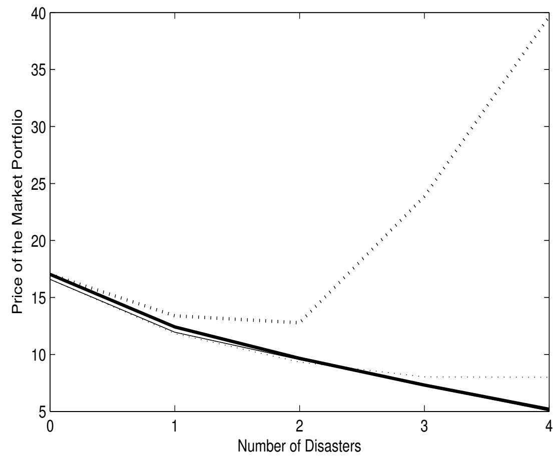

Figure 6. Price of the Market Portfolio with and without feed-backs.
In addition to those reported in Panel 1 of Figure 5, market prices have been plotted assuming that the probability of switching to a 'good' economic state, and viceversa, is independent of the disaster history of the economy (bold lines).

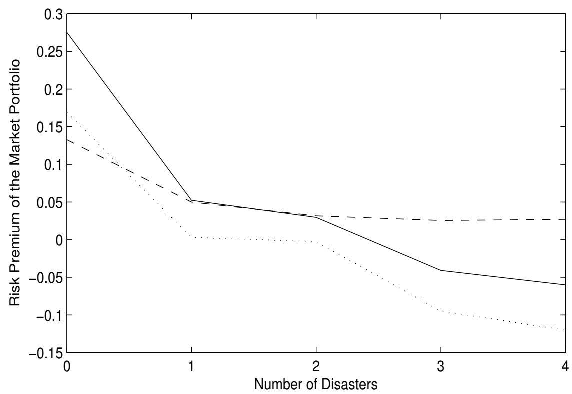

Figure 7. Market Risk Premium.
Equilibrium risk premium for the market portfolio plotted as a function of the number of disasters occurred in the economy (solid line). Its 'cash-flow beta' (dashed line) and 'valuation beta' (dotted line) components are also reported. 'Lowest rated' trees experience a disaster first.

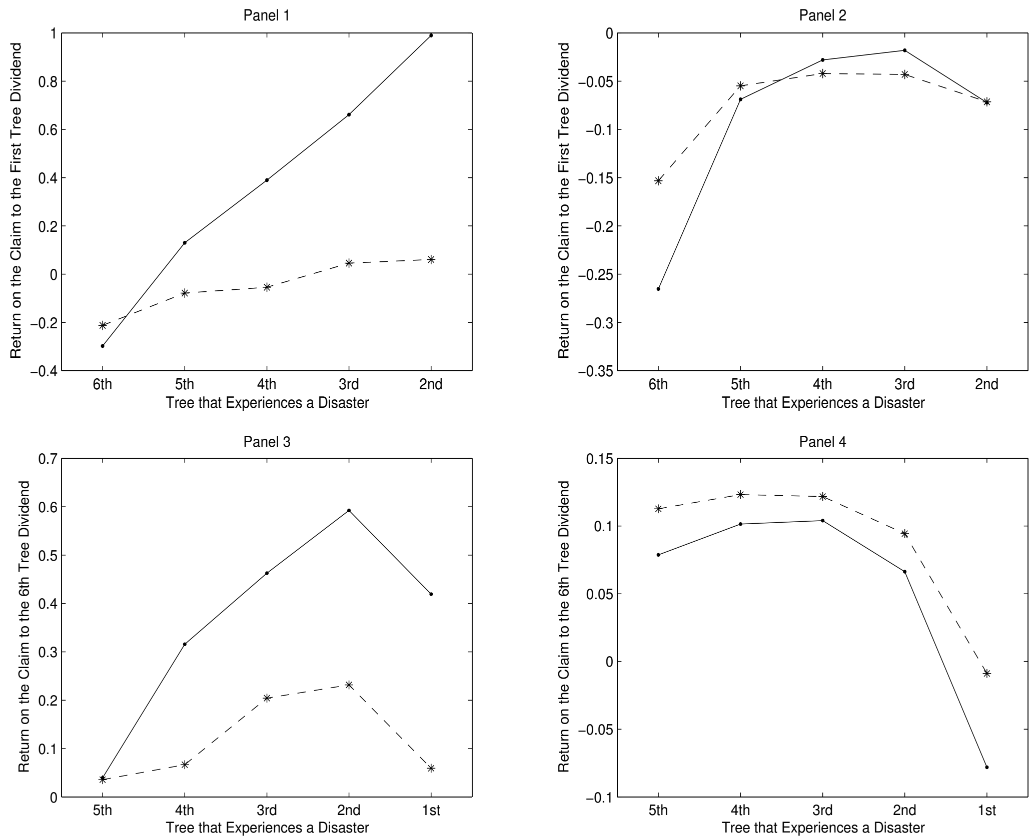

Figure 8. Returns of Individual Endowment Claims.
Panel 1 shows returns on the claim to the endowment with the lowest disaster risk, immediately following a disaster of a different tree. The x-axis reports the tree that has experienced a disaster. Initially there are 6 trees, and parameters are calibrated as described in Appendix B. Bullet points correspond to a scenario where the 'highest rated' endowment has a small market share (5%), whereas asterisks correspond to a market share of 40%. The RRA coefficient used is 7. Panel 2 reports the same exercise for a RRA coefficient of 3. Panel 3 reports the same quantities for returns on the claim to the endowment with highest disaster risk, for a RRA coefficient of 7. Panel 4 is as Panel 3, when the RRA coefficient is 3.

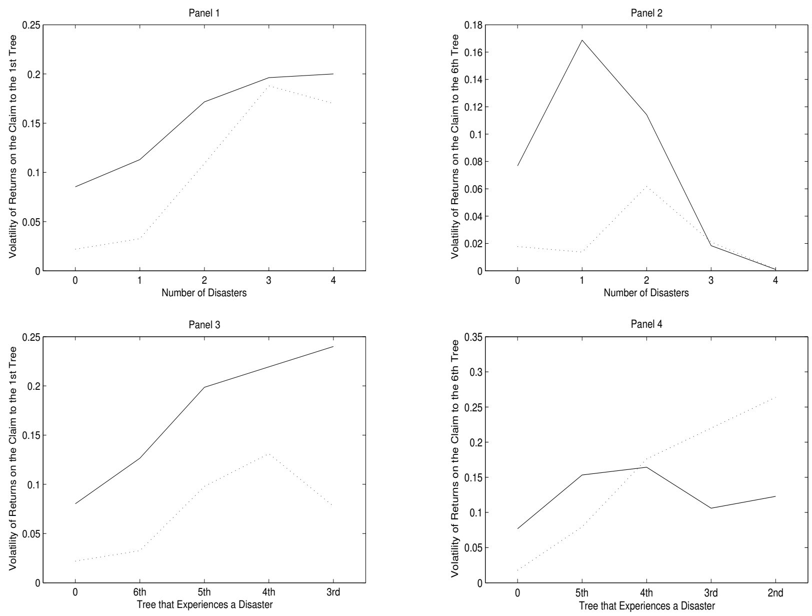

Figure 9. Instantaneous Volatilities of Asset Returns.
Equilibrium instantaneous volatilities for returns on the least risky (Panel 1 and 3) and most risky (Panel 2 and 4) endowment claims as a function of the number of disasters occurred (Panels 1 and 2), and of the tree that defaults next (Panels 3 and 4). In Panels 1 and 2, values corresponding to a higher number of disasters have been obtained assuming that riskier trees experince a disaster first. The initial number of trees is 6. Solid lines correspond to a small share (5%) for the evaluated endowment, while dotted lines correspond to a high (40%) market share.

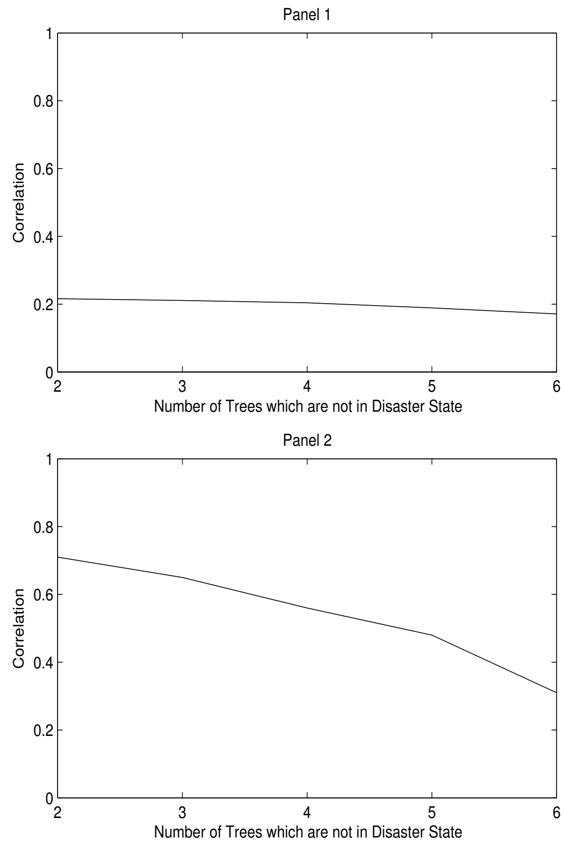

Figure 10. 'Volatility Leakage' Effect.
Panel 1 shows the correlation coefficient between instantaneous volatilities of asset returns for the tree with the highest and second to highest event risk. The correlation is plotted as a function of the number of trees which are not in disaster state. *Panel 2* shows the same quantity for the trees with the lowest and second to lowest disaster risk.

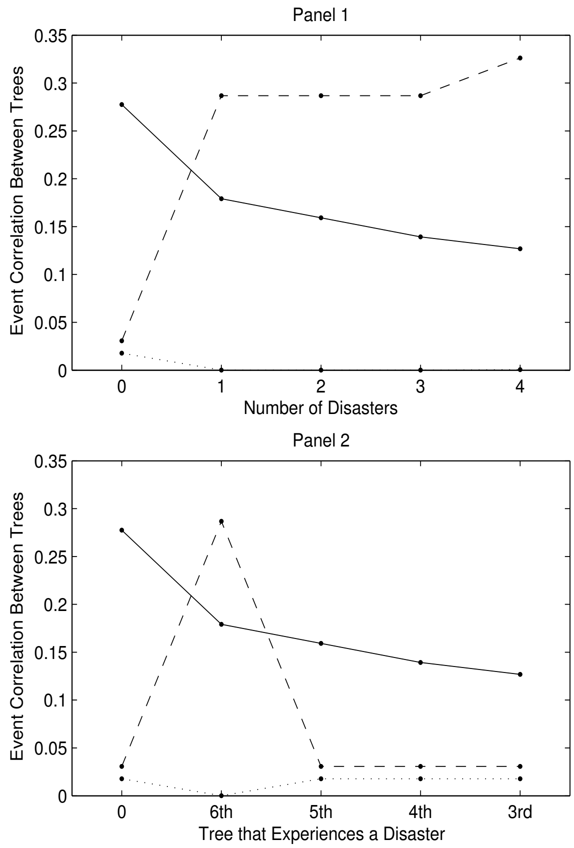

Figure 11. Behavior of Event Correlation.
Panel 1 shows the posterior 6-month event correlation of a disaster between two endowments with different disaster and recovery intensities, as a function of the number of disasters occurred in the remaining trees. *Panel 2* reports the same quantity after one disaster has occurred, as a function of the tree which experiences the disaster. The 6-th tree is the most risky, meaning that has the highest disaster risk and the lowest recovery intensity. Solid line connect partial information correlations, dotted (dashed) lines connect full information correlations conditional on a 'high' ('low') economic state. The number of currently supplying trees is 6.

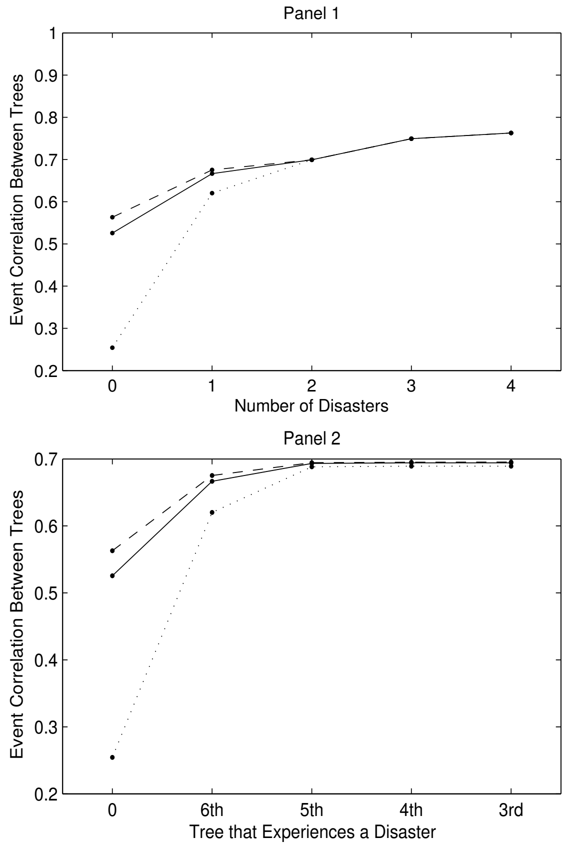

Figure 12. Behavior of Risk Neutral Event Correlation.
Panel 1 shows the posterior 6-month risk neutral event correlation of a disaster between two endowments with different disaster and recovery intensities, as a function of the number of disasters occurred in the remaining trees. *Panel 2* reports the same quantity after one disaster has occurred, as a function of the tree which experiences the disaster. The 6-th tree is the most risky, meaning that has the highest disaster risk and the lowest recovery intensity. Solid line connect partial information correlations, dotted (dashed) lines connect full information correlations conditional on a 'high' ('low') economic state. The number of currently supplying trees is 6.

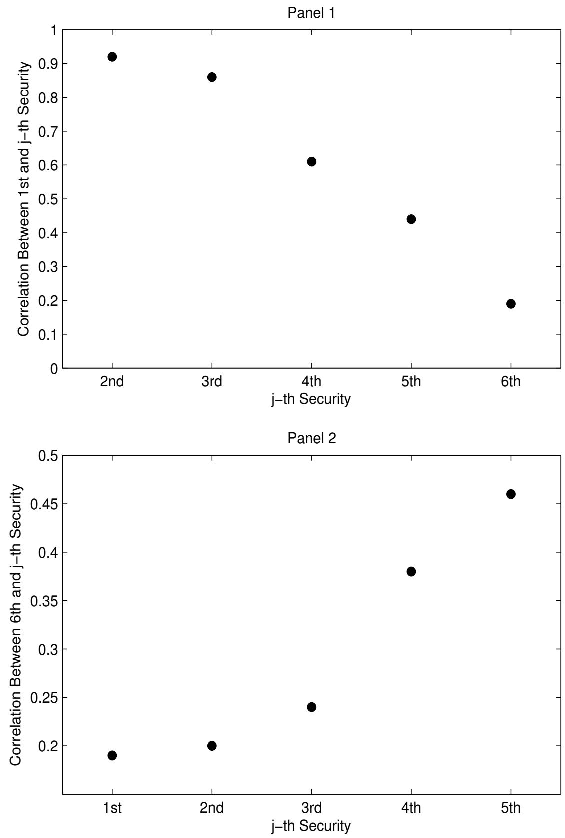

Figure 13. Instantaneous Correlation of Asset Returns.
Panel 1 shows the conditional correlation coefficient between returns on the security with the lowest event risk and the remaining securities, when 6 trees supply aggregate endowment and none has yet undergone a disaster. *Panel 2* shows the same quantity for returns on the security with the highest event risk. Each security supplies the same multiple x^i of unitary output Y.

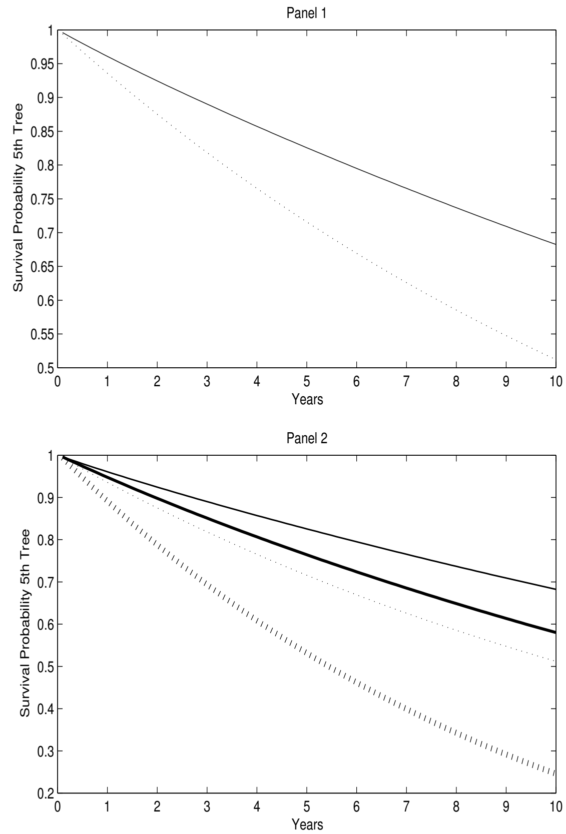

Figure 14. Contagion Effect for No-Disaster Probabilities.
Panel 1 plots the term structure of no disaster probabilities for the tree with the second to highest disaster risk. The solid line plots this term structure when 6 trees supply the aggregate endowment and parameters are calibrated as outlined in Appendix B. The dotted line plots the term structure after the disaster of a given tree (the third in terms of disaster risk), which is crucial for the economy, to the extent that after its disaster the probability of switching to a 'bad' economic state increases dramatically. *Panel 2* reports no-disaster probabilities of Panel 1, together with their risk neutral counterparts (bold lines).

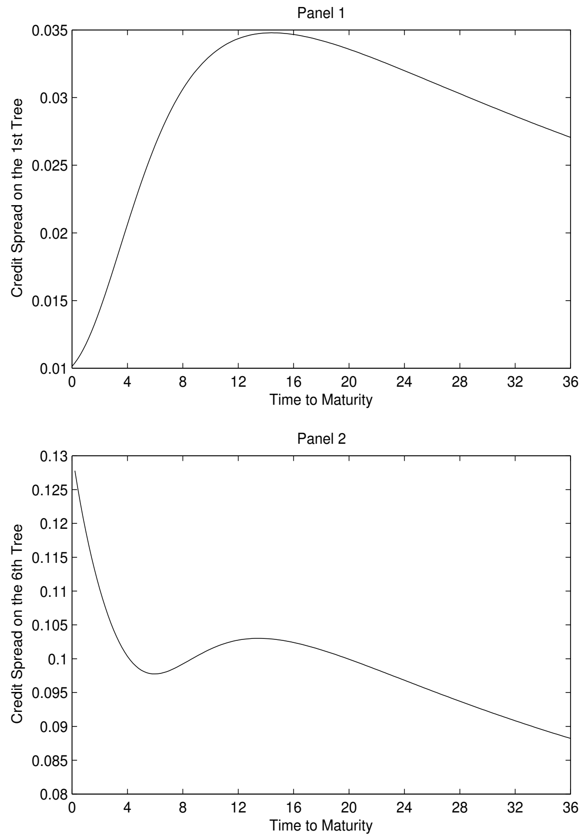

Figure 15. Term Structure of Equilibrium Credit-Spreads.
Term structure of credit spreads for the tree with the lowest event risk (first, *Panel 1*) and for the tree with the highest event risk (6th, *Panel 2*). The number of trees currently supplying the aggregate endowment is 6, none of which has yet undergone a disaster.

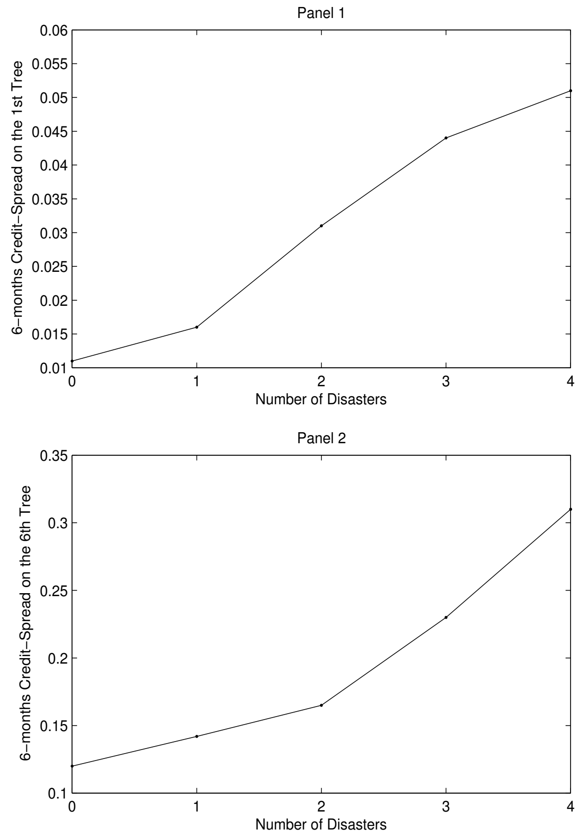

Figure 16. Behavior of Credit-Spreads wrt Market Diversification.
Credit spreads for the endowment with the lowest event risk (*Panel 1*) and for the tree with the highest event risk (*Panel 2*), plotted as a function of the number of disasters occurred, when initially there are 6 trees supplying aggregate consumption and the riskier endowments default first.

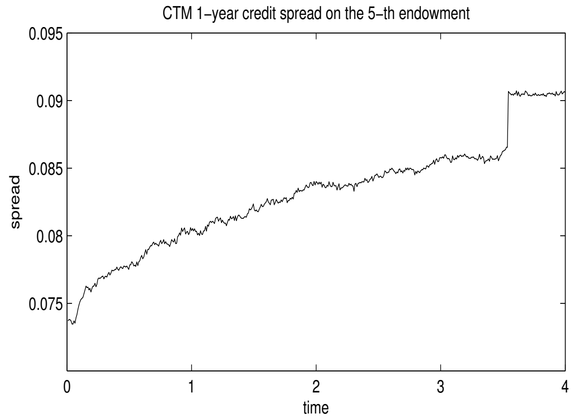

Figure 17. Evolution of Credit Spreads.
Simulated path of the equilibrium constant-maturity 1-year credit spread for the 5−th rated endowment. The number of trees currently supplying the aggregate endowment is 6