Asset Pricing in General Equilibrium with Constraints*

Georgy Chabakauri
London Business School
Institute of Finance and Accounting
Regent's Park
London NW1 4SA
United Kingdom
Tel: (44) 20 7000 8241
Fax: (44) 20 7000 8201
gchabakauri.phd2004@london.edu

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Abstract

We evaluate the impact of portfolio constraints on financial markets in a dynamic equilibrium pure exchange economy with one consumption good and heterogeneous investors. Despite numerous applications, portfolio constraints are notoriously difficult to incorporate into dynamic equilibrium analysis unless constrained investors are assumed to have logarithmic preferences. Our solution method yields new insights on the impact of constraints on stock prices without relying on this assumption. We compute the equilibrium when both investors have (identical for simplicity) CRRA preferences, one of them is unconstrained while the other faces an upper bound constraint on the proportion of wealth invested in stocks. We show that tighter constraints lead to higher price-dividend ratios and lower stock-return volatilities when the intertemporal elasticity of substitution (IES) is less than one, and lower price-dividend ratios and higher volatilities when IES is greater than one. Moreover, in the latter case the model generates countercyclical market prices of risk and stock return volatilities, procyclical price-dividend ratios, excess volatility and other patterns consistent with empirical findings. Finally, the baseline analysis is extended to study the impact of various portfolio constraints when investors disagree on mean dividend growth rates.

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1. Introduction

Portfolio constraints and market frictions have long been considered among key contributors towards understanding investor behavior and equilibrium asset prices. In particular, dynamic equilibrium models with heterogeneous investors facing portfolio constraints have extensively been employed by financial economists to confront a wide range of phenomena such as the equity premium puzzle, mispricing of redundant assets, role of arbitrageurs, impact of heterogeneous beliefs on asset prices, and stock comovements [e.g., among others, Detemple and Murthy (1997), Basak and Cuoco (1998), Basak and Croitoru (2000, 2006), Kogan, Makarov and Uppal (2007), Gallmeyer and Hollifield (2008), Pavlova and Rigobon (2008)]. However, tractable characterizations of equilibria are only obtained assuming that a constrained investor has logarithmic preferences which simplifies the analysis at the cost of assuming investor's myopia.[1] Despite recent developments in portfolio optimization, such as duality method of Cvitanic and Karatzas (1992), portfolio constraints are notoriously difficult to incorporate into general equilibrium analysis as well as portfolio choice when constrained investors have more general preferences inducing hedging demands.

The assumption of logarithmic preferences is not innocuous and impedes the evaluation of the impact of constraints on stock prices and stock return volatilities. Thus, in economic settings with two logarithmic investors and single consumption good [e.g., Detemple and Murthy (1997), Basak and Cuoco (1998), Basak and Croitoru (2000, 2006)] stock prices and hence stock return volatilities are unaffected by constraints since the income and substitution effects perfectly offset each other. When the constrained investor is logarithmic, the volatility effects of constraints have been studied in specific settings where the other (unconstrained) investor has different preferences [e.g., Gallmeyer and Hollifield (2008)], which requires further justification. To our best knowledge, this paper is the first to study the effect of different constraints on stock return volatility in a continuous-time economy without relying on the assumption of logarithmic investors. As a result, our solution method yields new insights on the impact of portfolio constraints on stock prices and, in particular, highlights the role of constraints in explaining empirically observed procyclical variation of price-dividend ratios and countercyclical variation of stock return volatilities (i.e., positive shocks to dividend growth rates lead to higher price-dividend ratios and lower stock return volatilities).

We solve for the equilibrium in a continuous-time pure exchange economy with one consumption good and two heterogeneous investors facing portfolio constraints. First, for general preferences and constraints we provide a characterization of interest rates and market prices of risk which highlight the role of constraints and risk sharing, and in specific economic settings

[1] The assumption that one investor has logarithmic preferences is also commonly made for tractability in models with unconstrained investors who differ in risk aversions. Thus, Dumas (1989) studies dynamic equilibrium in a production economy, where one investor has logarithmic while the other general CRRA preferences. Wang (1996) studies an exchange economy where one investor has logarithmic while the other square-root preferences. One notable exception is Bhamra and Uppal (2009), who study the effect of introducing non-redundant securities on the volatilities of asset returns in an exchange economy with CRRA investors not restricted to being logarithmic.

can explicitly be characterized in terms of empirically observable quantities such as stock returns and consumption volatilities. Based on these results, we specialize to settings with two CRRA investors one of whom is unconstrained while the other faces portfolio constraints. Specifically, we first derive the equilibrium when the constrained investor faces an upper bound on the proportion of wealth invested in stocks.[2] Then, we study the impact of short-sale constraints on equilibrium when investors have different beliefs about mean dividend growth. The methodological contribution of the paper is a solution method for the efficient computation of equilibria in economies with constraints. Specifically, we derive stock price-dividend ratios, stock return volatilities and other parameters in terms of wealth-consumption ratios that can be computed numerically via a simple iterative procedure with fast convergence.

At the first step of our analysis when we allow for general preferences, we demonstrate that the riskless rates and market prices of risk include new terms that capture the effects of constraints and risk sharing. In specific settings we obtain the expressions for interest rates and market prices of risk in terms of intuitive and empirically observable parameters such as stock return and consumptions volatilities. The tractability of our results allows to compare interest rates in constrained and unconstrained economies for a given allocation of consumption among investors and demonstrate that for various constraints interest rates will be lower in constrained economies whenever both investors have the same prudence-risk aversion ratios.

Using the insights from the case with general preferences we show that when investors have (identical for simplicity) CRRA preferences, one of them faces an upper bound on the proportion of wealth invested in stocks, and dividends follow a geometric Brownian motion, the interest rates and market prices of risk can explicitly be expressed in terms of marginal utility ratios, their volatilities and the volatilities of stock returns. We completely characterize the equilibrium by computing these volatilities numerically. While in models with two logarithmic investors price-dividend ratios and stock return volatilities are deterministic functions of time, in our setting these parameters depend on constrained investor's consumption share which evolves stochastically. The effect of constraints on price-dividend ratios and stock-return volatilities depends on the relative strength of classical income and substitution effects. When the intertemporal elasticity of substitution (IES) is less than one and hence the income effect dominates, price-dividend ratios increase while stock return volatilities decrease with tighter constraints, and vice versa

[2] Srinivas, Whitehouse and Yermo (2000) in a survey of pension fund regulations show that limits on both domestic and foreign equity holdings of pension funds are in place in a number of OECD countries such as Germany (30% on EU and 6% on non-EU equities), Switzerland (30% on domestic and 25% on foreign equities) and Japan (30% on domestic and 30% on foreign equities), among others. Moreover, our approach allows to study the impact of passive investors that hold a fixed fraction of their wealth in stocks, as in Chien, Cole and Lustig (2008). Samuelson and Zeckhouser (1988) document the popularity of this strategy using as an example the participants of popular TIAA/CREF retirement plan who choose a fraction of wealth to be invested in stocks and rarely change it due to "status quo bias", while Campbell (2006) points out that households may limit their participation in stock market and invest cautiously due to the lack of necessary skills. Important special case of our framework is stock market non-participation which in year 2002 accounted for 50% of U.S. households [e.g., Guvenen (2006)].

when IES is greater than one and the substitution effect is stronger.[3] Moreover, the effects of constraints are more pronounced in bad times, when dividends are hit by adverse shocks, than in good times.

To understand the intuition, we first evaluate the impact of portfolio constraints on investment opportunity sets and demonstrate that interest rates decrease while market prices of risk increase with tighter constraints, and that the effects of constraints are stronger in bad times. When the portfolio constraint binds, negative shocks to dividends shift the distribution of the aggregate wealth and consumption to the constrained investor since she is less exposed to stock market fluctuations. Thus, in bad times, when the constrained investor holds a significant fraction of aggregate wealth and consumption, the price-dividend ratio is approximately equal to her wealth-consumption ratio. With tighter constraints the investment opportunities of the constrained investor deteriorate since interest rates fall and she is unable to benefit from the increase in market prices of risk. As a result, her wealth-consumption ratio, and hence the price-dividend ratio, increases when the income effect dominates and decreases when the substitution effect dominates. The effect of constraints is weaker in good times since as the share of the unconstrained investor in aggregate wealth and consumption increases, all the economic parameters, including price-dividend ratios, converge to the parameters in the unconstrained economy.

Thus, when the substitution effect dominates, price-dividend ratios turn out to be procyclical (lower in bad times than in good times) while stock return volatilities exceed the volatility of dividends and are countercyclical (higher in bad times than in good times), consistently with the empirical evidence [e.g., Schwert (1989), Campbell and Cochrane (1999)]. Moreover, irrespective of investors' intertemporal elasticities of substitution, market prices of risk turn out to be countercyclical [e.g., Ferson and Harvey (1991)] since in bad times unconstrained investors lose wealth and require higher compensation for risk taking, causing market prices of risk to go up. We also study the survival of constrained investors in equilibrium and demonstrate that their impact on financial markets is gradually eliminated in the course of time but is significant even after one hundred years.

Finally, we extend our baseline analysis to economic settings with heterogeneous beliefs and multiple stocks. In both cases, for general preferences we derive expressions for interest rates and market prices of risk similar to those in the baseline model. In the case of heterogeneous beliefs we solve for equilibrium in a model where two investors have the same CRRA utilities and disagree on the dividend growth rate. The optimist is unconstrained while the pessimist faces a constraint on the proportion of wealth that can be held in short positions in stocks. We demonstrate that tighter short-sale constraints imply higher price-dividend ratios since they increase the constrained investor's demand for stocks. We also find that stock return volatility in the constrained economy can be both higher or lower than the volatility in an unconstrained

[3] When the investment opportunities worsen, the income effect induces investors to decrease consumption and save more while the substitution effect induces them to consume more and save less due to cheaper current consumption. For CRRA preferences with risk aversion γ, IES$=1/\gamma$, the income effect dominates for IES < 1 and the substitution effect dominates for IES > 1 while for IES $= 1$ both effects perfectly offset each other.

economy, depending on whether the latter is higher or lower than the volatility of dividend growth. This is because the short-sale constraints do not allow the investor to trade on her pessimism making her stockholding closer to what it would be in the case of homogeneous beliefs, and hence, the stock return volatility shifts towards volatility in an unconstrained homogeneous economy, given by the volatility of dividends.

Our solution method is based on a combination of the duality approach and dynamic programming. First, following Cvitanic and Karatzas (1992) we derive optimal consumptions in terms of the state price densities in equivalent unconstrained fictitious economies in which the interest rates and market prices of risk are adjusted to account for the difference in investors' behavior in constrained economies. Then, market clearing for consumption yields expressions for equilibrium parameters in terms of the adjustment parameters that solve a certain fixed point problem. Moreover, in our specific examples these adjustments to interest rates and market prices of risk can be derived in terms of instantaneous volatilities of stock returns and the ratios of marginal utilities of the two investors. Next, these volatilities and all the equilibrium parameters are explicitly characterized in terms of investors' wealth-consumption ratios that satisfy a system of quasilinear Hamilton-Jacobi-Bellman equations. We solve this system of equations numerically via a simple iterative procedure that requires solving a simple system of linear equations at each step.

There is a growing literature studying dynamic equilibria in continuous-time economies with heterogeneous investors and portfolio constraints assuming that constrained investors have logarithmic preferences. Basak and Cuoco (1998) consider a model in which one investor is unconstrained and guided by a general time-additive utility function while the other investor cannot invest in the stock market and has logarithmic preferences. They derive the riskless rates and market prices of risk in this economy and characterize all the equilibrium parameters explicitly when both investors are logarithmic. Detemple and Murthy (1997), Basak and Croitoru (2000, 2006) present equilibrium models with two logarithmic investors, heterogeneous beliefs and portfolio constraints. Hugonnier (2008) considers a similar model and shows that under restricted participation the stock prices implied by market clearing may contain a bubble and in the setting with multiple stocks the equilibrium might not be unique. In contrast to our work all the above papers do not find the impact of constraints on stock prices and their moments.

Jarrow (1980) studies the equilibrium effect of short-sale constraints in a one-period economy with mean-variance investors that have heterogeneous beliefs. Dumas and Maenhout (2002) develop an approach with two central planners for solving incomplete-market equilibrium with two CRRA investors. However, in their analysis the variance-covariance matrix of returns is taken as given and hence they do not study the impact of constraints on volatility. Kogan, Makarov and Uppal (2007) derive equilibrium parameters in an economy with borrowing constraints when one investor is logarithmic while the other has general CRRA utility and find that all the moments of asset returns are deterministic and stock return volatilities are unaffected by constraints. When little borrowing is permitted they numerically find interest rates and market prices of risk as functions of wealth distributions but do not consider the volatilities of stock returns. He and Krishnamurthy (2008) consider a model of intermediated asset pricing in which individual households are logarithmic and invest into stock only via an intermediary guided by CRRA utility. Wu (2008) studies the equilibrium in a setting with one unconstrained and one buy-and-hold CRRA investors. Gallmeyer and Hollifield (2008) study the asset pricing with short-sale constraints in the presence of heterogeneous beliefs when the pessimist and optimist have logarithmic and CRRA utilities respectively. They study equilibrium parameters by employing Monte-Carlo simulations and derive conditions for stock return volatilities to be larger or lower than in the unconstrained case assuming that investors have the same share of aggregate wealth at the initial date.

Bhamra (2007) analyzes the effect of liberalization on emerging markets' cost of capital in a model with two logarithmic investors, two stocks and one consumption good. Pavlova and Rigobon (2008) and Schornick (2009) consider models with constrained logarithmic investors and two consumption goods in international finance framework and derive various asset-pricing implications assuming that investors face preference shocks. Longstaff (2009) develops a two-asset economy where one of the assets is non-tradable for a certain period and logarithmic investors are heterogeneous in time discount parameter.

There are a number of papers that solve models with heterogeneous investors and portfolio constraints numerically in discrete time. Cuoco and He (2001) consider a model with general utilities and derive equilibrium asset prices in terms of stochastic weights of a representative investor's utility which are obtained numerically from a nonlinear system of equations. Guvenen (2006) solves numerically a model with restricted market participation when investors are guided by recursive utilities. Chien, Cole and Lustig (2008) also in a discrete-time framework consider a model with non-participants, passive and active investors guided by CRRA preferences, where passive investors hold fixed portfolios while active ones adjust them each period. Gomes and Michaelides (2008) study numerically the equilibrium with incomplete markets and investors subject to fixed cost of stock market participation and by calibration generate high equity premium and match observed market participation rate. Dumas and Lyasoff (2008) solve for equilibrium in various incomplete market settings in discrete time by employing binomial trees. These works do not study the impact of constraints on conditional stock return volatilities and do not provide expressions for equilibrium parameters in terms of observable quantities as we do in this paper by employing considerable flexibility of continuous-time methods.

The remainder of the paper is organized as follows. In Section 2, we derive interest rates and market prices of risk for general utility functions under the assumption that the dual optimization problem has a solution and discuss their properties. In Section 3 we illustrate our solution method by computing the equilibrium in a model with two CRRA investors where one investor is unconstrained while the other faces an upper bound on the fraction of wealth invested in stocks. Section 4 extends our baseline analysis to the settings with heterogeneous beliefs and multiple stocks. We also solve for equilibrium in a model with heterogeneous beliefs in which one of the investors faces short-sale constraints. Section 5 concludes, Appendix A provides the proofs and Appendix B provides further details for our numerical method.

2. General Equilibrium with Constraints

2.1. Economic Setup

We consider a continuous-time economy with one consumption good and an infinite horizon. The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, on which is defined a Brownian motion w. All the stochastic processes that appear in the paper are adapted to $\{\mathcal{F}_t, t \in [0, \infty)\}$, the augmented filtration generated by w.

The investors trade continuously in two securities, a riskless bond in zero net supply with instantaneous interest rate r and a stock in a positive net supply, normalized to one unit. The stock is a claim to an exogenous strictly positive stream of dividends δ following the dynamics

$$d\delta_t = \delta_t[\mu_{\delta t}dt + \sigma_{\delta t}dw_t], \tag{1}$$

where the dividend mean-return, μ_δ, and volatility, σ_δ, are stochastic processes. The dividend process (1) and its moments are assumed to be well-defined, without explicitly stating the regularity conditions. We consider equilibria in which bond prices, B, and stock prices, S, follow processes

$$dB_t = B_t r_t dt, \tag{2}$$

$$dS_t + \delta_t dt = S_t[\mu_t dt + \sigma_t dw_t], \tag{3}$$

where the interest rate r, the stock mean return μ and volatility σ are stochastic processes determined in equilibrium, and bond price at time 0 is normalized so that $B_0 = 1$.

There are two investors in the economy. Investor 1 is endowed with s units of stock and $-b$ units of bond, while investor 2 is endowed with $1 - s$ units of stock and b units of bond. The investors choose consumption, c_i, and an investment policy, $\{\alpha_i, \theta_i\}$, where α_i and θ_i denote the fractions of wealth invested in bonds and stocks, respectively, and hence, $\alpha_i + \theta_i = 1$. Investor i's wealth process W evolves as

$$dW_{it} = \Big[W_{it}\Big(r_t + \theta_{it}(\mu_t - r_t)\Big) - c_{it}\Big]dt + W_{it}\theta_{it}\sigma_t dw_t, \tag{4}$$

and her investment policies are subject to portfolio constraints

$$\theta_i \in \Theta_i, \quad i = 1, 2, \tag{5}$$

where $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$. We also assume that initial endowments of stocks are such that θ_i at time 0 belong to sets Θ_i. Thus, the financial market in our economy is incomplete due to the presence of portfolio constraints (5).

Each investor i ($i = 1, 2$) is guided by an expected utility over a stream of consumption c. In particular, her dynamic optimization is given by

$$\max_{c_i,\,\theta_i} E\Big[\int_0^\infty e^{-\rho t} u_i(c_{it})dt\Big], \tag{6}$$

subject to the budget constraint (4), no-bankruptcy constraint $W_t \geq 0$ and portfolio constraints (5), for some discount parameter $\rho > 0$. The utility functions $u_i(c)$ are assumed to be increasing, concave, three times continuously differentiable, satisfying Inada's conditions

$$\lim_{c\downarrow 0} u_i'(c) = \infty, \quad \lim_{c\uparrow\infty} u_i'(c) = 0, \quad i = 1, 2. \tag{7}$$

By A_{it} and P_{it} we denote absolute risk aversion and prudence parameters of investor i, given by

$$A_{it} = -\frac{u_i''(c)}{u_i'(c)}, \qquad P_{it} = -\frac{u_i'''(c)}{u_i''(c)}, \tag{8}$$

and assume that both are strictly positive for each investor.

Next, we define an *equilibrium* in this economy as a set of parameters $\{r_t, \mu_t, \sigma_t\}$ and of consumption and investment policies $\{c_{it}^*, \alpha_{it}^*, \theta_{it}^*\}_{i=1}^2$ such that consumption and investment policies solve dynamic optimization problem (6) for each investor, given price parameters $\{r_t, \mu_t, \sigma_t\}$, and consumption and financial markets clear, i.e.,

$$\begin{aligned} c_{1t}^* + c_{2t}^* &= \delta_t, \\ \alpha_{1t}^* W_{1t}^* + \alpha_{2t}^* W_{2t}^* &= 0, \\ \theta_{1t}^* W_{1t}^* + \theta_{2t}^* W_{2t}^* &= S_t, \end{aligned} \tag{9}$$

where W_{1t}^* and W_{2t}^* denote optimal wealths of investors 1 and 2 under optimal consumption and investment policies.

2.2. Characterization of Equilibrium

This Section characterizes the parameters of equilibria and studies their properties in economies with constrained investors. In particular, by employing the duality method of Karatzas and Cvitanic (1992), we recover expressions for interest rates and market prices of risk in equilibrium in terms of the parameters of equivalent fictitious unconstrained economies. These expressions are intuitive and highlight the impact of risk-sharing and attitude towards risk on equilibrium parameters. Moreover, they form a basis for an efficient methodology for computing equilibria, which we develop in Section 3.

We start by noting that since the market is incomplete due to the presence of portfolio constraints, a Pareto optimal allocation may not be feasible and hence, the ratio of the marginal utilities of consumption of the investors follows a stochastic process. This ratio can be interpreted as a stochastic weight in the construction of a representative-investor preferences in an equivalent economy, and serves as a state variable in terms of which the equilibrium can be characterized [e.g., Basak and Cuoco (1998), Cuoco and He (2001)]. By employing the methodology of Cvitanic and Karatzas (1992) we obtain optimal consumptions and then derive the equilibrium parameters from the market clearing conditions.[4] This approach is similar to the approach in Basak (2000), who characterizes the equilibrium in an economy where investors have heterogeneous beliefs, but in contrast to our work are unconstrained.

We start by characterizing optimal consumptions of constrained investors in a partial equilibrium in which the investment opportunities are taken as given, and then obtain the interest rate r, and the market price of risk κ, from the consumption clearing condition. For each investor i, following the approach of Cvitanic and Karatzas (1992), we characterize the optimality conditions for consumption by embedding investor i's partial equilibrium economy into an equivalent fictitious complete-market economy with bond and stock prices following dynamics with adjusted parameters:

$$dB_t = B_t[r_t + f(\nu_{it}^*)]dt, \tag{10}$$

$$dS_t + \delta_t dt = S_t[(\mu_t + \nu_{it}^* + f(\nu_{it}^*))dt + \sigma_t dw_t], \tag{11}$$

where $f_i(\nu)$ are *support functions* for the sets of portfolio constraints Θ_i, defined as

$$f_i(\nu) = \sup_{\theta \in \Theta_i} (-\nu\theta), \tag{12}$$

and ν_{1t}^* and ν_{2t}^* solve so called dual optimization problem, defined in Cvitanic and Karatzas (1992), and lie in the *effective domains* for support functions, given by

$$\Upsilon_i = \{\nu \in \mathbb{R} : f_i(\nu) < \infty\}. \tag{13}$$

It follows from the dynamics of bond and stock prices in fictitious economy (10)–(11) that the corresponding state prices ξ_{it} evolve as

$$d\xi_{it} = -\xi_{it}[r_{it}dt + \kappa_{it}dw_t], \tag{14}$$

where r_{it} and κ_{it} denote the adjusted riskless rate and market price of risk in fictitious economy i, given by

$$r_{it} = r_t + f_i(\nu_{it}^*), \qquad \kappa_{it} = \kappa_t + \frac{\nu_{it}^*}{\sigma_t}, \tag{15}$$

where $\kappa = (\mu - r)/\sigma$ is the *market price of risk* in the original constrained economy.

Throughout this Section we assume that the solutions to dual optimization problems exist and since the fictitious economies are complete, the marginal utilities of optimal consumption are given by

$$e^{-\rho t}u_i'(c_{it}^*) = \psi_i \xi_{it}, \quad i = 1, 2, \tag{16}$$

[4] Cuoco (1997) studies consumption-portfolio choice of constrained investors, mainly at a partial equilibrium level, and extends the results of Cvitanic and Karatzas to the case of more general utility functions and forms of market incompleteness. He derives a CAPM in an economy with portfolio constraints but does not study interest rates and other parameters of equilibrium.

for some constants $\psi_i > 0$. The first order conditions (16) and state prices (14) demonstrate that consumption and investment decisions of the constrained investor are equivalent to those of an unconstrained one, who faces interest rates and market prices of risk adjusted to account for the constraints. Moreover, optimality conditions in (16) allow to express consumptions c^*_{it} in terms of state prices in fictitious economies as follows:

$$c^*_{it} = I_i(\psi_i e^{\rho t}\xi_{it}), \quad i = 1, 2, \tag{17}$$

where $I_i(\cdot)$ denote inverse functions for marginal utilities $u'_i(\cdot)$.

The expressions for marginal utilities in (16) also imply that the *ratio of investors' marginal utilities*, defined as

$$\lambda_t = \frac{u'_1(c^*_{1t})}{u'_2(c^*_{2t})}, \tag{18}$$

is stochastic in equilibrium, and not a constant as in complete markets [e.g., Karatzas and Shreve (1998)] where consumption allocations are Pareto efficient. Basak and Cuoco (1998) and Cuoco and He (2001) demonstrate that the process λ serves as a convenient state variable in terms of which the equilibrium parameters can be expressed. Moreover, in an equivalent complete-market economy with a representative investor, parameter λ can be interpreted as a *stochastic weight* in the utility $u(c;\lambda)$ of a *representative investor*, given by

$$u(c;\lambda) = \max_{c_1+c_2=c} u_1(c_1) + \lambda u_2(c_2), \tag{19}$$

and follows a stochastic process

$$d\lambda_t = -\lambda_t[\mu_{\lambda t}dt + \sigma_{\lambda t}dw_t]. \tag{20}$$

The parameters μ_λ and σ_λ are determined in equilibrium and quantify the violation of Pareto-optimality in the economy.

Next we characterize the parameters of our economy in equilibrium in terms of adjustments ν^*_{it} from the market clearing in consumption. To determine the interest rate r and market price of risk κ we substitute optimal consumptions (17) into consumption clearing condition in (9), apply Itô's Lemma to both sides and recover equilibrium parameters by matching the drift and volatility terms. Similarly, from optimality conditions (16), by applying Itô's Lemma to equation (18) for λ_t and comparing the result with the process for λ_t in (20) we recover parameters μ_λ and σ_λ. The following Proposition summarizes the results.

Proposition 1. *If there exists an equilibrium, the riskless interest rate r, market price of risk κ, drift μ_λ and volatility σ_λ of weighting process λ that follows (20) are given by*

$$r_t = \bar{r}_t - \frac{A_t}{A_{1t}}f_1(\nu^*_{1t}) - \frac{A_t}{A_{2t}}f_2(\nu^*_{2t}) - \frac{A_t^3(P_{1t}+P_{2t})}{2A_{1t}^2A_{2t}^2}\sigma^2_{\lambda t} - \frac{A_t^3}{A_{1t}A_{2t}}\Big(\frac{P_{1t}}{A_{1t}} - \frac{P_{2t}}{A_{2t}}\Big)\delta_t\sigma_{\delta t}\sigma_{\lambda t}, \tag{21}$$

$$\kappa_t = \bar{\kappa}_t - \frac{A_t}{A_{1t}}\frac{\nu^*_{1t}}{\sigma_t} - \frac{A_t}{A_{2t}}\frac{\nu^*_{2t}}{\sigma_t}, \tag{22}$$

$$\mu_{\lambda t} \;=\; A_t\delta_t\sigma_{\delta t}\sigma_{\lambda t} + f_1(\nu^*_{1t}) - f_2(\nu^*_{2t}) - \frac{A_t}{A_{1t}}\sigma^2_{\lambda t}, \qquad \sigma_{\lambda t} = \frac{\nu^*_{1t} - \nu^*_{2t}}{\sigma_t}, \tag{23}$$

where $\bar{r}$ is the riskless rate and $\bar{\kappa}$ is the market price of risk in an unconstrained economy, given by

$$\bar{r}_t = \rho + A_t\delta_t\mu_{\delta t} - \frac{A_tP_t}{2}\delta^2_t\sigma^2_{\delta t}, \quad \bar{\kappa}_t = A_t\delta_t\sigma_{\delta t} \tag{24}$$

A_{it}, P_{it}, and A_t and P_t are absolute risk aversions and prudence parameters of investor i and a representative investor with utility (19), respectively.[5]

Optimal consumptions c^*_i, wealths W_i, stock S and optimal investment policies θ^*_i are given by

$$c^*_{it} \;=\; g_i(\delta_t, \lambda_t), \tag{25}$$

$$W^*_{it} \;=\; \frac{1}{\xi_{it}}E_t\Big[\int_0^\infty \xi_{is}c^*_{is}ds\Big], \tag{26}$$

$$S_t \;=\; W^*_{1t} + W^*_{2t}, \tag{27}$$

$$\theta^*_{it} \;=\; \frac{1}{\sigma_t}\Big(W^*_{it}\Big(\kappa_t + \frac{\nu^*_{it}}{\sigma_t}\Big) + \frac{\phi_{it}}{\xi_{it}}\Big), \tag{28}$$

where functions $g_i(\delta_t, \lambda_t)$ are such that c^*_{1t} and c^*_{2t} satisfy consumption clearing in (9) and equation (18) for process λ, state prices ξ_{it} follow processes (14) and ϕ_i are such that

$$M_{it} \equiv E_t\Big[\int_0^\infty \xi_{is}c^*_{is}ds\Big] = M_{i0} + \int_0^t \phi_{is}dw_s.$$

Initial value λ_0 is such that the budget constraints at time 0 are satisfied:

$$s_iS_0 + b_i = W^*_{i0}, \tag{29}$$

where $s_1 = s$, $s_2 = 1 - s$, $b_1 = -b$ and $b_2 = b$. Moreover, adjustments ν^*_{it} satisfy complementary slackness condition

$$f_i(\nu^*_{it}) + \theta^*_{it}\nu^*_{it} = 0. \tag{30}$$

Proposition 1 provides the characterization of equilibrium parameters in terms of adjustments ν^*_i in fictitious economy. Expression (21) decomposes interest rates r into groups of terms that separate the effects of constraints and the inefficiency of risk sharing. The first term in (21) is the riskless rate in the unconstrained economy with the representative investor. The next two terms capture the effect of binding constraints on interest rates and tend to increase or decrease

[5]As demonstrated in Basak (2000), the risk aversion, A, and prudence, P, of the representative investor can be obtained from the following expressions:

$$\frac{1}{A_t} = \frac{1}{A_{1t}} + \frac{1}{A_{2t}}, \qquad \frac{P_t}{A^2_t} = \frac{P_{1t}}{A^2_{1t}} + \frac{P_{2t}}{A^2_{2t}}.$$

them depending on the signs of support functions $f_i(\nu)$. In particular, these terms are positive in economic settings with binding portfolio constraints when investors buy more bonds. This is due to the fact that the investors behave as if their subjective interest rates r_{it} in their fictitious economy were higher than in the real one, and hence positive adjustments $f_i(\nu_i^*)$. Finally, the last two terms in expression (21) capture the effect of risk sharing, quantified by volatility σ_λ. The weight λ acts as a state variable that gives rise to specific hedging demands that can push interest rates in either direction.

Similarly, the expression (22) for the market price of risk is comprised of the market price of risk in an unconstrained economy [first term in (22)] and the effects of constraints [second and third terms in (22)]. Expressions for the drift μ_λ and volatility σ_λ parameters of the stochastic weighting process λ in (23) demonstrate that this process, in general, is no longer a local martingale as in works assuming logarithmic constrained investor [e.g., Basak and Cuoco (1998), Gallmeyer and Hollifield (2008), Pavlova and Rigobon (2008)]. Finally we observe that optimal consumptions, wealths, stock prices and investments can be obtained from expressions (25) – (28) when the parameters of equilibrium, and hence all state prices, are known.

The results in Proposition 1 can also be used to compute the equilibrium parameters numerically. On one hand, Proposition 1 expresses equilibrium parameters and investment policies in terms of adjustments ν_i^*, and on the other, the adjustments can be obtained from the complementary slackness condition (30). Thus, finding the adjustments becomes essentially a fixed point problem, which can potentially be solved by the method of successive iterations. Moreover, as demonstrated in Huang and Pages (1992), under certain conditions optimal wealths (26) satisfy linear PDEs with coefficients determined by equilibrium parameters while optimal policies (28) can be expressed in terms of derivatives of wealths W_i^*. Hence, the adjustments can be expressed in terms of derivatives of W_{it} from conditions (30) and substituted back into the PDEs for optimal wealths. Thus, the characterization of equilibrium reduces to solving a system of quasilinear PDEs which, as we demonstrate in Section 3, can efficiently be solved numerically for specific constraints.

2.3. Further Properties of Equilibrium

We here explore the implications of Proposition 1 by noting that in various economic settings the signs of adjustments ν_i^* and support functions $f_i(\nu)$ can easily be determined explicitly from the definitions of support functions and effective domains in (12) and (13). Moreover, the interest rates r and market prices of risk κ can be expressed in terms of empirically observed quantities, such as stock and consumption volatilities, thus providing empirical implications of the model.

Table 1 presents the effective domains and the signs of the support functions for plausible constraints and allows to analyze their effect on equilibrium parameters. For example, when investors face constraints on the proportion of wealth invested in stocks [case (d) in Table 1] the results in Proposition 1 and Table 1 imply that these constraints tend to decrease the interest

Table 1
Effective Domains and Support Functions

Case	Constraint	Υ	$f(\nu)$
(a)	$\theta \in \mathbb{R}$	0	0
(b)	$\theta = 0$	$\mathbb{R}$	0
(c)	$\underline{\theta} \leq \theta \leq \overline{\theta},\ \underline{\theta} \leq 0$	$\mathbb{R}$	$+$
(d)	$\theta \leq \overline{\theta},\ \overline{\theta} > 0$	$\nu \leq 0$	$+$
(e)	$\theta \geq \underline{\theta},\ \underline{\theta} < 0$	$\nu \geq 0$	$+$
(f)	$\theta \geq \underline{\theta},\ \underline{\theta} > 0$	$\nu \geq 0$	$-$

rates and increase the market prices of risk relative to an unconstrained model if stock volatility σ is strictly positive. Hence, these constraints work in the right direction for explaining the equity premium puzzle [e.g., Mehra and Prescott (1985)]. The overall effect of constraints on interest rates is convoluted by the risk sharing captured by the last two terms in the expression for interest rates (21). The following Corollary to Proposition 1 establishes simple sufficient conditions under which the interest rate r will be lower than the interest rate $\bar{r}$ in a representative-investor unconstrained economy.

Corollary 1. *If the utility functions and the allocation of consumption are such that $P_1/A_1 = P_2/A_2$ and the sets of portfolio constraints have positive support functions $f_i(\nu)$ then the interest rate in a constrained economy, r, is lower than in an unconstrained one, $\bar{r}$, and the following upper bound for rate r holds:*

$$r_t \leq \bar{r}_t - \frac{A_t^3(P_{1t} + P_{2t})}{2A_{1t}^2 A_{2t}^2}\sigma_\lambda^2. \tag{31}$$

The Corollary demonstrates that the inability to share risks contributes to the decrease of interest rates by creating hedging needs against fluctuating ratios of marginal utilities λ. The condition that investors have the same prudence-risk aversion ratio is in particular satisfied when both investors have identical HARA preferences.[6] In the case of two logarithmic investors when one of them is unconstrained the result in Corollary 1 has also been pointed out in the literature [e.g., Basak and Cuoco (1998)].

Conveniently, in various economic settings interest rates and market price of risk can be expressed only in terms of the parameters of utility functions and empirically observed parameters. For example, when investor 1 is unconstrained and investor 2 faces a constraint allowing her to invest in stock no more than a certain fraction of wealth [case (d) of Table 1], it can be observed

[6]For HARA utility function absolute risk aversion is given by $-u''(c)/u'(c) = \gamma/(\gamma_0 + c)$. Differentiating both sides of this expression and then dividing by $-u''(c)/u'(c)$ we obtain that $P_i/A_i = 1 + \gamma$, and hence, the prudence-risk aversion ratio is the same for both investors.

that parameters r and κ are given by:

$$r_t = \bar{r}_t - \frac{A_t}{A_{2t}}\bar{\theta}\sigma_t\sigma_{\lambda t} - \frac{A_t^3(P_{1t}+P_{2t})}{2A_{1t}^2A_{2t}^2}\sigma_{\lambda t}^2 - \frac{A_t^3}{A_{1t}A_{2t}}\Big(\frac{P_{1t}}{A_{1t}} - \frac{P_{2t}}{A_{2t}}\Big)\delta_t\sigma_{\delta t}\sigma_{\lambda t}, \quad \kappa_t = \bar{\kappa}_t + \frac{A_t}{A_{2t}}\sigma_{\lambda t}, \tag{32}$$

where stock return volatility σ can easily be obtained from the data, while the weighting process volatility σ_λ can be obtained in terms of utility parameters and the parameters of the consumption processes for each investor. In particular, assuming that consumption processes c_i for each investor follow Itô's processes

$$dc_{it} = c_{it}[\mu_{c_it}dt + \sigma_{c_it}dw_t], \tag{33}$$

applying Itô's Lemma to the definition of weighting process λ in (18) we find that

$$\sigma_{\lambda t} = A_{1t}c_{1t}\sigma_{c_1t} - A_{2t}c_{2t}\sigma_{c_2t}. \tag{34}$$

In specific frameworks the volatilities of consumption growth can potentially be estimated from the data. In particular, for the model with restricted participation ($\bar{\theta} = 0$) Malloy, Moskowitz and Vissing-Jorgensen (2009) estimate consumption volatilities of stock market participants and non-participants to be 3.6% and 1.4% respectively, while Mankiw and Zeldes (1991) and Guvenen (2006) show that the share of consumption of non-participants in aggregate consumption is 0.68. As a result, the expressions for r and κ in (32) can potentially be used for identifying the parameters of the utility functions of investors as well as for quantifying the impact of risk sharing inefficiencies on the interest rates and market prices of risk.

3. Equilibrium with Proportional Constraints

This Section applies the results of Section 2 to compute and analyze the equilibrium in a specific economic setting in which investor 1 is unconstrained while investor 2 faces a constraint allowing her to invest in stock no more than a certain fraction of wealth. For simplicity we assume that dividends follow a geometric Brownian motion and both investors have identical CRRA preferences. Using the results of Section 2, in Section 3.1 we present a simple solution method for finding an equilibrium in this economy, and in Section 3.2 we study the impact of constraints on the equilibrium. In our setting with fully rational investors we also study the survival of constrained investors in the long run and demonstrate that it takes a long time to eliminate their impact on financial markets.

3.1. Characterization and Computation of Equilibrium

In this Section we present a solution method which allows to compute the equilibrium in an efficient way. This method does not rely on a widely used assumption of a logarithmic constrained investor [e.g., Detemple and Murthy (1997), Basak and Cuoco (1998), Basak and Croitoru (2000, 2006), Kogan, Makarov and Uppal (2003), Bhamra (2007), Gallmeyer and Hollifield (2008), Hugonnier (2008), Pavlova and Rigobon (2008), Schornick (2009)] which allows to derive the adjustments ν_i^* in fictitious economy explicitly at the cost of investor's myopia inherent in logarithmic preferences. In discrete time, Cuoco and He (2001), Guvenen (2006), Chien, Cole and Lustig (2008) and Gomes and Michaelides (2008) study the models with constrained heterogeneous investors numerically without assuming that constrained investor is logarithmic. In contrast to these works, in settings with two CRRA investors the flexibility of continuous-time analysis allows us to recover tractable expressions for interest rates and market prices of risk and to find new insights on the impact of constraints on price-dividend ratios and stock return volatilities.

Finding an equivalent unconstrained economy is a challenging problem which so far has only been solved for logarithmic investors [e.g., Cvitanic and Karatzas (1992), Karatzas and Shreve (1998)] or CRRA investors but assuming constant investment opportunity sets [e.g., Tepla (2000)]. We tackle this problem by first expressing the parameters of the fictitious economy in terms of the stochastic weighting process λ, and the volatilities of λ and stock returns, which then are obtained in terms of the investors' wealth-consumption ratios satisfying Hamilton-Jacobi-Bellman equations. Even though in equilibrium the coefficients of HJB equations themselves depend on the sensitivities of wealth-consumption ratios with respect to parameter λ, we demonstrate that the time-independent solutions can easily be obtained via an iterative procedure that at each step requires solving a simple system of linear algebraic equations.

Throughout Section 3 we assume for simplicity that dividends follow a geometric Brownian motion

$$d\delta_t = \delta_t[\mu_\delta dt + \sigma_\delta dw_t], \tag{35}$$

both investors have CRRA utilities with relative risk aversion parameter γ, given by[7]

$$u_i(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}, \quad i = 1, 2, \tag{36}$$

and solve optimization problem in (6) subject to budget constraint (4), no-bankruptcy constraint $W_t \geq 0$, and portfolio constraint $\theta \leq \bar{\theta}$ for investor 2, while investor 1 is unconstrained. By $J_i(W_t, \lambda_t, t)$ we denote the indirect utility function of investor i.

For convenience, we solve the optimization problem of constrained investor 2 in an equivalent fictitious unconstrained economy in which she maximizes objective function (6) subject to budget constraint

$$dW_{2t} = \Big[W_{2t}\Big(r_t + f_2(\nu_{2t}^*) + \theta_{2t}(\mu_t - r_t + \nu_{2t}^*)\Big) - c_{2t}\Big]dt + W_{2t}\theta_{2t}\sigma_t dw_t, \tag{37}$$

where ν_{2t}^* and $f_2(\nu_{2t}^*)$ are adjustments to stock mean returns and riskless rates respectively. By applying dynamic programming we find that the indirect utility functions should satisfy the

[7]The assumption that investors have identical risk aversions is made for simplicity. More general case can be considered along the same lines.

following HJB equations:

$$
\begin{aligned}
0 = \max_{c_i,\theta_i} \quad & \Big\{ e^{-\rho t}\frac{c_{it}^{1-\gamma}}{1-\gamma} + \frac{\partial J_{it}}{\partial t} + \Big[W_{it}\Big(r_t + f_i(\nu_{it}^*) + \theta_{it}(\mu_t - r_t + \nu_{it}^*) \Big) - c_{it} \Big] \frac{\partial J_{it}}{\partial W_{it}} \\
& - \lambda_t \mu_{\lambda t}\frac{\partial J_{it}}{\partial \lambda_t} + \frac{1}{2}\Big[W_{it}^2 \theta_{it}^2 \sigma_t^2 \frac{\partial^2 J_{it}}{\partial W_{it}^2} - 2W_{it}\theta_{it}\lambda_t\sigma_t\sigma_{\lambda t}\frac{\partial^2 J_{it}}{\partial W_{it}\partial \lambda_t} + \lambda_t^2\sigma_{\lambda t}^2 \frac{\partial^2 J_{it}}{\partial \lambda_t^2} \Big] \Big\},
\end{aligned}
\tag{38}
$$

with transversality condition $E_t[J_{iT}] \to 0$ as $T \to \infty$, which guarantees the convergence of the integral in investors' optimization (6). We next obtain expressions for ν_i^* and $f_i(\nu_i^*)$ without solving the dual problem by noting that since investor 1 is unconstrained $\nu_1^* = 0$ [case (a) in Table 1] while ν_2^* can be obtained from equilibrium expression for $\sigma_{\lambda t}$ in (23), and hence,

$$
\nu_{1t}^* = 0, \qquad f_1(\nu_{1t}^*) = 0, \qquad \nu_{2t}^* = -\sigma_t\sigma_{\lambda t}, \qquad f_2(\nu_{2t}^*) = \bar{\theta}\sigma_t\sigma_{\lambda t}. \tag{39}
$$

The HJB equations in (38) are standard except for the fact that the equation for investor 2 is in terms of parameters of fictitious economy, which allows to formulate her problem as an unconstrained one. We conjecture that the indirect utility functions are given by

$$
J_i(W_i, \lambda, t) = e^{-\rho t}\frac{W_i^{1-\gamma}}{1-\gamma} H_i(\lambda, t)^\gamma, \quad i = 1, 2. \tag{40}
$$

Then, from the first order conditions with respect to consumption we obtain

$$
c_{it}^* = \frac{W_{it}}{H_{it}}, \quad i = 1, 2, \tag{41}
$$

where H_{it} is a shorthand notation for $H_i(\lambda, t)$, and hence, functions H_{it} can be interpreted as the *wealth-consumption ratio* of investor i. By substituting indirect utility functions (40) into HJB equations it can be verified that wealth-consumption ratios satisfy the following PDEs:

$$
\frac{\partial H_{it}}{\partial t} + \frac{\lambda_t^2\sigma_{\lambda t}^2}{2}\frac{\partial^2 H_{it}}{\partial \lambda_t^2} - \lambda_t\Big(\mu_{\lambda t} + \frac{1-\gamma}{\gamma}\kappa_{it}\sigma_{\lambda t}\Big)\frac{\partial H_{it}}{\partial \lambda_t} + \Big(\frac{1-\gamma}{2\gamma}\kappa_{it}^2 + (1-\gamma)r_{it} - \rho\Big)\frac{H_{it}}{\gamma} + 1 = 0, \quad i = 1, 2, \tag{42}
$$

where r_{it} and κ_{it} denote riskless rate and price of risk in a fictitious economy and are defined in (15) in terms of adjustments given in (39). Moreover, optimal investment policies for investors 1 and 2 are given by

$$
\theta_{it} = \frac{1}{\gamma\sigma_t}\Big(\kappa_{it} - \gamma\sigma_{\lambda t}\frac{\partial H_{it}}{\partial \lambda_t}\frac{\lambda_t}{H_{it}}\Big), \quad i = 1, 2. \tag{43}
$$

Since the horizon is infinite we will look for *time-independent* and bounded solutions of equations (42). Moreover, throughout this Section we assume that $\bar{\theta} \le 1$. We note that if investor 2 faces borrowing constraint, i.e. $\bar{\theta} \ge 1$, the equilibrium coincides with the equilibrium in an unconstrained economy in which the investors, being identical, optimally choose $\theta_{it}^* = 1$.

Conveniently, since the fictitious economy is complete, the equations for wealth-consumption ratios in (42) are linear if volatilities σ and σ_λ are known. However, in equilibrium these volatilities themselves depend on wealth-consumption ratios H_i. The stock return volatility σ can be obtained by applying Itô's Lemma to stock price $S_t = R_t\delta_t$, where R_t is a shorthand notation for the *stock price-dividend ratio* which can be expressed in terms of wealth-consumption ratios from the market clearing conditions in (9). Furthermore, the volatility σ_λ can be obtained from the complementary slackness condition (30). The following Proposition 2 summarizes our results and provides a characterization of equilibrium in terms of wealth-consumption ratios.

Proposition 2. *If there exists an equilibrium, the riskless interest rate r, market price of risk κ and drift μ_λ of weighting process λ that follows (20) are given by*

$$\begin{aligned} r_t &= \bar{r} - \frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\bar{\theta}\sigma_t\sigma_{\lambda t} - \frac{1+\gamma}{2\gamma}\frac{\lambda_t^{1/\gamma}}{(1+\lambda_t^{1/\gamma})^2}\sigma_{\lambda t}^2, && (44)\\ \kappa_t &= \bar{\kappa} + \frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\sigma_{\lambda t}, && (45)\\ \mu_{\lambda t} &= \gamma\sigma_\delta\sigma_{\lambda t} - \bar{\theta}\sigma_t\sigma_{\lambda t} - \frac{1}{1+\lambda_t^{1/\gamma}}\sigma_{\lambda t}^2, && (46)\end{aligned}$$

where $\bar{r}$ is the riskless rate and $\bar{\kappa}$ is the market price of risk in an unconstrained economy, given by

$$\bar{r} = \rho + \gamma\mu_\delta - \frac{\gamma(1+\gamma)}{2}\sigma_\delta^2, \quad \bar{\kappa} = \gamma\sigma_\delta. \tag{47}$$

Optimal consumptions c_i^*, wealths W_i^*, stock price-dividend ratio R and optimal investment policies θ_i^* are given by

$$\begin{aligned} c_{1t}^* &= \frac{1}{1+\lambda_t^{1/\gamma}}\delta_t, \qquad c_{2t}^* = \frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\delta_t, && (48)\\ W_{1t}^* &= H_{1t}\frac{1}{1+\lambda_t^{1/\gamma}}\delta_t, \quad W_{2t}^* = H_{2t}\frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\delta_t, && (49)\\ R_t &= H_{1t}\frac{1}{1+\lambda_t^{1/\gamma}} + H_{2t}\frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}, && (50)\\ \theta_{1t}^* &= \frac{1}{\gamma\sigma_t}\Big(\kappa_t - \gamma\sigma_{\lambda t}\frac{\partial H_{1t}}{\partial\lambda_t}\frac{\lambda_t}{H_{1t}}\Big), \quad \theta_{2t}^* = \bar{\theta}, && (51)\end{aligned}$$

while the volatilities of the stock returns, σ, and weighting process, σ_λ, are given by

$$\sigma_t = \sigma_\delta - \sigma_{\lambda t}\frac{\partial R_t}{\partial\lambda_t}\frac{\lambda_t}{R_t}, \qquad \sigma_{\lambda t} = \frac{(1-\bar{\theta})\gamma\sigma_\delta}{\frac{1}{1+\lambda_t^{1/\gamma}} + \gamma\frac{\partial H_{2t}}{\partial\lambda_t}\frac{\lambda_t}{H_{2t}} - \bar{\theta}\gamma\frac{\partial R_t}{\partial\lambda_t}\frac{\lambda_t}{R_t}}, \tag{52}$$

where wealth-consumption ratios H_{1t} and H_{2t} satisfy equations (42). Moreover, the initial value λ_0 for the weighting process (20) solves equation

$$sH_2(\lambda_0, 0)\frac{\lambda_0^{1/\gamma}}{1+\lambda_0^{1/\gamma}}\delta_0 - (1-s)H_1(\lambda_0, 0)\frac{1}{1+\lambda_0^{1/\gamma}}\delta_0 = b. \tag{53}$$

The expressions for riskless rate r and price of risk κ in Proposition 2 are in terms of the of volatilities σ and σ_λ, as well as parameter $\lambda^{1/\gamma}$ which in our economic setting can be interpreted as the ratio of consumptions of investors 2 and 1, as it follows from the expressions in (48). As in the general case in Proposition 1, interest rates are comprised of three terms, where the first term is a riskless rate in an unconstrained economy, while the second and third terms highlight the impact of constraints and risk sharing. Moreover, the effect of risk sharing, as captured by volatility σ_λ, can be expressed in terms of consumption volatilities. In particular, from expression (34) it follows that

$$\sigma_{\lambda t} = \gamma(\sigma_{c_1 t} - \sigma_{c_2 t}). \tag{54}$$

It will be demonstrated later that volatility σ_λ is positive in equilibrium since investor 1 is more exposed to risk and hence her consumption growth is more volatile.

Proposition 2 also demonstrates that when $\bar{\theta} < 1$ the portfolio constraint of investor 2 is always binding since otherwise, having identical preferences, both investors should find optimal to invest $\theta_i < 1$ which contradicts market clearing conditions (9). Moreover, Proposition 2 provides expressions for equilibrium volatilities σ and σ_λ in terms of the elasticities of wealth-consumption and price-dividend ratios with respect to weighting process λ, given by

$$\epsilon_{H_2 t} = \frac{\partial H_{2t}}{\partial \lambda_t}\frac{\lambda_t}{H_{2t}}, \qquad \epsilon_{Pt} = \frac{\partial R_t}{\partial \lambda_t}\frac{\lambda_t}{R_t}. \tag{55}$$

From the expression for the volatility σ_λ in (52) it follows that σ_λ is decreasing in elasticity ϵ_{H_2} and increasing in ϵ_P. The effect of elasticities in (55) on volatility σ_λ then determines their impact on all the other parameters in equilibrium.

To understand the effect of these elasticities on volatility σ_λ we observe that elasticity ϵ_{H_2} is proportional to the stock hedging demand of investor 2 given by the second term in the expression for optimal policy (43). Moreover, since σ_λ is positive, it follows from this expression that higher elasticity ϵ_{H_2} tends to decrease optimal investment in stock. Thus, higher ϵ_{H_2} makes the stock less attractive, and hence reduces the cost of being constrained. Therefore, σ_λ also decreases to reflect decreased risk sharing distortions of the constraint. Moreover, as follows from the expressions for volatilities (52) the increase in elasticity ϵ_P tends to decrease stock volatility σ since the dividends and weighting process are negatively correlated. Hence, if volatility σ decreases, the stock becomes more attractive for both investors. However, since investor 2 is constrained, her ideal unconstrained holding moves further away from her constrained holding $\bar{\theta}$ and hence the risks are shared in a less optimal way and σ_λ increases.

Proposition 2 also allows to explicitly identify the coefficients of PDEs (42) for wealth-consumption ratios H_i, which depend on equilibrium parameters identified in expressions (44)–(52). Moreover, it appears that the coefficients themselves depend on ratios H_i and hence, we obtain a system of quasilinear PDEs the solutions to which completely characterize the equilibrium. We next solve for time-independent solutions of PDEs (42) which correspond to the infinite horizon case. To solve the equations (42), we first fix a large horizon parameter T, choose a starting value for $H_i(\lambda, T)$ and then solve the equation backwards using a modification of Euler's finite-difference method until the solution converges to a stationary one. This approach is similar to the subsequent iterations method for solving Bellman equations in discrete time [e.g., Ljungqvist and Sargent (2004)] when at a distant time in the future the value function is set equal to some function (usually zero) and then the value functions at earlier dates are obtained by solving equations backwards.

Since weight λ varies from zero to infinity, we first perform a change of variable and rewrite the PDEs (42) as well as the equilibrium parameters in Proposition 2 in terms of *constrained investor's share in aggregate consumption*, given by

$$y_t = \frac{\lambda_t^{1/\gamma}}{1 + \lambda_t^{1/\gamma}}. \tag{56}$$

Variable y takes values in the interval $[0, 1]$ and provides one-to-one mapping to variable λ. The solution of PDEs in terms of new variable we label as $\tilde{H}_i(y, t)$. Assuming that the solutions to new PDEs are continuous and twice continuously differentiable, setting in those equations $y = 0$ and $y = 1$ we recover boundary conditions for $\tilde{H}_i(y, t)$. Next, we replace the derivatives by their finite-difference analogues letting the time and state variable increments denote $\Delta t \equiv T/M$ and $\Delta y \equiv 1/N$, where M and N are integer numbers. Solving the equation backwards, sitting at time t we compute the coefficients of finite-difference analogues of PDEs (42) using the solutions $\tilde{H}_i(y, t + \Delta t)$ obtained from the previous step $t + \Delta t$. As a result, the coefficients of equations for $\tilde{H}_i(y, t)$ are known at time t and hence $\tilde{H}_i(y, t)$ can be found by solving a system of linear finite-difference equations with three-diagonal matrix. Appendix B provides further details of the numerical algorithm. The wealth-consumption ratios then allow us to derive all the parameters of equilibrium.

Remark 1 (Bond prices). Proposition 2 allows to determine the instantaneous interest rate r_t. Therefore, the bond price B_t can be obtained by solving numerically the equation for the bond price dynamics (2).

Remark 2 (Existence of Equilibrium). The numerical analysis shows that the function on the left-hand side of the equation for λ_0 in (53) is a monotone function of λ_0 and maps interval $[0, \infty)$ into $[C_0, C_1)$, where C_0 and C_1 are some constants, and hence, if $b \in [C_0, C_1)$ there always exists the unique solution λ_0 that satisfies the equation. Given the existence of λ_0 and the solutions to HJB equations (42), expressions (44)–(52) fully characterize the equilibrium in the economy.

3.2. Analysis of Equilibrium

We now study the impact of constraints on various equilibrium parameters. Important implication of our model is that in contrast to models with logarithmic investors the constraints do affect

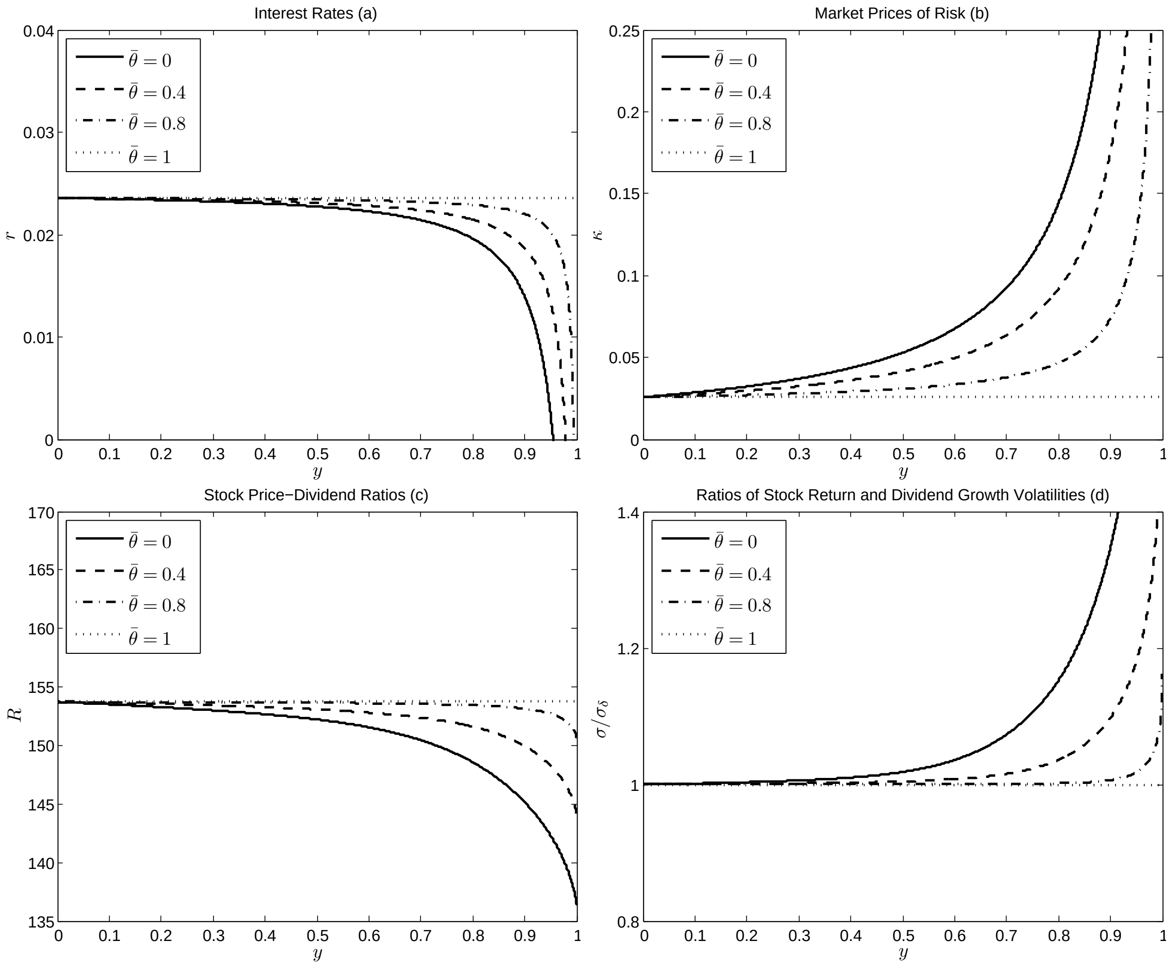

Figure 1: Parameters of Equilibrium with Constraints, $\gamma < 1$.

The figure plots interest rates r, market prices of risk κ, price-dividend ratios R and ratios of stock return and dividend growth volatilities σ/σ_δ as functions of constrained investor's consumption share y. Dividend mean growth rate $\mu_\delta = 1.8\%$ and volatility $\sigma_\delta = 3.2\%$ are from the estimates in Campbell (2003), based on consumption data in 1891–1998, while risk aversion and time discount are set to $\gamma = 0.8$ and $\rho = 0.01$.

the price-dividend ratios and stock return volatilities. Figures 1 and 2 present equilibrium interest rates, market prices of risk, price-dividend ratios and the ratios of stock return and dividend growth volatilities as functions of constrained investor's consumption share y for different levels of the tightness of constraints $\bar{\theta}$ when risk aversions are less than unity ($\gamma = 0.8$) and greater than unity ($\gamma = 3$), respectively. The equilibrium is derived under plausible parameters for the dividend process.[8] We note that in our model the instantaneous changes in the dividend growth $d\delta/\delta$ and constrained investor's consumption share dy are negatively correlated since negative shocks to dividends shift relative consumption to constrained investors, due to the fact that the latter are less affected by adverse stock market fluctuations. Hence, higher consumption share

[8] In particular, the parameters for the dividend process ($\mu_\delta = 1.8\%, \sigma_\delta = 3.2\%$) are taken from the estimates in Campbell (2003), based on consumption data in 1891–1998 years, and the discounting parameter is set to $\rho = 0.01$.

y is associated with bad times while lower y is associated with good times. Following the literature [e.g., Chan and Kogan (2002)] we label economic variables as *procyclical* if they increase in good times (when dividend growth rate shocks are positive) and decrease in bad times (when dividend growth rate shocks are negative), and as *countercyclical* if they decrease in good times and increase in bad times.

For risk aversion less than unity Figure 1 demonstrates that tighter constraints decrease interest rates and price-dividend ratios, and increase market prices of risk and stock return volatilities. For risk aversion greater than unity, Figure 2 shows that tighter constraints decrease interest rates and stock return volatilities, and increase market prices of risk and price-dividend ratios. In both cases the impact of constraints is asymmetric and is more pronounced in bad times, when consumption share y is larger. We first analyze the equilibrium parameters for the case $\gamma < 1$, presented on Figure 1, and then for the case $\gamma > 1$, presented on Figure 2.

Panel (a) of Figure 1 presents interest rates when $\gamma < 1$ and demonstrates that in line with the results of Section 2 interest rates in constrained economy are lower than in an unconstrained one for a given consumption share y. Moreover, they become lower with tighter constraints and are decreasing functions of constrained investor's share of consumption y. Intuitively, constrained investor invests more in bonds driving interest rates down. Moreover, constraints prevent the investor from sharing risks efficiently and smoothing consumption over time. As a result, when her current consumption is high the price of future consumption increases making her more willing to lend at a lower interest causing interest rates to fall.

Panel (b) of Figure 1 shows that the prices of risk are higher in the constrained than in the unconstrained economies and increase as constraint becomes tighter. When the constrained investor invests only a fraction $\bar{\theta} < 1$ of her wealth in the stock, for the markets to clear investor 1 should be leveraged so that $\theta_1^* > 1$. This, however, implies that the unconstrained investor should be more exposed to risk as the constraint tightens, and hence, the market price of risk should be higher. Moreover, market price of risk also increases with constrained investor's consumption share y since in those states in which the unconstrained investor consumes less and possesses less wealth she is more risk averse and requires market prices of risk to increase for the stock market to clear. Thus, the market price of risk is countercyclical, consistently with the empirical literature [e.g., Ferson and Harvey (1991)].

Panel (c) of Figure 1 demonstrates that the price-dividend ratios become lower with tighter constraints and the effect of constraints is more pronounced in states with higher constrained investor's consumption share y. To understand the patterns of price-dividend ratios we first observe that in equilibrium the price-dividend ratio can be interpreted as the ratio of aggregate wealth over aggregate consumption since the market clearing conditions (9) imply that the sock price equals aggregate wealth while the aggregate consumption equals the dividend. As a result, the price-dividend ratio will be close to wealth-consumption ratio of unconstrained or constrained investor depending on which of them dominates in the market by holding larger fraction of consumption and wealth. When the unconstrained investor dominates (y is low), the equilibrium will

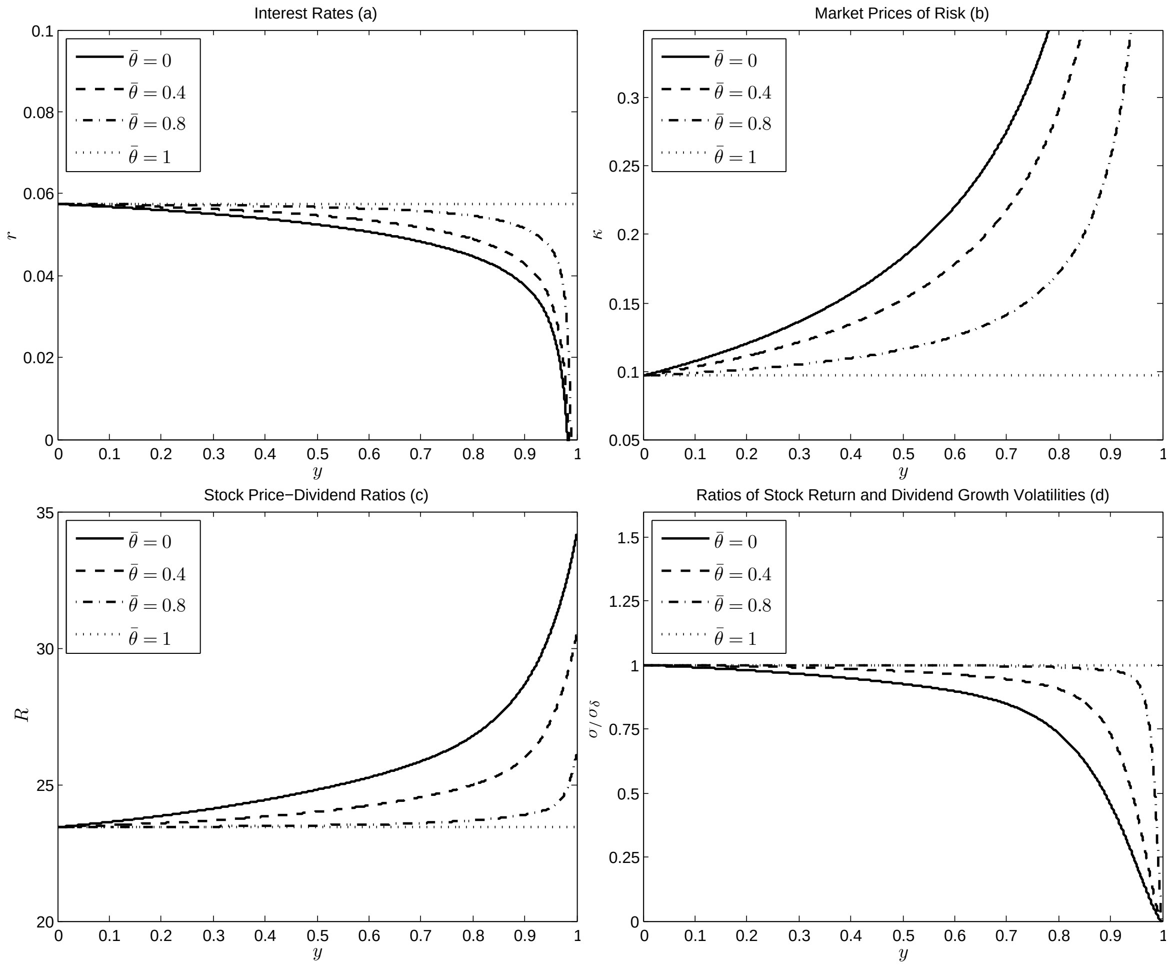

Figure 2: Parameters of Equilibrium with Constraints, $\gamma > 1$.

The figure plots interest rates r, market prices of risk κ, price-dividend ratios R and ratios of stock return and dividend growth volatilities σ/σ_δ as functions of constrained investor's consumption share y. Dividend mean growth rate $\mu_\delta = 1.8\%$ and volatility $\sigma_\delta = 3.2\%$ are from the estimates in Campbell (2003), based on consumption data in 1891–1998, while risk aversion and time discount are set to $\gamma = 3$ and $\rho = 0.01$.

be close to that in the benchmark unconstrained economy in which case all equilibrium parameters, including price-dividend ratios, are constant (dotted lines in Figures 1 and 2). However, in states with dominating constrained investor (y is high) the price-dividend ratio is close to constrained investor's wealth-consumption ratio, which increases or decreases with tighter constraints depending on the relative strength of classical income and substitution effects. When the investment opportunities worsen, the income effect induces investors to decrease consumption and save more while the substitution effect induces them to consume more and save less due to cheaper current consumption. For CRRA preferences the intertemporal elasticity of substitution (IES) equals $1/\gamma$, the income effect dominates for IES < 1 and the substitution effect dominates for IES > 1 while in the case of IES $= 1$ both effects perfectly offset each other. With tighter constraints the investment opportunities for constrained investor worsen due to the decline in interest rates and inability to fully benefit from the increase in market prices of risk, and hence her wealth-consumption ratios decrease for $\gamma < 1$ via the substitution effect.[9] As a result, the price-dividend ratios decrease with tighter constraints and the effect is stronger in bad times, when constrained investor dominates the market and the decline in interest rates is sharper.

The stock return volatilities on panel (d) of Figure 1 increase with tighter constrains and are higher in bad times (when y is high) than in good times (when y is low). This is due to the fact that the instantaneous changes in price-dividend ratio R and dividend δ are positively correlated due to the fact that ratio R is a decreasing function of consumption share y, which is negatively correlated with changes in dividend δ. Consequently, since the stock price is the product of price-dividend ratio and the dividend, stock return volatility should be higher in constrained economy. Moreover, this effect is stronger in bad times (when y is high) due to the concavity of ratio R, and when $\bar{\theta}$ is low, due to the higher sensitivity of ratio R to changes in y. Thus, for $\gamma < 1$ consistently with the empirical literature [e.g., Schwert (1989), Campbell and Cochrane (1999)] our model generates procyclical price-dividend ratios, countercyclical stock return volatilities exceeding the volatility of dividends, as well as negative correlation between changes in stock returns and their volatilities. Moreover, the results on Figure 1 demonstrate that lower price-dividend ratios R predict higher market prices of risk κ as well as higher risk premia (given by $\mu - r = \kappa\sigma$).

Turning to the case $\gamma > 1$ we observe from the results shown on Figure 2 that the constraints affect the interest rates and market prices of risk in the same directions as in the case $\gamma < 1$. However, by contrast with the case of $\gamma < 1$, due to the dominance of income effect, price-dividend ratios increase while stock return volatilities decrease with tighter constraints, and the effects are stronger in bad times.[10] One might think that the results in the case $\gamma > 1$ are more plausible than in the case $\gamma < 1$ given the evidence [e.g., Mehra and Prescott (1985)] that risk aversion is greater than unity. However, we note, that the intuition for the dynamics of price-dividend ratios and stock return volatilities in our model is driven by the relative strength of income and substitution effect and not by the risk aversion per se. It is well known that CARA utility does not allow to separate IES from the risk aversion and hence, in our setting IES > 1 is necessarily associated with $\gamma < 1$.

We also note that since lower $\bar{\theta}$ decreases interest rates and increases market prices of risk,

[9]The relation between wealth-consumption ratios and the attractiveness of investment opportunities can conveniently be illustrated in an unconstrained partial equilibrium economy with constant interest rate r and market price of risk $\kappa = (\mu - r)/\sigma$, and an investor maximizing her objective (6) subject to budget constraint (4) and no-bankruptcy constraint. It can easily be verified that when condition $\rho - (1-\gamma)(r + 0.5\kappa^2/\gamma) > 0$ is satisfied, the investor's wealth-consumption ratio is given by:

$$\frac{W}{c} = \frac{\gamma}{\rho - (1-\gamma)(r + 0.5\kappa^2/\gamma)},$$

Hence, if investment opportunities deteriorate due to decrease of r or κ, the wealth-consumption ratio increases if the income effect dominates ($\gamma > 1$) and decreases if the substitution effect dominates ($\gamma < 1$).

[10]In our model when $\gamma > 1$ the instantaneous volatility of stock returns is lower than that of dividend growth and hence there is no excess volatility. Bhamra and Uppal (2009) demonstrate a significant excess volatility in a complete-market exchange economy with CRRA investors that differ in risk aversions. Thus, excess volatility is likely to be present in the extension of our model to the case where investors have different risk aversions.

irrespective of risk aversion γ, the case of restricted participation which corresponds to $\bar{\theta} = 0$ better explains the levels of observed interest rates and market prices of risk. In particular, in our model with plausible parameters described above and $\gamma = 3$, when we set $y = 0.7$ [e.g., Mankiw and Zeldes (1991), Guvenen (2006)] we obtain $r = 4.8\%$ and $\kappa = 28\%$, while the volatilities of individual consumptions are $\sigma_{c_1} = 9\%$ and $\sigma_{c_2} = 0.7\%$. The estimates in Campbell (2003) show that $r = 2\%$ and $\kappa = 36\%$, while Malloy, Moskowitz, and Vissing-Jorgensen (2009) show that $\sigma_{c_1} = 3.6\%$ and $\sigma_{c_2} = 1.4\%$. Thus, our model implies riskless rates and market prices of risk sufficiently close to those in the data for such a simple model.

Remark 3 (Duffie-Epstein preferences). The discussion above demonstrates that for risk aversion $\gamma < 1$ the model generates empirically plausible patterns for price-dividend ratios and stock return volatilities while for $\gamma > 1$ it generates high market prices of risk and low interest rates close to those observed in the data. We note that the intuition for price-dividend ratios and stock return volatilities only relies on the relative strength of income and substitution effects. As pointed out above, for CRRA preferences the intertemporal elasticity of substitution (IES) equals $1/\gamma$ and hence high IES leading to the dominance of substitution effect is only possible for $\gamma < 1$. However, more general Duffie-Epstein recursive preferences allow for IES independent of risk aversion parameter γ [Duffie and Epstein (1992)]. Our results lead to a conjecture that in a model with Duffie-Epstein preferences with both IES and risk aversion exceeding unity [as in Bansal and Yaron (2004)] it might be possible to match interest rates and market prices of risk, as well as generate procyclical price-dividend ratios and countercyclical stock return volatilities which exceed the volatility of dividends, consistently with the empirical literature.[11]

Our results also allow to obtain the expressions for consumption growth volatilities of investors, which also capture the effect of risk sharing between them. The expressions for the volatilities can be obtained by applying Itô's Lemma to optimal consumptions (48) and are reported in the following Corollary 2.

Corollary 2. *The optimal consumption growth volatilities of unconstrained and constrained investors are given by*

$$\sigma_{c_1 t} = \sigma_\delta + \frac{1}{\gamma}\frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\sigma_{\lambda t}, \qquad \sigma_{c_2 t} = \sigma_\delta - \frac{1}{\gamma}\frac{1}{1+\lambda_t^{1/\gamma}}\sigma_{\lambda t}. \tag{57}$$

It can be shown in our example that the volatility σ_λ is positive, and hence, consumption volatilities in (57) imply that unconstrained investor, being exposed to more risk, has larger volatility of consumption than the constrained one. Basak and Cuoco (1998) show in the case of restricted participation and $\gamma = 1$ that the volatility σ_{c_2} of constrained investor is zero and all

[11] Campbell and Cochrane (1999) and Chan and Kogan (2002) present the models with habit formation and "catching up with the Joneses" preferences respectively, that explain the patterns for price-dividend ratios and stock return volatilities.

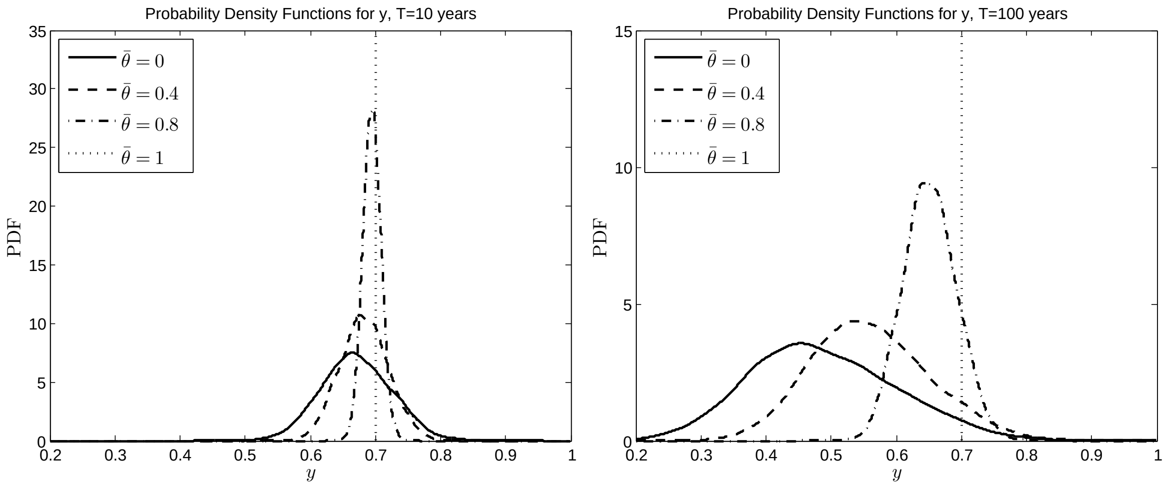

Figure 3: Probability Density Functions for Constrained Investor's Share in Aggregate Consumption, $\gamma = 3$.

The figure plots interest rates r, market prices of risk κ, price-dividend ratios R and ratios of stock return and dividend growth volatilities σ/σ_δ as functions of constrained investor's consumption share y. Dividend mean growth rate $\mu_\delta = 1.8\%$ and volatility $\sigma_\delta = 3.2\%$ are from the estimates in Campbell (2003), based on consumption data in 1891–1998, while risk aversion and time discount are set to $\gamma = 3$ and $\rho = 0.01$.

the risk is borne by the unconstrained investor. However, in our case with $\gamma > 1$ volatility σ_{c_2} is greater than zero, as also in the data for non-stockholders [e.g. Malloy, Moskowitz, Vissing-Jorgensen (2009)].

Finally, we address the question of how the constraints affect the distribution of consumption between the investors. So far we have compared the parameters of equilibria with different constraint $\bar{\theta}$ for a given level of consumption share y. This comparison does not account for the fact that share y itself depends on $\bar{\theta}$. Figure 3 shows probability density functions of y for $\gamma = 3$, different constraints $\bar{\theta}$ and time horizons equal to ten and one hundred years respectively. The probability densities imply that consumption share y tends to decline, and hence, the impact of constrained investor becomes smaller in the course of time even though it is still significant even after hundred years. As discussed in Hong, Kubik and Stein (2004) stock market participation depends on person-specific characteristics such as social integrations and education. Thus, specializing to the case of restricted participation ($\bar{\theta} = 0$) our model demonstrates that these characteristics lead to gradual, although slow, elimination of non-stockholders' impact on financial markets via natural selection.[12]

[12]In unconstrained economic settings the survival of irrational investors has been studied in Kogan, Ross, Wang and Westerfield (2004), Berrada (2009), Dumas, Kurshev and Uppal (2009) and Yan (2008), among others. The results in the latter three works suggest that it takes a long time to eliminate the impact of irrational investors that have wrong beliefs about mean dividend growth rates. Hugonnier (2008) considers survival of constrained logarithmic investor and demonstrates that their impact can quickly be eliminated. However, in his calibration the volatility of dividends is 20% while we set this parameter to the volatility of aggregate consumption 3.2% taken from Campbell(2003). When in the calibration we choose $\gamma = 1$ and $\sigma_\delta = 20\%$ consistently with Hugonnier our results also imply fast elimination of constrained investor's impact.

4. Extensions and Ramifications

In this Section we demonstrate that our model is extendable to different alternative economic settings. Section 4.1 extends the results of Section 2 to the case of heterogeneous beliefs and provides a numerical solution to the model with CRRA investors with heterogeneous beliefs when one of them faces short-sale constraints. Section 4.2 demonstrates that the results of Section 2 generalize to the environments with multiple assets.

4.1. Heterogeneous Beliefs Formulation

We now consider an economy in which investors are constrained and have different beliefs about mean dividend growth rate in the economy. We first generalize the results of Section 2 and derive expressions for the parameters of equilibrium in terms of adjustments in fictitious economy and the differences in beliefs. Then, we specialize to a framework in which both investors have identical CRRA preferences and the pessimist faces short-sale constraints. We solve this model numerically by employing the approach of Section 3 and discuss some properties of the equilibrium parameters.

Basak (2000, 2005) derives expressions for equilibrium parameters for general utility functions in the economy in which investors face heterogeneous belief but does not study the impact of constraints as we do in this work. Our model is also related to the model of Gallmeyer and Hollifield (2008) in which the pessimist has logarithmic preferences and faces short-sale constraints while the investor with general CRRA is optimistic and unconstrained. By contrast, our model does not rely on the assumption of a logarithmic constrained investor.

The economic setting is similar to that of Section 2. In particular, investors trade in two securities, a riskless bond and stock, and dividends follow process (1). They agree on dividends, bond and stock prices and the dividend growth rate volatility σ_δ but disagree on the growth rate μ_δ. Throughout this Section we will be using superscript i to denote quantities on which investors disagree, while by subscript i investor-specific quantities on which there is no disagreement. Investors update their beliefs $\mu^i_{\delta t}$ in a Bayesian fashion:

$$\mu^i_{\delta t} = E^i[\mu_{\delta t}|\mathcal{F}^\delta_t], \quad i \in \{o, p\}, \tag{58}$$

where $E^i[\cdot]$ denotes the expectation under the subjective probability measure of investor i and $\mathcal{F}^\delta_t$ is the augmented filtration generated by δ_t. Both investors have different priors $\mu^i_{\delta 0}$ and investor 1 is optimistic ($i = o$) while investor 2 is pessimistic ($i = p$) about the dividend growth. From the point of view of investor i the dividends and stock prices follow the processes

$$d\delta_t = \delta_t[\mu^i_{\delta t} + \sigma_{\delta t} dw^i_t], \tag{59}$$

$$dS_t + \delta_t dt = S_t[\mu^i_t dt + \sigma_t dw^i_t], \tag{60}$$

where w^i_t denotes Brownian motions under the *subjective probability measure* of investor i.

From the filtering theory in Lipster and Shiryayev (1977) it follows that Brownian motions w_t^i are given by

$$dw_t^i = \frac{\mu_\delta - \mu_{\delta t}^i}{\sigma_\delta} dt + dw_t, \qquad i \in o, p. \tag{61}$$

By $\Delta\mu_{\delta t}$ we denote the *disagreement process* defined as

$$\Delta\mu_{\delta t} = \frac{\mu_{\delta t}^o - \mu_{\delta t}^p}{\sigma_{\delta t}}. \tag{62}$$

Moreover, if dividends follow geometric Brownian motion (35) and investors' initial priors are normally distributed with parameters

$$\mu_\delta^i \sim N(\hat{\mu}_{\delta 0}^i, \hat{\sigma}_{\delta 0}^i),$$

then $\mu_{\delta t}^i$ is also normally distributed and the processes for $\mu_{\delta t}^i$ and $\Delta\mu_{\delta t}$ are given by

$$d\mu_{\delta t}^i = \frac{\hat{\sigma}_{\delta t}^i}{\sigma_\delta} dw_t^i, \tag{63}$$

$$d\Delta\mu_{\delta t} = -\frac{\hat{\sigma}_{\delta t}^p}{\sigma_\delta} \Delta\mu_{\delta t} dt + \frac{\hat{\sigma}_{\delta t}^o - \hat{\sigma}_{\delta t}^p}{\sigma_\delta} dw_t^i, \tag{64}$$

where

$$\hat{\sigma}_{\delta t}^i = \frac{\hat{\sigma}_{\delta 0}^i \sigma_\delta^2}{\hat{\sigma}_{\delta 0}^i t + \sigma_\delta^2}. \tag{65}$$

The budget constraint for each investor is given by (4) in which Brownian motion w and stock mean-return μ are replaced by investor's subjective Brownian motion w^i and mean-return μ^i. Each investor solves optimization problem (6) in which now expectation operator $E[\cdot]$ is replaced by operator $E^i[\cdot]$ under investor's subjective beliefs, subject to the budget constraint, no-bankruptcy constraint $W_t \geq 0$ and portfolio constraints (5).

The equilibrium in this economy is a set of parameters $\{r_t, \mu_t^o, \mu_t^p, \sigma_t\}$ and of consumption and investment policies $\{c_{it}^*, \alpha_{it}^*, \theta_{it}^*\}_{i\in\{o,p\}}$ which solve investor i's dynamic optimization problem and satisfy market clearing conditions in (9).

As in Section 2, the parameters of equilibrium are characterized in terms of adjustments ν_i^* and support functions $f_i(\nu_i^*)$. We first characterize investor's marginal utilities in terms of state prices that follow processes as in (14) but with Brownian motions under subjective probability measures. Then, we introduce the ratio of their marginal utilities λ, which follows the process

$$d\lambda_t = -\lambda_t[\mu_{\lambda t}^i dt + \sigma_{\lambda t} dw_t^i]. \tag{66}$$

By employing market clearing conditions we obtain the parameters of equilibrium. Proposition 3 summarizes our results.

Proposition 3. *If there exists an equilibrium, the riskless interest rate r, perceived market prices of risk κ^i, drifts μ^i_λ and volatility σ_λ of weighting process (66) are given by*

$$\begin{aligned} r_t &= \bar{r}_t - \frac{A_t}{A_{ot}} f_o(\nu^*_{ot}) - \frac{A_t}{A_{pt}} f_p(\nu^*_{pt}) - \frac{A^3_t(P_{ot}+P_{pt})}{2A^2_{ot}A^2_{pt}}\sigma^2_{\lambda t} - \frac{A^3_t}{A_{ot}A_{pt}}\Big(\frac{P_{ot}}{A_{ot}} - \frac{P_{pt}}{A_{pt}}\Big)\delta_t\sigma_{\delta t}\sigma_{\lambda t} \\ &\quad - \frac{A^2_t}{A_{pt}}\delta_t\sigma_{\delta t}\Delta\mu_{\delta t} + \frac{A^2_t}{A_{ot}A_{pt}}\sigma_{\lambda t}\Delta\mu_{\delta t}, \end{aligned} \tag{67}$$

$$\kappa^o_t = \bar{\kappa}_t + \frac{A_t}{A_{pt}}\sigma_{\lambda t}, \quad \kappa^p_t = \bar{\kappa}_t - \frac{A_t}{A_{ot}}\sigma_{\lambda t}, \tag{68}$$

$$\mu^o_{\lambda t} = A_t\delta_t\sigma_{\delta t}\sigma_{\lambda t} + f_o(\nu^*_{ot}) - f_p(\nu^*_{pt}) - \frac{A_t}{A_{ot}}\sigma^2_{\lambda t} - \Delta\mu_\delta\kappa^p_t, \quad \mu^p_{\lambda t} = \mu^o_{\lambda t} - \sigma_{\lambda t}\Delta\mu_{\delta t}, \tag{69}$$

$$\sigma_{\lambda t} = \Delta\mu_{\delta t} + \frac{\nu^*_{ot} - \nu^*_{pt}}{\sigma_t}, \tag{70}$$

where $\bar{r}$ is the riskless rate and $\bar{\kappa}$ is the market price of risk in an unconstrained economy populated by optimists, given by

$$\bar{r}_t = \rho + A_t\delta_t\mu^o_{\delta t} - \frac{A_t P_t}{2}\delta^2_t\sigma^2_{\delta t}, \quad \bar{\kappa}_t = A_t\delta_t\sigma_{\delta t}, \tag{71}$$

A_{it}, P_{it}, and A_t and P_t are absolute risk aversions and prudence parameters of investor i and a representative investor with utility (19), respectively.

Expressions for optimal consumption c^*_i and stock price S are as in Proposition 1. Optimal wealths W^*_i and optimal investment policies θ^*_i are given by expressions (26) and (28) in which expectation operator $E[\cdot]$ and market prices of risk κ are replaced by subjective operator $E^i[\cdot]$ and price of risk κ^i. Initial value λ_0 for weighting process (66) is such that budget constraint at time zero (29) is satisfied. Moreover, adjustments ν^*_i satisfy complementary slackness conditions (30), as in Proposition 1.

The expressions for interest rates in Proposition 3 demonstrate the impact of heterogeneous beliefs on interest rates and subjective market prices of risk. In particular, the expression for interest rates have additional terms [last two terms in (67)] which demonstrate the direct effect of disagreement process $\Delta\mu_\delta$. Since the disagreement process is positive, its impact depends on the sign of volatility σ_λ. Moreover, the expression for volatility σ_λ in (70) demonstrates that this parameter itself depends on $\Delta\mu_\delta$ since the disagreement affects the efficiency of the risk sharing, quantified by σ_λ. Unlike the setup of Section 2, investors now disagree also on the market prices of risk, which are given in (68).

We now consider a modification of the model in Section 3 in which now investors have heterogeneous beliefs about the dividend growth rate. In particular, investor 1 is optimistic and unconstrained while investor 2 is pessimistic and faces constraints that impose a limit on the short-sales $\theta \geq \underline{\theta}$, where $\underline{\theta} < 0$. For simplicity, as in Yan (2008) we assume that investors do not update their beliefs and believe that dividends follow a GBM

$$d\delta_t = \delta_t[\mu^i_\delta dt + \sigma_\delta dw^i_t], \tag{72}$$

and their difference in beliefs we denote by $\Delta\mu_\delta$. This assumption can further be justified by noting that under plausible parameters it takes very long time for the beliefs to converge.[13]

As in Section 3 we characterize the equilibrium in terms of the wealth-consumption ratios of investors which satisfy HJB equations (42) in which the drift parameter μ_λ is now investor-specific and should be replaced by μ^i_λ. Our results are summarized in Proposition 4.

Proposition 4. *If there exists an equilibrium, the riskless interest rate r, perceived market price of risk κ^i and drifts μ^i_λ of weighting process λ that follows (20) are given by*

$$\begin{aligned} r_t &= \bar{r} + \frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\underline{\theta}\sigma_t(\Delta\mu_\delta - \sigma_{\lambda t}) - \frac{1+\gamma}{2\gamma}\frac{\lambda_t^{1/\gamma}}{(1+\lambda_t^{1/\gamma})^2}\sigma^2_{\lambda t} \\ &\quad -\gamma\frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\sigma_\delta\Delta\mu_\delta + \frac{\lambda_t^{1/\gamma}}{(1+\lambda_t^{1/\gamma})^2}\sigma_{\lambda t}\Delta\mu_\delta, \qquad (73)\\ \kappa^o_t &= \bar{\kappa} + \frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\sigma_{\lambda t}, \quad \kappa^p_t = \bar{\kappa} - \frac{1}{1+\lambda_t^{1/\gamma}}\sigma_{\lambda t}, \qquad (74)\\ \mu^o_{\lambda t} &= \gamma\sigma_\delta\sigma_{\lambda t} - \frac{1}{1+\lambda_t^{1/\gamma}}\sigma^2_{\lambda t} - \Delta\mu_\delta\kappa^p_t + \underline{\theta}\sigma_t(\Delta\mu_\delta - \sigma_{\lambda t}), \quad \mu^p_{\lambda t} = \mu^o_{\lambda t} - \Delta\mu_\delta\sigma_{\lambda t}, \qquad (75)\end{aligned}$$

where $\bar{r}$ is the riskless rate and $\bar{\kappa}$ is the market price of risk in an unconstrained economy populated by optimists, given by

$$\bar{r} = \rho + \gamma\mu^o_\delta - \frac{\gamma(1+\gamma)}{2}\sigma^2_\delta, \quad \bar{\kappa} = \gamma\sigma_\delta. \qquad (76)$$

Optimal consumptions c^*_i, wealths W^*_i, stock price-dividend ratio R and optimal investment policies θ^*_i are given by

$$\begin{aligned} c^*_{ot} &= \frac{1}{1+\lambda_t^{1/\gamma}}\delta_t, \qquad c^*_{pt} = \frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\delta_t, \qquad (77)\\ W^*_{ot} &= H_{ot}\frac{1}{1+\lambda_t^{1/\gamma}}\delta_t, \quad W^*_{pt} = H_{pt}\frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}\delta_t, \qquad (78)\\ R_t &= H_{ot}\frac{1}{1+\lambda_t^{1/\gamma}} + H_{pt}\frac{\lambda_t^{1/\gamma}}{1+\lambda_t^{1/\gamma}}, \qquad (79)\\ \theta^*_{ot} &= \frac{1}{\gamma\sigma_t}\Big(\kappa^o_t - \gamma\sigma_{\lambda t}\frac{\partial H_{ot}}{\partial\lambda_t}\frac{\lambda_t}{H_{ot}}\Big), \quad \theta^*_{pt} = \frac{1}{\gamma\sigma_t}\Big(\kappa^p_t - \gamma\sigma_{\lambda t}\frac{\partial H_{pt}}{\partial\lambda_t}\frac{\lambda_t}{H_{pt}}\Big), \qquad (80)\end{aligned}$$

[13] In particular, assuming that investors have the same variances for the prior belief, $\hat{\sigma}^i_{\delta 0} = \hat{\sigma}_{\delta 0}$, equations for the disagreement and estimation error processes in (64) and (65) imply that

$$\Delta\mu_{\delta t} = \Delta\mu_{\delta 0}\left(\frac{\sigma^2_\delta}{\hat{\sigma}_{\delta 0}t + \sigma^2_\delta}\right)^{\sigma_\delta}.$$

Assuming further that $\hat{\sigma}_{\delta 0} = \sigma_\delta$ and taking $\sigma_\delta = 3.2\%$, as in Campbell (2003), we obtain that it takes 100 years for the disagreement $\Delta\mu_\delta$ to decrease by 20%.

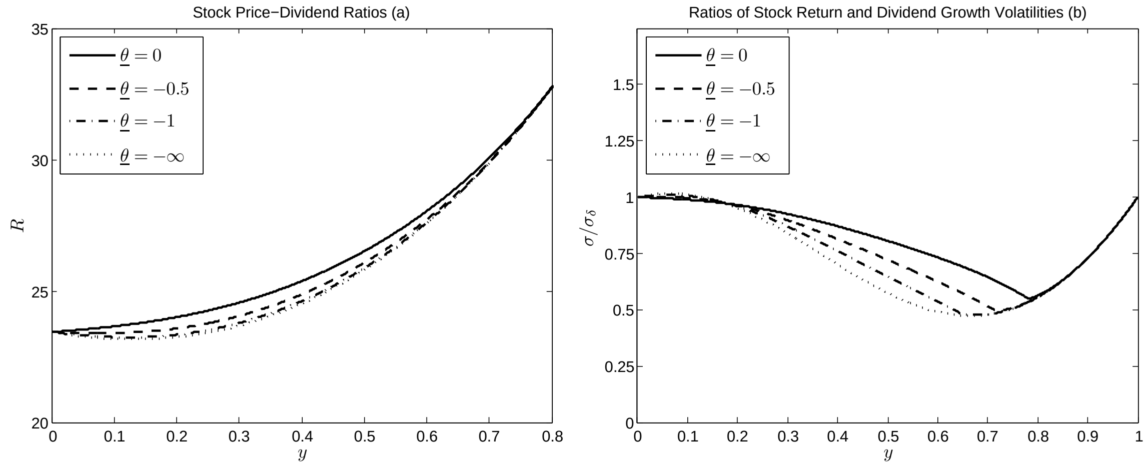

Figure 4: Price-Dividend Ratios and Ratios of Stock Return and Dividend Growth Volatilities with Heterogeneous Beliefs, $\gamma > 1$.

The figure plots interest rates r, market prices of risk κ, price-dividend ratios R and ratios of stock return and dividend growth volatilities σ/σ_δ as functions of constrained investor's consumption share y. Dividend mean growth rate $\mu_\delta = 1.8\%$ and volatility $\sigma_\delta = 3.2\%$ are from the estimates in Campbell (2003), based on consumption data in 1891–1998, while risk aversion and time discount are set to $\gamma = 3$ and $\rho = 0.01$.

while the volatilities of the stock returns, σ, and weighting process, σ_λ, are given by

$$\sigma_t = \sigma_\delta - \sigma_{\lambda t}\frac{\partial R_t}{\partial \lambda_t}\frac{\lambda_t}{R_t}, \qquad \sigma_{\lambda t} = \min\Big\{\frac{(1-\underline{\theta})\gamma\sigma_\delta}{\frac{1}{1+\lambda_t^{1/\gamma}} + \gamma\frac{\partial H_{pt}}{\partial \lambda_t}\frac{\lambda_t}{H_{pt}} - \underline{\theta}\gamma\frac{\partial R_t}{\partial \lambda_t}\frac{\lambda_t}{R_t}}, \Delta\mu_\delta\Big\}, \tag{81}$$

where wealth-consumption ratios H_{ot} and H_{pt} satisfy equations (42). Moreover, the initial value λ_0 for the weighting process (20) solves equation

$$sH_p(\lambda_0, 0)\frac{\lambda_0^{1/\gamma}}{1+\lambda_0^{1/\gamma}}\delta_0 - (1-s)H_o(\lambda_0, 0)\frac{1}{1+\lambda_0^{1/\gamma}}\delta_0 = b. \tag{82}$$

Proposition 4 characterizes equilibrium parameters in terms of wealth-consumption ratios and highlights the effects of heterogeneous beliefs and short-sale constraints. Crucial difference form the results of Proposition 2 is that now market prices of risk (74) and the drifts of weighting process (75) are investor-specific due to investors' disagreement on the dividend growth. Moreover, the short-sale constraint will not always be binding in equilibrium since when constrained investor's share of aggregate consumption is large she becomes more willing to smooth consumption over time and invests more in stock.

By calibrating our economy to plausible parameters we find that constraints have little effect on riskless rates, while market prices of risk are investor-specific. Therefore, we here focus on price-dividend ratios and stock return volatilities which are presented on Figure 4 for different levels of $\underline{\theta}$. We here consider only the case $\gamma > 1$ and note that the case $\gamma < 1$ can be analyzed in a similar way. The dotted lines correspond to quantities in an unconstrained economy ($\underline{\theta} = -\infty$) which are computed using explicit formula for stock prices in terms of weighting process λ, available in Yan (2008). We assume that the optimist has correct beliefs about mean dividend growth while the pessimist underestimates it by 40%. The first picture on Figure 4 demonstrates that tighter short-selling constraints (higher $\underline{\theta}$) increase price-dividend ratios. In the presence of short-sale constraints the optimist should hold less stocks in equilibrium which decreases her perceived market price of risk. As a result, investment opportunities deteriorate and her wealth-consumption ratio increases due to the dominance of substitution effect. Thus, when the optimist dominates in the market, the price-dividend ratio should go up for the similar reasons as in Section 3. When the pessimist dominates, the constraint does not bind and the price-dividend ratio becomes closer to that in the unconstrained case.

It can also be observed that the price-dividend ratios on panel (a) of Figure 4 are U-shaped when $\underline{\theta}$ is low, even though this effect is not economically significant. To understand the intuition, we observe that when the optimist dominates in the market, when pessimist's consumption and wealth share gradually increases, she shorts more in proportion of her wealth. As a result, the optimist should hold more stocks in equilibrium which requires higher market prices of risk, and hence, better investment opportunities. Therefore, the income effect decreases the optimist's wealth-consumption ratio. However, as the pessimist's consumption share increases further, at some point the price-dividend ratio should start increasing again since when the pessimist dominates, the optimist's wealth is low and she becomes unable to hold large amount of stock. As a result, shorting becomes less attractive for the pessimist in equilibrium and her subjective market price of risk increases pushing up the wealth-consumption ratio and hence the price-divided ratio.

Panel (b) of Figure 4 demonstrates that stock return volatility can both be higher and lower than the volatility of dividends, which is due to the U-shaped form of the price-dividend ratio. Moreover, as short-sale constraints become tighter the volatility of stock returns decreases for small consumption shares y, increases for medium y, and is almost unchanged for values of y close to unity when the constraint does not bind. Intuitively, short-sale constraints limit the ability of the pessimist to trade on her pessimism and hence her stockholding look as if she had smaller disagreement with the unconstrained investor. As a result, the economic parameters should become closer to the values in the unconstrained economy without disagreement. In particular, stock return volatilities should move closer to the volatility of dividends σ_δ, which we observe on Figure 4. This effect can also be formally demonstrated by observing that adjustment parameters for unconstrained and constrained investors are such that $\nu_o^* = 0$ [case (a) in Table 1] and $\nu_p^* \geq 0$ [case (e) in Table 1], and hence the volatility σ_λ given by (70) decreases towards zero since the volatility of stock returns σ is positive. Then, from the expression for volatility σ in (81) it follows that the difference between σ and dividend growth volatility σ_δ becomes smaller.

In a similar model with a logarithmic constrained pessimist Gallmeyer and Hollifield (2008) find that the stock return volatility increases when the unconstrained optimist has risk aversion $\gamma > 1$ and each investor is initially endowed with 50% of the market portfolio. By contrast with their work we present the analysis of price-dividend ratios and stock return volatilities as functions of both the pessimist's consumption share y and the tightness of the short-sale constraint. Moreover, we show that the volatility σ can decrease with tighter constraints even though the economic magnitude of this effect is small. Finally, our numerical method relies only on solving linear algebraic equations at each step rather than employing Monte-Carlo simulations as in their work.

4.2. Multiple Stock Formulation

We now demonstrate that the baseline analysis of Section 2 with single stock can easily be generalized to the case of multiple stocks. The uncertainty is now generated by a multi-dimensional Brownian motion $w = (w_1, ..., w_N)$. The investors trade in a riskless bond and N stocks in a positive net supply, normalized to unity, each of which is a claim to an exogenous strictly positive stream of dividends δ_n following the dynamics

$$d\delta_{nt} = \delta_{nt}[\mu_{\delta_n t}dt + \sigma_{\delta_n t}^{\top}dw_t], \quad n = 1, ..., N, \tag{83}$$

where μ_{δ_n} and σ_{δ_n} are stochastic processes. We consider equilibria in which bond prices, B, and stock prices, S, follow processes

$$dB_t = B_t r_t dt \tag{84}$$

$$dS_{nt} + \delta_{nt}dt = S_{nt}[\mu_{nt}dt + \sigma_{nt}^{\top}dw_t], \quad n = 1, ..., N. \tag{85}$$

We let $\mu \equiv (\mu_1, ..., \mu_N)^{\top}$ denote the vector of stock mean returns and $\sigma \equiv (\sigma_1, ..., \sigma_N)^{\top}$ the volatility matrix, assumed invertible, with each component measuring the covariance between the stock return and Brownian motion w_n. By δ we denote the process for aggregate dividend, $\delta = \delta_1 + \delta_2 + ... + \delta_N$, which follows the process

$$d\delta_t = \delta_t[\mu_{\delta t} + \sigma_{\delta t}^{\top}dw_t], \tag{86}$$

where

$$\mu_{\delta t} = \frac{\delta_{1t}}{\delta_t}\mu_{\delta_1 t} + ... + \frac{\delta_{Nt}}{\delta_t}\mu_{\delta_N t}, \quad \sigma_{\delta t} = \frac{\delta_{1t}}{\delta_t}\sigma_{\delta_1 t} + ... + \frac{\delta_{Nt}}{\delta_t}\sigma_{\delta_N t}.$$

Investor 1 is endowed with s_n units of stock n and $-b$ units of bond, while investor 2 is endowed with $1 - s_n$ units of stock n and b units of bond. Investor i's wealth process W follows

$$dW_{it} = \Big[W_{it}\Big(r_t + \theta_{it}^{\top}(\mu_t - r_t)\Big) - c_{it}\Big]dt + W_{it}\theta_{it}^{\top}\sigma_t dw_t, \tag{87}$$

and her investment policies are subject to portfolio constraints

$$\theta_i \in \Theta_i, \quad i = 1, 2, \tag{88}$$

where Θ_i is a closed convex set in $\mathbb{R}^N$ and $\theta = (\theta_1, ..., \theta_N)^\top$ is the vector of wealth proportions invested in the N stocks. Each investor i solves her dynamic optimization (6) subject to budget constraint (87), no-bankruptcy constraint $W_t \geq 0$ and portfolio constraints (88).

Following the approach of Section 2 we first embed the optimization problem for each investor into an equivalent fictitious complete-market economy in which stock prices evolve as

$$d\xi_{it} = -\xi_{it}[r_{it}dt + \kappa_{it}^\top dw_t]. \tag{89}$$

Assuming that dual problems in Cvitanic and Karatzas (1992) have solutions we obtain that riskless rates r_{it} and market prices of risk κ_{it} in fictitious economy are given by

$$r_{it} = r_t + f_i(\nu_{it}^*), \qquad \kappa_{it} = \kappa_t + \sigma_t^{-1}\nu_{it}^*, \tag{90}$$

where κ is the market price of risk in the original economy, $f_i(\nu)$ are support functions for the sets Θ_i, defined as

$$f_i(\nu) = \sup_{\theta \in \Theta_i}(-\nu^\top \theta), \tag{91}$$

ν_{1t}^* and ν_{2t}^* solve duality optimization problem in Cvitanic and Karatzas (1992) and belong to the effective domains for support functions, given by

$$\Upsilon_i = \{\nu \in \mathbb{R}^N : f_i(\nu) < \infty\}. \tag{92}$$

Proposition 5 characterizes the equilibrium in terms of the adjustments ν_{it}^* and $f(\nu_{it}^*)$ in fictitious economies and the parameters of the process for the ratio of marginal utilities of consumption, λ_t, which evolves as

$$d\lambda_t = -\lambda_t[\mu_{\lambda t}dt + \sigma_{\lambda t}^\top dw_t]. \tag{93}$$

Proposition 5. *If there exists an equilibrium, the riskless interest rate r, market price of risk κ, drift μ_λ and volatility σ_λ of weighting process λ that follows (93) are given by*

$$r_t = \bar{r}_t - \frac{A_t}{A_{1t}} f_1(\nu_{1t}^*) - \frac{A_t}{A_{2t}} f_2(\nu_{2t}^*) - \frac{A_t^3(P_{1t} + P_{2t})}{2A_{1t}^2 A_{2t}^2} \sigma_{\lambda t}^\top \sigma_{\lambda t} - \frac{A_t^3}{A_{1t}A_{2t}} \Big(\frac{P_{1t}}{A_{1t}} - \frac{P_{2t}}{A_{2t}}\Big) \delta_t \sigma_{\delta t}^\top \sigma_{\lambda t}, \tag{94}$$

$$\kappa_t = \bar{\kappa}_t - \frac{A_t}{A_{1t}} \sigma_t^{-1} \nu_{1t}^* - \frac{A_t}{A_{2t}} \sigma_t^{-1} \nu_{2t}^*, \tag{95}$$

$$\mu_{\lambda t} = A_t \delta_t \sigma_{\delta t}^\top \sigma_{\lambda t} + f_1(\nu_{1t}^*) - f_2(\nu_{2t}^*) - \frac{A_t}{A_{1t}} \sigma_{\lambda t}^\top \sigma_{\lambda_t}, \qquad \sigma_{\lambda t} = \sigma_t^{-1}(\nu_{1t}^* - \nu_{2t}^*), \tag{96}$$

where $\bar{r}$ is the riskless rate and $\bar{\kappa}$ is the market price of risk in an unconstrained economy, given by

$$\bar{r}_t = \rho + A_t \delta_t \mu_{\delta t} - \frac{A_t P_t}{2} \delta_t^2 \sigma_{\delta t}^\top \sigma_{\delta t}, \quad \bar{\kappa}_t = A_t \delta_t \sigma_{\delta t}, \tag{97}$$

A_{it}, P_{it}, and A_t and P_t are absolute risk aversions and prudence parameters of investor i and a representative investor with utility (19), respectively. Optimal consumptions c_i^*, wealths W_i and optimal investment policies θ_i^* are given by

$$\begin{aligned} c_{it}^* &= g_i(\delta_t, \lambda_t), & (98) \\ W_{it}^* &= \frac{1}{\xi_{it}} E_t\Big[\int_0^\infty \xi_{is} c_{is}^* ds\Big], & (99) \\ \theta_{it}^* &= \sigma_t^{-1}\Big(W_{it}^*(\kappa_t + \sigma_t^{-1}\nu_{it}^*) + \frac{\phi_{it}}{\xi_{it}}\Big), & (100) \end{aligned}$$

where functions $g_i(\delta_t, \lambda_t)$ are such that c_{1t}^* and c_{2t}^* satisfy consumption clearing in (9) and equation (18) for process λ, state prices ξ_{it} follow processes (14) and ϕ_i are such that

$$M_{it} \equiv E_t\Big[\int_0^\infty \xi_{is} c_{is}^* ds\Big] = M_{i0} + \int_0^t \phi_{is}^\top dw_s.$$

Initial value λ_0 is such that budget constraints at time 0 are satisfied:

$$s_{i1} S_{10} + \ldots + s_{iN} S_{N0} + b_i = W_{i0}^*, \tag{101}$$

where $s_{1n} = s_n$, $s_{2n} = 1 - s_n$, $b_1 = -b$ and $b_2 = b$. Moreover, adjustments ν_{it}^* satisfy complementary slackness condition

$$f_i(\nu_{it}^*) + \theta_{it}^{*\top} \nu_{it}^* = 0. \tag{102}$$

The expression for interest rates (94) can again be decomposed into three groups of terms that represent riskless rate in an unconstrained economy, the impact of constraints and the effect of risk sharing. The last term in (94) also shows that in the case of heterogeneous utility functions the interest rates depend on the covariance between aggregate dividend and weighting process λ, captured by $\sigma_\delta^\top \sigma_\lambda$. The expression for equilibrium interest rates also allows to formulate a simple sufficient condition under which the equilibrium interest rates in the constrained economy are lower than in the unconstrained one.

Corollary 3. *If the utility functions and the allocations of consumption are such that $P_1/A_1 = P_2/A_2$ and the sets of portfolio constraints Θ_i contain the origin, i.e. $0 \in \Theta_i$, then the interest rate in a constrained economy, r, is lower than in an unconstrained one, $\bar{r}$, and the following upper bound for rate r holds:*

$$r_t \le \bar{r}_t - \frac{A_t^3(P_{1t} + P_{2t})}{2A_{1t}^2 A_{2t}^2} \sigma_\lambda^\top \sigma_\lambda. \tag{103}$$

The expressions for the market price of risk now reflect the impact of multiple constraints. By contrast with the single stock case, market clearing conditions can only determine the aggregate value of all stocks and not the values of individual ones. Moreover, as demonstrated in Hugonnier (2008) if the weighting process is not a martingale then there might be multiple equilibria with different stock prices but unique riskless rates and market prices of risk. The application of the methodology developed in Section 3 for finding equilibria in a multi-stock economy we leave for the future research.

5. Conclusion

Despite numerous applications of dynamic equilibrium models with heterogeneous investors facing portfolio constraints, little is known about the equilibrium when we depart from the assumption of logarithmic preferences. In various frameworks we provide explicit expressions for interest rates and market prices of risk in terms of instantaneous volatilities of stock returns and consumptions as well as risk aversions and prudence parameters. We then consider an economic setting where one investor is unconstrained while the other faces upper bound on the proportion that can be invest in stocks, and both investors have identical CRRA utilities. We completely characterize the equilibrium in terms of investors' wealth-consumption ratios satisfying a pair of differential equations that we solve numerically by employing a simple iterative algorithm. We further demonstrate that the direction in which portfolio constraints change price-dividend ratios and stock returns volatilities crucially depends on the intertemporal elasticity of substitution (IES). In particular, when the IES is greater than unity the model generates countercyclical market prices of risk and stock return volatilities, procyclical price-dividend ratios, excess volatility and other patterns consistent with empirical findings. We also find that the impact of constrained investor diminishes in the course of time but is still significant even after one hundred years. Our approach is then extended to the case of heterogeneous beliefs and multiple assets. Given the tractability of our analysis we believe that our approach for finding equilibria in economies with constraints may find applications in various models with heterogeneous investors and incomplete financial markets as well as in solving portfolio choice problems with constraints at a partial equilibrium level.

Appendix A: Proofs

Proof of Proposition 1. First, we obtain a system of equations for parameters of the fictitious economy by substituting the expressions for optimal consumption (17) into consumption clearing condition in (9), applying Itô's Lemma to both sides and matching the coefficients. Noting from the properties of inverse functions that

$$I_i'(\psi_i e^{\rho t}\xi_{it}) = \frac{1}{u_i'(c_{it}^*)}, \quad I_i''(\psi_i e^{\rho t}\xi_{it}) = -\frac{u_i''(c_{it}^*)}{u_i'(c_{it}^*)}\frac{1}{(u_i'(c_{it}^*))^2},$$

we obtain the following equations

$$\frac{r_t-\rho}{A_t} + \frac{f_1(\nu_{1t}^*)}{A_{1t}} + \frac{f_2(\nu_{2t}^*)}{A_{2t}} + \frac{1}{2}\Big(P_{1t}\Big(\frac{\kappa_{1t}}{A_{1t}}\Big)^2 + P_{2t}\Big(\frac{\kappa_{2t}}{A_{2t}}\Big)^2\Big) = \delta_t\mu_{\delta t}, \tag{A.1}$$

$$\frac{\kappa_{1t}}{A_{1t}} + \frac{\kappa_{2t}}{A_{2t}} = \delta_t\sigma_{\delta t}. \tag{A.2}$$

By applying Itô's Lemma to both sides of the definition of λ in (18) and noting that marginal utilities $u_i'(c_i^*)$ are given by (16), matching the terms we obtain the drift μ_λ and volatility σ_λ of the weighting process (20):

$$\mu_{\lambda t} = \sigma_{\lambda t}\kappa_{2t} + f_1(\nu_{1t}^*) - f_2(\nu_{2t}^*), \quad \sigma_{\lambda t} = \kappa_{1t} - \kappa_{2t}. \tag{A.3}$$

Taking into account the definition of κ_{it} in terms of adjustments in (15) from equations (A.1)–(A.3) we obtain expressions (21)–(23) in Proposition 1. Analogously, it can be shown that in the unconstrained economy the interest rate is given by (24).

Optimal consumptions c_{it}^* are obtained from consumption clearing and the equation for weight λ in (18). Expressions for optimal wealths and optimal policy (26) and (28) follow from the results in Cox and Huang (1989), Huang and Pages (1992) and Karatzas and Shreve (1998), while stock prices (3) are derived from the market clearing conditions in (9). The complementary slackness condition in (30) is established in Chapter 6.3 of Karatzas and Shreve (1998). *Q.E.D.*

Proof of Corollary 1. The proof directly follows from Proposition 1 by noting that the last term in the expression for r in (21) disappears. *Q.E.D.*

Proof of Proposition 2. We obtain expressions (44)–(48) for equilibrium parameters from expressions (21)–(25) in Proposition 1 by substituting adjustment parameters (39) and risk-aversion and prudence parameters for CRRA preferences, given by

$$\begin{aligned} A_{1t} &= \frac{\gamma}{c_{1t}}, & A_{2t} &= \frac{\gamma}{c_{2t}}, & A_t &= \frac{\gamma}{\delta_t}, \\ P_{1t} &= \frac{1+\gamma}{c_{1t}}, & P_{2t} &= \frac{1+\gamma}{c_{2t}}, & P_t &= \frac{1+\gamma}{\delta_t}. \end{aligned} \tag{A.4}$$

We first demonstrate that the constraint for investor 2 should always be binding in equilibrium. The complementary slackness condition (30), given expressions for adjustments (39), takes the form $(\bar{\theta} - \theta^*_{2t})\nu^*_{2t} = 0$. Therefore, if constraint does not bind it follows that $\nu^*_{2t} = 0$. Hence, from (23) we obtain that $\sigma_{\lambda t} = 0$ and $\mu_{\lambda t} = 0$ and the economy will permanently remain in a Pareto-efficient unconstrained equilibrium. As a result, since the investors have identical preferences and the equilibrium investment opportunity sets are constant when $\sigma_{\lambda t} = 0$ and $\mu_{\lambda t} = 0$, it can easily be verified that the investors will choose $\theta^*_{it} = 1$, which violates constraint $\theta_{2t} \leq \bar{\theta} < 1$. Therefore, the constraint should always be binding in equilibrium.

Expressions for wealths W^*_{it} follow from the first order condition for consumption in (41), while the expression for price-dividend ratio R follows from the expression for stock price (3), derived from consumption clearing, and the expressions for wealths in (49). Optimal policy for investor 1, θ^*_{1t}, in (51) is obtained by solving an HJB equation, while policy for investor 2 equals $\bar{\theta}$ since the investor always binds on her constraint, as demonstrated below. Stock return volatility σ in (52) is derived by applying Itô's Lemma to stock price, given by $S_t = R_t\delta_t$.

From the definition of κ_{it} in (15), expression for κ_t in (45) and expressions for adjustments in (39) we find that

$$\kappa_{2t} = \gamma\sigma_\delta - \frac{1}{1+\lambda_t^{1/\gamma}}\sigma_\lambda. \tag{A.5}$$

Substituting κ_{2t} from (A.5) into expression for optimal investment policy (43) and noting that constraint $\theta_{2t} \leq \bar{\theta}$ is always binding we obtain the following equation for σ_λ:

$$\frac{1}{\gamma\sigma_t}\Big(\gamma\sigma_\delta - \sigma_{\lambda t}\Big(\frac{1}{1+\lambda_t^{1/\gamma}} + \gamma\frac{\partial H_{2t}}{\partial \lambda_t}\frac{\lambda_t}{H_{2t}}\Big)\Big) = \bar{\theta}. \tag{A.6}$$

Substituting volatility σ given by first expression in (52) into equation (A.6) and solving it yields σ_λ given by second expression in (52). Finally, the equation for λ_0 is obtained so as to satisfy time-0 budget constraints (29). By substituting W^*_{10}, W^*_{20} and $S_0 = R_0\delta_0$ from Proposition 2 into the budget constraints (29) it can easily be observed that both constraints are satisfied whenever equation (53) for λ_0 holds.[14] *Q.E.D.*

[14] We also note that the results of Proposition 2 can be derived without relying on the methodology in Cvitanic and Karatzas (1992) by solving the HJB for investor 2 directly in constrained economy. Since the constraint is always binding the problem is equivalent to the one with constraint $\theta_{2t} = \bar{\theta}$. The HJB equation is then given by (38) in which $\theta_{2t} = \bar{\theta}$ and $\nu^*_{it} = 0$, since we solve in constrained economy. Then, conjecturing that J_{2t} has form (40) yields the equation for H_{2t}. From the first order condition (41) we obtain $e^{-\rho t}W_{2t}^{-\gamma}H_{2t}^{\gamma} = \xi_{2t}$, where ξ_{2t} is the marginal utility of investor 2 which follows the process (14). Applying Itô's Lemma to both sides shows that

$$\bar{\theta}\sigma_t = \frac{\kappa_{2t}}{\gamma} - \sigma_{\lambda t}\frac{\partial H_{2t}}{\partial \lambda_t}\frac{\lambda_t}{H_{2t}}.$$

Substituting this expression into HJB after some algebra we obtain equation (42) for investor 2. Price of risk κ_2 can be found from (A.2)–(A.3) while r_2 can be found by applying Itô's Lemma to $\xi_{2t}W_{2t} = e^{-\rho t}W_{2t}^{1-\gamma}H_{2t}^{\gamma}$, noting that the right-hand side satisfies HJB equation, $\theta_{2t} = \bar{\theta}$, and matching the terms.

Moreover, since investor 1 faces complete market, in the derivation of r_t and κ_t to obtain equations (A.1)–(A.2) we assume that $u'(c^*_{1t}) = \psi_1 e^{\rho t}\xi_{1t}$ where

$$\xi_{1t} = -\xi_{1t}[r_t dt + \kappa_t dw_t].$$

Proof of Corollary 2. Applying Itô's Lemma to both sides of the first order conditions for consumption (16) and matching the terms we find that

$$c_{it}\sigma_{c_i t} = \frac{\kappa_{it}}{A_{it}}. \tag{A.7}$$

Since investor 1 is unconstrained, $\kappa_1 = \kappa$ and is given by (45) while κ_2 is given by (A.5). Substituting κ_1 and κ_2 into (A.7) and noting that for CRRA utility $A_i = \gamma/c_i$ we obtain expressions (57) for volatilities σ_{c_i}. *Q.E.D.*

Proof of Proposition 3. From expression (61) we first express Brownian motion w^p in terms of Brownian motion w^o as follows:

$$dw_t^p = \Delta\mu_{\delta t}dt + dw_t^o, \tag{A.8}$$

and then rewrite all subsequent stochastic processes in terms of Brownian motion w^o under the optimist's probability measure. Then, state prices ξ_{it} in fictitious economies follow processes:

$$d\xi_{ot} = -\xi_{ot}[r_{ot}dt + \kappa_t^o dw_t^o], \quad d\xi_{pt} = -\xi_{pt}[(r_{pt} + \Delta\mu_{\delta t}\kappa_t^p)dt + \kappa_t^p dw_t^o]. \tag{A.9}$$

Optimal consumptions in fictitious economies are given by (17). Substituting them into consumption clearing condition in (9), applying Itô's Lemma to both sides and matching terms as in the proof of Proposition 1 after some algebra we obtain:

$$\frac{r_t - \rho}{A_t} + \frac{f_o(\nu_{ot}^*)}{A_{ot}} + \frac{f_p(\nu_{pt}^*)}{A_{pt}} + \frac{1}{2}\Big(P_{ot}\Big(\frac{\kappa_t^o}{A_{ot}}\Big)^2 + P_{pt}\Big(\frac{\kappa_t^p}{A_{pt}}\Big)^2\Big) = \frac{\kappa_t^o}{A_{ot}}\frac{\mu_{\delta t}^o}{\sigma_{\delta t}} + \frac{\kappa_t^p}{A_{pt}}\frac{\mu_{\delta t}^p}{\sigma_{\delta t}}, \tag{A.10}$$

$$\frac{\kappa_t^o}{A_{ot}} + \frac{\kappa_t^p}{A_{pt}} = \delta_t\sigma_{\delta t}. \tag{A.11}$$

By applying Itô's Lemma to both sides of the definition of λ in (18) and noting that marginal utilities $u_i'(c_i^*)$ are given by (16) and state prices follow (A.9), matching the terms we obtain the drift μ_λ and volatility σ_λ of the weighting process (66) for the optimist:

$$\mu_{\lambda t}^o = \sigma_{\lambda t}\kappa_t^p - \Delta\mu_{\delta t}\kappa_t^p + f_o(\nu_{ot}^*) - f_p(\nu_{pt}^*), \quad \sigma_{\lambda t} = \kappa_t^o - \kappa_t^p. \tag{A.12}$$

Using equations (A.10), (A.11) and the second equation in (A.12) we obtain expressions for r and κ in Proposition 3.

To obtain drift $\mu_{\lambda t}^p$ we rewrite the process for λ_t given by (66) under the Brownian motion of the optimist as follows

$$d\lambda_t = -\lambda_t[(\mu_{\lambda t}^p + \sigma_{\lambda t}\Delta\mu_{\delta t})dt + \sigma_{\lambda t}dw_t^o].$$

Huang and Pages (1992) derive this result assuming that $\int_0^t |r_\tau|d\tau < \infty$ a.s., and $\kappa_t < \bar{K}$ a.s., where $\bar{K}$ is a constant. It is difficult to check these conditions analytically. However, the graphs on Figure 3 demonstrate that the states with y close to 1, where r_t and κ_t are unbounded, have zero probability, and hence, the conditions are likely to be satisfied. We also check numerically that the integrals in investor's optimization (6) converge to J_{it} derived in Section 3.

Matching the drift parameters for the processes for λ_t from both optimist's and pessimist's points of view yields expression for $\mu^p_{\lambda t}$ in Proposition 3. To obtain expression for σ_λ we note first that by the definition of prices of risk in fictitious economies

$$\kappa^o_t = \frac{\mu^o_t - r_t}{\sigma_t} + \frac{\nu^o_{it}}{\sigma_t}, \quad \kappa^p_t = \frac{\mu^p_t - r_t}{\sigma_t} + \frac{\nu^p_{it}}{\sigma_t}. \tag{A.13}$$

Moreover, rewriting the process for stock prices S_t for both the optimist and pessimist in terms of Brownian motion w^o

$$\begin{aligned} dS_t &= S_t[\mu^o_t dt + \sigma_t dw^o_t] \\ &= S_t[(\mu^p_t + \Delta\mu_{\delta t}\sigma_t)dt + \sigma_t dw^o_t], \end{aligned}$$

and matching the terms we obtain

$$\frac{\mu^o_t - \mu^p_t}{\sigma_t} = \Delta\mu_{\delta t}. \tag{A.14}$$

The expression for σ_λ in (A.12) along with equations (A.13) and (A.14) gives σ_λ reported in Proposition 3. The rest of the proof is as in Proposition 1. *Q.E.D.*

Proof of Proposition 4. From the definition of the support function in (12) applied to $\theta \geq \underline{\theta}$ and the expression (70) for volatility σ_λ we obtain the adjustment parameters:

$$\nu^*_{1t} = 0, \quad f(\nu^*_{1t}) = 0, \quad \nu^*_{2t} = \sigma_t(\Delta\mu_{\delta t} - \sigma_{\lambda t}), \quad f(\nu^*_{2t}) = -\underline{\theta}\sigma_t(\Delta\mu_{\delta t} - \sigma_{\lambda t}). \tag{A.15}$$

Substituting adjustments (A.15) and risk-aversion and prudence parameters in (A.4), into the expressions (67)–(71) we obtain equilibrium parameters (73)–(76) reported in Proposition 4.

Consumptions (77) are obtained from the consumption clearing condition in (9) and definition of λ_t in (18). Wealth-consumption ratios H^o and H^p satisfy HJB equations (42) in which μ_λ is replaced by μ^o_λ and μ^p_λ respectively. Hence, from the first order condition for consumption in (41) and market clearing condition we obtain expressions for W^*_{it} and R_t. Expressions for optimal policies are obtained by solving HJB equations in fictitious economies, as in Section 3, while stock return volatility σ is obtained by applying Itô's Lemma to stock price $S_t = R_t\delta_t$.

The complementary slackness condition in (30) in our setting takes the form $(\underline{\theta} - \theta^*_{it})\nu^*_{it} = 0$. As a result, if constraint is not binding $\nu^*_{it} = 0$, and hence, from the expression in (70) it follows that $\sigma_\lambda = \Delta\mu_{\delta t}$. To solve for σ_λ when the constraint is binding we first substitute κ^p from (74) into the investment policy (80) and obtain

$$\theta^*_{pt} = \frac{1}{\gamma\sigma_t}\Big(\gamma\sigma_\delta - \sigma_{\lambda t}\Big(\frac{1}{1+\lambda_t^{1/\gamma}} + \gamma\frac{\partial H^p_t}{\partial\lambda_t}\frac{\lambda_t}{H^p_t}\Big)\Big). \tag{A.16}$$

Then, substituting σ from (81) into (A.16) and solving equation $\theta^*_{pt} = \underline{\theta}$ we obtain

$$\sigma_{\lambda t} = \frac{(1-\underline{\theta})\gamma\sigma_\delta}{\frac{1}{1+\lambda_t^{1/\gamma}} + \gamma\frac{\partial H_{pt}}{\partial\lambda_t}\frac{\lambda_t}{H_{pt}} - \underline{\theta}\gamma\frac{\partial R_t}{\partial\lambda_t}\frac{\lambda_t}{R_t}}. \tag{A.17}$$

Moreover, since $\nu^*_{2t} \geq 0$ [Table 1 case (e)] if the constraint binds σ_λ is given by (A.17) and should be lower than $\Delta\mu_{\delta t}$ which leads to expression for σ_λ in Proposition 4.[15] *Q.E.D.*

Proof of Proposition 5. The proof is a multi-dimensional version of the proof of Proposition 1. *Q.E.D.*

Proof of Corollary 3. From the definition of support functions in (12) it follows easily that $f_i(\nu) \geq 0$ if $0 \in \Theta_i$. Then, the proof follows from the fact that in the expression for interest rates r in Proposition 5 the second and third terms are positive while the last term vanishes. *Q.E.D.*

[15] Similarly to the discussion in the footnote in the proof of Proposition 2 it can be argued that the results in Proposition 4 can be obtained without relying on the methodology in Cvitanic and Karatzas (1992).

Appendix B: Numerical Method

We here present the details of our numerical solution method in Section 3 first for $\gamma > 1$ and then for $\gamma < 1$. Since variable λ takes values in the interval $(0, +\infty)$ we first rewrite the HJB equations (42) in terms of variable $y = \lambda^{1/\gamma}/(1 + \lambda^{1/\gamma})$. By $\tilde{H}_i(y, t)$ we denote the wealth-consumption ratios as functions of y so that

$$H_i(\lambda, t) = \tilde{H}_i(y(\lambda), t). \tag{B.1}$$

The derivatives of $H_i(\lambda, t)$ then can be expressed in terms of derivatives of $\tilde{H}_i(y, t)$ by differentiating both sides in (B.1) as follows:

$$\frac{\partial H_{it}}{\partial t} = \frac{\partial \tilde{H}_{it}}{\partial t}, \quad \lambda_t \frac{\partial H_{it}}{\partial \lambda_t} = \frac{y(1-y)}{\gamma} \frac{\partial \tilde{H}_{it}}{\partial y}, \tag{B.2}$$

$$\lambda_t^2 \frac{\partial^2 H_{it}}{\partial \lambda_t^2} = \frac{y^2(1-y)^2}{\gamma^2} \frac{\partial^2 H_{it}}{\partial y^2} + \frac{2y(1-y)((1-\gamma)/2 - y)}{\gamma^2} \frac{\partial \tilde{H}_{it}}{\partial y}. \tag{B.3}$$

Taking into account our change of variable and the expressions for derivatives in (B.2)–(B.3) from the expressions in Proposition 2, definitions of parameters r_{it} and κ_{it} in (15), and expressions for adjustment parameters in (39) we obtain the following expressions for equilibrium parameters in fictitious economies:

$$\begin{aligned} r_{1t} &= \bar{r} - \frac{y}{1-y}\bar{\theta}\sigma_t\sigma_{yt} - \frac{1+\gamma}{2\gamma}\frac{y}{1-y}\sigma_{yt}^2, \quad \kappa_{1t} = \gamma\sigma_\delta + \frac{y}{1-y}\sigma_{yt}, \\ r_{2t} &= \bar{r} + \bar{\theta}\sigma_t\sigma_{yt} - \frac{1+\gamma}{2\gamma}\frac{y}{1-y}\sigma_{yt}^2, \quad \kappa_{2t} = \gamma\sigma_\delta - \sigma_{yt}, \\ \mu_{\lambda t} &= \frac{\mu_{yt}}{1-y}, \quad \sigma_{\lambda t} = \frac{\sigma_{yt}}{1-y} \end{aligned} \tag{B.4}$$

where $\bar{r}$ is given by (47), μ_{yt}, σ_t and σ_{yt} are given by

$$\mu_{yt} = \gamma\sigma_\delta\sigma_{yt} - \bar{\theta}\sigma_t\sigma_{yt} - \sigma_{yt}^2, \quad \sigma_t = \sigma_\delta - \frac{\sigma_{yt}}{\gamma}\frac{\partial \tilde{R}_t}{\partial y_t}\frac{y_t}{\tilde{R}_t}, \quad \sigma_{yt} = \frac{(1-\bar{\theta})\gamma\sigma_\delta}{1 + \frac{\partial \tilde{H}_{2t}}{\partial y_t}\frac{y_t}{\tilde{H}_{2t}} - \bar{\theta}\frac{\partial \tilde{R}_t}{\partial y_t}\frac{y_t}{\tilde{R}_t}}, \tag{B.5}$$

and $\tilde{R}_t$ is a price-dividend ratio as a function of y. Substituting expressions for derivatives (B.2) and (B.3) into the HJB equations (42) we obtain the following PDEs for $\tilde{H}_{it}$:

$$\begin{aligned} &\frac{\partial \tilde{H}_{it}}{\partial t} + \frac{y_t^2\sigma_{yt}^2}{2\gamma^2}\frac{\partial^2 \tilde{H}_{it}}{\partial y_t^2} + \frac{y_t}{\gamma^2}\Big(\sigma_{yt}^2\frac{(1-\gamma)/2 - y_t}{1-y_t} - \gamma\mu_{yt} - (1-\gamma)\kappa_{it}\sigma_{yt}\Big)\frac{\partial \tilde{H}_{it}}{\partial y_t} \\ &+ \frac{1}{\gamma}\Big(\frac{1-\gamma}{2\gamma}\kappa_{it}^2 + (1-\gamma)r_{it} - \rho\Big)\tilde{H}_{it} + 1 = 0, \quad i = 1, 2. \end{aligned} \tag{B.6}$$

To find stationary, time-independent solutions of equations (B.6) we fix a large horizon T, pick two functions $\tilde{h}_1(y)$ and $\tilde{h}_2(y)$, specify terminal condition

$$\tilde{H}_i(y, T) = \tilde{h}_i(y), \quad i = 1, 2, \tag{B.7}$$

and solve HJB equations (B.6) backwards until the convergence to stationary solutions. We assume that functions $\tilde{h}_i$ are continuous and differentiable on the interval $[0,1]$ and satisfy conditions $\tilde{h}_1(1)=0$ and $\tilde{h}_2'(1)=(\gamma-1)\tilde{h}_2(1)$.

We assume that $\tilde{H}_i(y,t)$ are twice continuously differentiable in the interval $(0,1)$, have bounded first and second right derivatives at $y=0$, $\sigma_y^2>0$, and there exist limits $(1-y)^2\partial^2\tilde{H}_1(y,t)/\partial y^2\to 0$, $(1-y)\partial^2\tilde{H}_2(y,t)/\partial y^2\to 0$ and $(1-y)\partial\tilde{H}_1(y,t)/\partial y\to 0$, as $y\to 1$. After we compute the solutions we also verify numerically that these assumptions are satisfied for $\gamma>1$.

Passing to the limit $y\to 0$ in equations (B.6) we obtain simple ordinary differential equations for $H_i(0,t)$ solving which yields boundary conditions at $y=0$:

$$\tilde{H}_i(0,t)=\tilde{h}_i(0)e^{p_i(T-t)}+\frac{e^{p_i(T-t)}-1}{p_i},\quad i=1,2, \tag{B.8}$$

where

$$p_1=\frac{1-\gamma}{2}\bar{\theta}^2\sigma_\delta^2+\frac{(1-\gamma)\bar{r}-\rho}{\gamma},\quad p_2=\frac{1-\gamma}{2}\sigma_\delta^2+\frac{(1-\gamma)\bar{r}-\rho}{\gamma}. \tag{B.9}$$

Expressions in (B.8) and (B.9) demonstrate that conditions $p_i\le 0$ are necessary for the existence of stationary solutions of equations (B.6). To obtain boundary conditions at $y=1$ we multiply the equations for $H_1(y,t)$ and $H_2(y,t)$ by $(1-y)^2$ and $(1-y)$, respectively, and passing to the limit $y\to 1$ we obtain:

$$(1-\bar{\theta})(\gamma-1)\tilde{H}_1(1,t)=0,\quad \frac{\partial\tilde{H}_2(1,t)}{\partial y}=(\gamma-1)\tilde{H}_2(1,t). \tag{B.10}$$

The problem then becomes to solve HJB equations (B.6) subject to terminal condition (B.7) and boundary conditions (B.8) and (B.10).

For simplicity, in the description of the numerical method we omit subscript i. We let the time and state variable increments denote $\Delta t\equiv T/M$ and $\Delta y\equiv 1/N$, where M and N are integer numbers, and index time and state variables by $t=0,\Delta t,2\Delta t,...,T$ and $y=0,\Delta y,2\Delta y,...,1$, respectively. Next, we derive discrete-time analogues of HJB equations and boundary conditions replacing derivatives by their finite-difference analogues as follows:

$$\frac{\tilde{H}_{n,k+1}-\tilde{H}_{n,k}}{\Delta t}+a_{n,k+1}\frac{\tilde{H}_{n+1,k}-2\tilde{H}_{n,k}+\tilde{H}_{n-1,k}}{\Delta y^2}+b_{n,k+1}\frac{\tilde{H}_{n,k}-\tilde{H}_{n-1,k}}{\Delta y}+c_{n,k+1}\tilde{H}_{n,k}+1=0, \tag{B.11}$$

$$\tilde{H}_{n,M}=\tilde{h}_n,\quad \tilde{H}_{0,k}=d_{0,k},\quad \tilde{H}_{N,k}=e_{N,k}\tilde{H}_{N-1,k}, \tag{B.12}$$

where $n=1,2,...,N-1$, $k=1,2,...,M-1$, $\tilde{H}_{n,k}=\tilde{H}(n\Delta y,k\Delta t)$. The coefficients in (B.11) correspond to coefficients in equation (B.6) and are computed using the solution $\tilde{H}_{n,k+1}$, while coefficients in (B.11) are obtained by replacing terminal condition (B.7) and boundary conditions (B.8) and (B.10) by their finite-difference analogues. The system of equations in (B.11)–(B.12) is then solved backwards in time, starting at $k=M-1$. Given solution $H_{n,k+1}$ we compute all the coefficients in (B.11) at step $k+1$, and hence at step k function $H_{n,k}$ for fixed k solves a system of linear algebraic equations. We then iterate backwards until the process converges to a stationary time-independent solution.

Figure 1 shows the numerical solutions for wealth-consumption ratios plotted against constrained investors share of consumption, y, for plausible exogenous parameters. These numerical solutions have the appearance of bounded and twice continuously differentiable on interval $[0, 1]$ functions irrespective of the grid parameter Δy. Assuming that they are indeed twice continuously differentiable, and given that they satisfy finite-difference equations (B.11)–(B.12), passing to a limit $\Delta y \to 0$ indeed gives solutions to the HJB equations for wealth-consumption ratios.[16]

When risk aversion γ is less than unity wealth-consumption ratio H_{1t} and its derivatives become unbounded while σ_{yt} approaches zero, as y approaches unity. As a result, the assumptions under which the boundary conditions (B.8) and (B.10) are derived are violated. However, it turns out that function $(1-y)H_{1t}$ is bounded and equals zero when $y = 1$. Hence, we derive the differential equation for $(1-y)H_{1t}$ and solve it using the methodology described above.

The model with heterogeneous beliefs in Section 4.1 is solved in a similar way. First, we derive an HJB equation in terms of consumption share y, which is given by (B.6) in which μ_y is replaced by μ_y^i. Then, we obtain boundary conditions and solve the finite-difference equations numerically.

[16]As an additional check we also verify by Monte-Carlo simulations that for both investors integrals in their optimization problem (6) do not explode under optimal consumption policies in (48) and converge to the values obtained by our numerical method. The convergence of those integrals also implies that the transversality conditions for HJB equations (38) are satisfied.

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