Dynamic Margin Constraints*

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Abstract

This paper provides a theoretical analysis of margin requirements and their market-wide implications. The key feature of the model is time variation in margin constraints, which are endogenously determined by prevailing market conditions. I demonstrate that in the states when margins do not bind and investors are not financially constrained time variation in margins can have destabilizing effects: it reduces liquidity and increases volatility of returns as well as their correlations relative to the economy without margins. The effect is opposite when margin requirements are tight and investors optimally exhaust their debt capacity. A purely dynamic hedging demand produced by time variation in margins is responsible for the amplification effects. The destabilizing impact of margin requirements 1) is stronger for intermediately severe margin requirements 2) is more pronounced if individual assets are collateralized separately, and 3) is almost insensitive to the number of risky assets in the economy.

Keywords: margin requirements, collateral, hedging demand, volatility, liquidity
JEL classification: G11, G12

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1 Introduction

Margin debt is a pervasive phenomenon in the financial industry: many investors borrow money and construct leveraged portfolios. To protect themselves against losses caused by adverse price movements, lenders almost always require borrowers to put their portfolios as collateral and, in addition, impose certain limits (margin constraints) on maximum leverage. In many circumstances, the terms of margin borrowing are not fixed and depend on prevailing market conditions.[1] In this paper, I provide a theoretical analysis of time variation in margin constraints as well as its impact on volatility of returns, risk premium, liquidity, and correlation of returns.

Specifically, I demonstrate that time-varying margin constraints can have several important market-wide implications. First, the constraints can make returns more volatile when they do not bind and less volatile when they bind. Second, dynamic margins can make the market substantially less liquid when the probability that the constraints become binding in the nearest future is relatively high. Thus, variation in margins can explain why volatility of returns increases and liquidity "dries up" when market conditions deteriorate. In a multiple asset setting, time variation in margins can contribute to contagion: when investors are close to the point where they exhaust their debt capacity the correlation between returns on assets with independent fundamentals increases.

As a laboratory for analysis of these effects, I use a continuous time equilibrium model with infinite horizon and multiple risky assets. All assets are identical and pay stochastic dividends, which are uncorrelated across assets. Dividends are exogenous and represent fundamentals in the model. There are three types of agents in the economy dubbed as professional investors, long-term investors, and financiers. Professional investors and long-term investors have different preferences and demands and trade with each other. The role of financiers is to provide margin credit to professional investors and set margin constraints, which are endogenously determined by current market conditions.

The focus of this paper is the impact of hedging demand produced by time variation in margin requirements on the equilibrium. Even if margin constraints do not bind at a particular moment, the possibility of hitting the constraint in the future can substantially change the optimal portfolio policy of professional investors. A negative shock to fundamentals decreases prices and increases volatility. Financiers, who want to protect themselves against potential losses, tighten margin constraints. In response, because of hedging motives, current asset holders sell risky assets in advance driving the price even lower and volatility even higher. Further tightening of margins feeds back into further preventive selling, and this vicious circle amplifies the reaction of prices to changes in fundamentals and leads to high volatility. It is worth emphasizing that professional investors who sell assets to hedge the variation in margin constraints are not financially constrained and can, in principle, ignore changes in margins. Moreover, they do not face a risk to be constrained in the immediate future.

I start the analysis by considering a single asset setup. To identify the impact of dynamic margin constraints, I compare two otherwise identical economies with and without margins, where the economy without margins serves as a benchmark. I demonstrate that margin constraints significantly change the equilibrium: in the economy with margins, professional investors hold less assets; the Sharpe ratio

[1] For example, prime brokerage firms providing margin credit to hedge funds can unilaterally tighten margin requirements. In the year 2008 adverse changes in margins caused collapses of several high profile hedge funds. See additional evidence of time variation in margin requirements in Section 2.

is higher when margins bind; returns are more volatile and the liquidity is lower when investors are unconstrained but face the risk of hitting the constraint; returns are less volatile when margins bind.

To examine how hedging contributes to these results, I compare the economy with its modified analog, in which professional investors are replaced by myopic investors who do not anticipate changes in investment opportunities. Even without margin requirements the hedging demand plays an essential role and significantly affects the equilibrium making returns more volatile and markets less liquid. Moreover, I find that in the presence of hedging demand the impact of introducing margin requirements becomes more pronounced, i.e. they make volatility higher when margins do not bind and lower when they bind. Also, the hedging demand exacerbates the drop in liquidity caused by margin constraints.

The impact that margin constraints have in equilibrium depends on their *average* severity, which in turn is determined by how sensitive financiers are to uncertainty in the value of collateral. I show that the effect of margins is the strongest when they are on average at the intermediate level. Indeed, if financiers are very tolerant, margin requirements are loose and do not have a sizable effect on the economy. In the other extreme, financiers are very sensitive to potential losses. In this case margin constraints on average are very tight and bind almost all the time. Hence, professional investors typically do not hold a sizable amount of risky assets and their rebalancing has only a limited effect, which translates into a weak amplification of volatility.

A multiple asset version of the model provides additional insights making it possible to study the correlation of returns, which is endogenous in the model. Since it is assumed that fundamentals of individual assets are uncorrelated, the obtained correlation of returns is a purely equilibrium effect resulting from optimal behavior of professional investors. Even without margin constraints returns become correlated due to contagion caused by the wealth effect (Kyle and Xiong, 2001). While Kyle and Xiong (2001) consider only two assets, I extend their analysis to the case with an arbitrary number of assets and demonstrate that the wealth effect can become stronger as the number of assets available for investing increases.

If financiers impose margin requirements, their exact form depends on the collateralization scheme used by investors. I consider two types of collateralization. In the first scheme, portfolio is collateralized as a whole, i.e. each investor with a leveraged portfolio uses only one financier to get capital. In the second scheme, individual securities are collateralized separately and investors borrow money from several financiers such that the number of financiers coincides with the number of assets. I demonstrate that under both collateralization schemes margin requirements amplify volatility and contagion, i.e. increase correlation of returns despite independence of asset fundamentals. However, the effect is stronger if assets are collateralized individually. Thus, the model suggests that the tendency of hedge funds to use multiple prime brokerage firms can further magnify the destabilizing effect of dynamic margin constraints.

Finally, I analyze how the number of assets in the economy affects the impact of margin requirements on the volatility and correlation of returns. I find that the effect of introducing margins is almost insensitive to the number of assets. This result holds for both collateralization schemes and is especially surprising in the portfolio collateralization framework.

In equilibrium, agents simultaneously hedge two sorts of variation in investment opportunities: changes in the Sharpe ratio and changes in margin constraints. The first type of variation is conventional and was extensively studied in the literature. The second type is much less common, but it plays an important role in my analysis. To examine the hedging demand produced by time-varying margin requirements, I consider a separate portfolio problem in which it appears in a pure form. It is a standard continuous time portfolio problem with a constant Sharpe ratio, but with exogenously varying portfolio constraints. To maintain generality, I assume that margin requirements can either continuously change or experience sudden jumps. I show that if margins and asset prices are correlated, the optimal strategy as a function of the current tightness of margin requirements has two regions. If margins are relatively loose and margins and prices are positively (negatively) correlated, agents with relative risk aversion greater than one underinvest (overinvest) in the risky asset. Remarkably, the hedging demand of this type might have a significant magnitude: under reasonable assumptions on volatility of margins and their persistence an agent with a moderate risk aversion reduces investment in a risky asset by $10 - 15\%$. If margins are relatively tight, it is optimal to exhaust the credit limit and the optimal portfolio is on the margin constraint. The point where margins become binding is endogenously determined as a part of the solution.

In general, hedging is not the only possible channel through which trading activity can affect equilibrium prices and returns. Even without margin constraints volatility of returns can be different from the volatility of fundamentals due to the wealth effect (Xiong, 2001; Kyle and Xiong, 2001). The key idea is that a negative shock to fundamentals decreases the wealth of current asset holders and triggers portfolio rebalancing. As a result, they unwind some of their positions pushing the price down even further. This decline in price causes additional rebalancing and sell-offs amplifying the fundamental shock and increasing volatility of returns.

Another channel is directly related to margin constraints and can be called the collateral channel. An adverse shock to fundamentals reduces asset prices and, as a consequence, the value of collateral. If the shock is sufficiently large, investors might be forced to liquidate a part of their portfolio to meet margin calls even if the instantaneous Sharpe ratio is high and investors are willing to maintain higher leverage. The sell-off drives the price down even further instigating new margin calls. This channel works even if margin constraints are constant over time. If margin constraints can vary in response to changing market environment, financiers may tighten the terms of credit and make the sell-off even more pronounced. This is the liquidity spiral of Brunnermeier and Pedersen (2008).

While Brunnermeier and Pedersen (2008) is the closest paper to this study, there are several important differences between their analysis and this paper. First, the focus of this paper is the hedging demand produced by time-varying margin constraints and its equilibrium implications whereas Brunnermeier and Pedersen (2008) emphasize the importance of funding liquidity, i.e. availability of funding to risk neutral speculators. In my model, an amplification effect is produced by *anticipation* of limited funding, but not the lack of funding per se. Second, in Brunnermeier and Pedersen (2008) margins are destabilizing under certain conditions only. In particular, an asymmetry of information between financiers and speculators plays an important role: margins increase in price volatility because financiers cannot distinguish between fundamental shocks and liquidity shocks. In my model, there is no asymmetric information and the possibility of hitting margin constraints always has a destabilizing effect. Third, in Brunnermeier and Pedersen (2008) an important input of the model is autoregressive conditional heteroscedasticity of fundamentals. In this paper, fundamentals are homoscedastic and heteroscedasticity of returns arises endogenously.

This paper is related to several strands of literature. First, it compliments the literature analyzing the impact of *constant* margin requirements on stock prices in a general equilibrium. Using an OLG framework with heterogeneous agents, Kupiec and Sharpe (1991) show that the relation between constant margin requirements and stock price volatility is ambiguous and depends on the source of variation in the aggregate risk-bearing capacity. Detemple and Murthy (1997) focus on the price levels in the presence of portfolio constraints and demonstrate the existence of "collateral value premium". Chowdhry and Nanda (1998) show how in a two-period model constant margin requirements can produce market instability, which is understood as a multiplicity of prices clearing the market. Aiyagari and Gertler (1999) and Mendoza and Smith (2002) demonstrate that constant margin requirements can amplify fundamental shocks and increase the volatility of stock prices. Coen-Pirani (2005) emphasizes the importance of endogeneity of risk-free rate and the assumption that the risk-free asset is in zero net supply. If the risk-free rate is allowed to adjust in response to shocks, in general it will absorb them when margin constraints are binding and stock prices will be almost unaffected. Saha (2007) argues that stock return volatility decreases when the margin constraint binds.

Although time variation of margin constraints is common in financial markets, it is surprisingly underexplored in the academic literature. Yuan (2005) constructs a one-period rational expectation model in which borrowing constraint is a function of price. Risk-sensitive regulation can be interpreted as a form of dynamic margins (e.g., Basak and Shapiro, 2001; Daníelsson, Shin, and Zigrand, 2004; Daníelsson and Zigrand, 2008). Pavlova and Rigobon (2008) study the impact of portfolio constraints on stock prices and exchange rates in a three-good three-country general equilibrium model, in which agents have logarithmic preferences.

The analysis of the hedging demand generated by margin constraints fits into the vast literature studying the effect of changes in investment opportunities on the optimal portfolio policy. Following a seminal paper by Merton (1971), this literature examined variations in the risk-free rate (e.g. Brennan, Schwartz, and Lagnado, 1997; Campbell and Viceira, 2001; Liu, 2007), expected returns (e.g. Kim and Omberg, 1996; Campbell and Viceira, 1999; Barberis, 2000; Wachter, 2002), volatility (e.g. Chacko and Viceira, 2005; Liu, 2007), correlations of returns (e.g. Ang and Bekaert, 2002), and simultaneous variations in the interest rate and the market price of risk (Detemple, Garcia, and Rindisbacher, 2003). My paper emphasizes the role of a relatively non-standard hedging demand produced by variation in margin constraints.

This paper is also close to the literature studying the impact of portfolio constraints on the optimal portfolio policy (e.g. Cvitanić and Karatzas, 1992; Grossman and Vila, 1992; Vila and Zariphopoulou, 1997; Cuoco and Liu, 2000; Teplá, 2000; Schroder and Skiadas, 2003; Detemple and Rindisbacher, 2005). In my model, a portfolio problem with constraints is an important part of an equilibrium.

This paper is also related to the growing literature on the impact of margin constraints on the existence and limits of arbitrage. Basak and Croitoru (2000) consider a model with heterogeneous investors facing constant portfolio constraints and argue that in such framework mispricing can be maintained in an equilibrium. Gromb and Vayanos (2002) focus on welfare implications of arbitrageurs' financial constraints and show that arbitrageurs might fail to invest at a Pareto optimal level.

The rest of the paper is organized as follows. Section 2 summarizes anecdotal evidence on time variation of margin requirements and claims that it is a widespread phenomenon. In Section 3 a general equilibrium model is formulated. In this model, margin constraints are endogenous and determined by changing volatility of returns. Sections 4 and 5 provide the analysis of single asset and multiple asset economies, respectively. These sections contain the main results of the paper. Section 6 concludes by discussing potential directions for future research. Appendix A provides an additional analysis of the portfolio choice problem under time-varying margin requirements. This is a partial equilibrium analysis, in which dynamics of prices and margin constraints are exogenously specified. Appendix B collects all proofs and Appendix C provides details on numerical techniques used in the paper.

2 Time-varying margin requirements: anecdotal evidence

Portfolio constraints in the form of margin requirements (minimum capital requirements) are typical features of almost any investment environment. In particular, margin requirements play a fundamental role in the relationship between hedge funds and prime brokerage firms. Pursuing highly leveraged strategies, hedge funds significantly rely on availability of external credit. Provision of such credit is one of the most prominent functions of prime brokers, who typically structure loans to hedge funds as repurchase agreements (repos). The terms of repo agreements substantially vary across hedge funds and over time. The last fact is particularly important: the tightness of margin constraints is the outcome of private continuous negotiations between hedge funds and prime brokerage firms. Typically, the prime broker is permitted to tighten the margin requirement (modify collateral haircut) at its own discretion and even without advance notice.[2] The increase in the margin can be substantial and might have a huge impact on the fund's liquidity. It is natural to expect that when markets are strained or excessively volatile, prime brokers are more likely to use their discretionary right to adjust margin requirements.

There is abundant anecdotal evidence that during the recent subprime mortgage crisis prime brokerage firms indeed widely used their right to change margin requirements. For instance, according to Citigroup data, for leveraged investors buying AA-rated corporate bonds, the average down payment demanded by prime brokers has increased from 1-3% of the value of the leveraged portfolio in March 2007 to 8-12% in March 2008. This corresponds to the decline of maximum leverage from over 30 times to 8.3 times. Similarly, the haircut on high-yield bonds rated BB has increased from 10-15% to 25-40%. On equities the haircut has increased from 15% to 20%, which means leverage has fallen from 6.7 times in March 2007 to 5 times in March 2008 (Financial News, March 6, 2008). The most substantial change in margin requirements occurred for AAA CLOs: the haircut jumped from 4% in March 2007 to 20% in March 2008.

Tightening of margin requirements played an essential role in several recent hedge fund failures. For instance, Peloton ABS Fund, which collapsed in February 2008, right before its debacle made desperate attempts to comply with requests of banks (Goldman Sachs and UBS) to increase its collateral from 10% to 25% (Bloomberg, March 10). In fact, Peloton's margin requirements were changed many times before the end of February (HedgeWorld Daily News, May 9, 2008). Also, increasing margin requirements contributed to the collapse of Carlyle Capital in March 2008 (MarketWatch, March 7, 2008). A couple of weeks before its failure the fund issued a statement saying that it received "substantial additional margin calls and additional default notices from its lenders" and that "these

[2]Sometimes large and well-established hedge funds manage to negotiate lock-up agreements: the prime broker is not allowed to change margin requirements without advance notice if the fund meets certain conditions. A length of a lock-up period varies from one to three months.

additional margin calls and increased collateral requirements could quickly deplete its liquidity and impair its capital."

Trading on margin is regulated not only by brokerage firms providing the credit, but also by several agencies including the Federal Reserve, the SEC, and several more specialized associations. For instance, Regulation T of the Federal Reserve sets initial margin requirements, i.e. restricts the amount of credit that investors can get from brokers for the purpose of buying stocks. These requirements were introduced in October 1934, and at that time the required margin was set at 45%. From 1934 till 1974 this level was changed 22 times, fluctuating from 40% to 100%. Since January 1974 the initial margin requirement is 50%. Although margins have been constant over a long period of time, there is an ongoing discussion of this policy.

Buying on margin or entering into repo transactions are direct ways to construct a leveraged portfolio. However, leverage can also be increased indirectly by using various derivatives. For example, the Regulation T can be partially circumvented by using futures. Noteworthy, indirect leverage also creates exposure to variation in margin constraints. Indeed, the availability of derivatives and their liquidity can also vary over time implicitly producing time variation in margin requirements faced by investors. For exchange-traded derivatives margins are set by exchanges (brokerage firms might request additional collateral that exceeds exchange margins) and can be adjusted periodically in response to changing market conditions. For example, CME Group uses Standard Portfolio Analysis of Risk Performance (SPAN) methodology to compute margins. The major variable in this approach is volatility of each futures market. Typically, when volatility increases the margin requirements become tighter.

In addition, certain macroeconomic events might lead to implicit abrupt changes in margin requirements. For instance, consider the situation when the Federal Reserve introduces a new lending facility to inject liquidity in the financial system. It is quite likely that such a step would alleviate the liquidity needs of large investment banks, and their prime brokerage arms would be able to provide more capital to hedge funds. As a result, margin requirements might effectively become less tight. Also, margin requirements implicitly change when lenders decide to modify the list of securities they accept as a collateral.

Overall, there is abundant anecdotal evidence that margin requirements faced by different investors change over time. This fact might have important asset pricing implications, which are studied in the next sections.

3 Model

In this section, I construct an equilibrium model featuring endogenous time-varying margin constraints. This is a continuous-time framework with an infinite horizon.

3.1 Assets

There are $K + 1$ assets in the economy. The first one is a risk-free asset with an exogenously specified rate of return r. This asset is assumed to have an infinitely elastic supply. The other K assets are risky; an aggregate supply of each asset is normalized to $\bar{X}$. Cash flows generated by asset k follow an Ornstein-Uhlenbeck process

$$dD_t^k = \phi(\bar{D} - D_t^k)dt + \sigma_D dB_t^k, \tag{1}$$

where B_t^k, $k = 1, ..., K$ are independent standard Brownian motions defined on a probability space (Ω, F, P) and ϕ, $\bar{D}$, and σ_D are constants. For simplicity, it is assumed that all risky assets are identical with ϕ, $\bar{D}$, and σ_D being the same for all assets. In general, the price of the risky asset S_t^k is a function of the current level of cash flows D_t^k and state variables. It is convenient to define excess return on one unit of the asset k: $dQ_t^k = dS_t^k + D_t^k dt - rS_t^k dt$. In the vector form, the stochastic differential equation for Q_t can be represented as

$$dQ_t = \mu_Q dt + \sigma_Q dB_t,$$

where the risk premium μ_Q is a $k \times 1$ matrix and volatility of returns σ_Q is a $k \times k$ matrix. Both μ_Q and σ_Q are functions of state variables and are determined by equilibrium conditions stated below.

3.2 Agents

There are three types of agents in the model: professional investors, long-term investors, and financiers. Although the first two types trade risky assets, their objectives are completely different. Professional investors use all available information and choose the portfolio policy and consumption stream maximizing their utility function. They can be thought of as hedge funds who have a good understanding of market conditions and can rationally exploit fluctuations in asset prices. There is an infinite number of identical professional investors, and they behave competitively. Each individual investor has an infinitesimally small demand and they are all located on a unit interval $[0, 1]$.

Each professional agent i has Epstein-Zin preferences (Epstein and Zin, 1989; Weil, 1989) over an infinite stream of consumption C_{it}, which is financed by portfolio X_{it}. In continuous time, Epstein-Zin preferences can be described by stochastic differential utility with the utility process J_{it}, which in the Duffie and Epstein (1992a, b) parametrization is specified as[3]

$$J_{it} = E_t \left[\int_t^{\infty} f(C_{is}, J_{is}) ds \right], \tag{2}$$

where

$$f(C_{is}, J_{is}) = \beta(1-\gamma) J_{is} \left(\log C_{is} - \frac{1}{1-\gamma} \log((1-\gamma) J_{is}) \right) \tag{3}$$

In order to make the wealth process stationary it is assumed that $\beta > r$. Eq. (3) corresponds to Epstein - Zin preferences with the elasticity of intertemporal substitution (EIS) equal to one and the coefficient of risk aversion γ. The major benefit of unit EIS is that it simplifies the model due to a well-known property of Epstein - Zin preferences with unit EIS: the consumption-wealth ratio is constant and is equal to β (Giovannini and Weil, 1989; Schroder and Skiadas, 1999). The assumption of unit EIS is not critical for the intuition and it is quite likely that all qualitative results of the paper

[3]For proof of existence of stochastic differential utility in the infinite horizon case see Appendix C of Duffie and Epstein (1992b) and Duffie and Lions (1992).

would also be valid for other values of EIS. It is known that the elasticity of intertemporal substitution mostly affects the consumption policy, but not the portfolio strategy (Campbell and Viceira, 1999). Since the major focus of this paper is on the investment decisions, setting EIS equal to one does not significantly restrict the analysis allowing to examine the effect of risk aversion on the optimal portfolio allocation.

The second class of agents is formed by long-term investors. They are price takers with an exogenous aggregate demand for the risky asset k

$$X_t^{Lk} = \frac{F_t^k - S_t^k}{\bar{\gamma}}, \tag{4}$$

where S_t^k is the current price and F_t^k is the fundamental value of the asset k defined as a present value of future cash flows

$$F_t^k = E_t \int_t^{\infty} \exp(-rs) D_{t+s}^k ds = \frac{\bar{D}}{r} + \frac{D_t^k - \bar{D}}{r + \phi}. \tag{5}$$

The volatility of F_t^k is related to the volatility of dividends D_t^k and is equal to $\sigma_F = \sigma_D/(r + \phi)$. Demand for each asset is a downward-sloping function, so when there is a selling pressure on the market long-term investors absorb it only at a discounted price. Eq. (4) can be considered as a reduced form of demand arising in models with risk-averse market makers (Grossman and Miller, 1988). The constant $\bar{\gamma}$ can be interpreted as a measure of risk tolerance of long-term investors. It is convenient to think about long-term investors as pension plans, sovereign funds, or large individual investors.[4]

Long-term investors serve two functions in the model. On the one hand, they create trading opportunities for professional investors and allow them to earn high return by keeping risky assets. Long-term investors neither try to predict future changes in prices, nor exploit the knowledge of model state variables leaving these opportunities to professional investors. On the other hand, long-term investors are buyers of last resort. If professional investors become financially constrained and desperately need to sell a part of their risky portfolio, long-term investors will take the other side of the trade, but at deeply discounted prices.

The third group of agents consisting of financiers is relatively passive and does not trade on the market. Their role is to provide margin credit to professional investors and impose margin constraints.

3.3 Margin constraints

3.3.1 Motivation for margin requirements

The key feature of the model is margin constraints faced by professional investors. If existing investment opportunities are highly attractive, professional investors use borrowed money for investing in risky assets. Financiers, who provide capital, put restrictions on how leveraged the position of professional investors can be (i.e. they impose margin requirements). To determine the exact form of margins, it is necessary to pin down the reason why margin constraints exist in the first place.

[4] In extreme cases, even firms themselves can be buyers of last resort for their own equity (Hong, Wang, and Yu, 2008).

Although their existence is tantamount to credit rationing extensively studied in the corporate finance literature, the origin of margins in the financial industry is likely to be different.

To be specific, I assume that margin constraints result from two market imperfections. First, similar to Geanakoplos (1997, 2003) there are no penalty for defaulting, i.e. a borrower can go away without any legal or reputational consequences. This extreme assumption captures the idea of limited contract enforcement (Kehoe and Levine, 1993). This is especially plausible if a lender has to deal with a big number of relatively small borrowers. Although sometimes costs of enforcement can be recovered by setting a high interest rate, it is much more efficient to overcome the enforcement inefficiency by asking a borrower to put collateral. If the collateral has a well defined market value and can be easily sold, the lender is fully protected even if the borrower is highly leveraged.

The second imperfection assumed in the model is impossibility to sell collateral immediately at the prevailing market price if a borrower does not meet a margin call. Although the time the lender needs to complete the sell-off can be relatively short, under volatile market conditions the lender is exposed to risk of getting a price below the value of loan. Hence, if the equity part of collateral is small, the lender might not recover the loan in the case of default. To get some protection, the lender imposes restrictions on the maximum leverage the borrower is allowed to have and these restrictions are determined by current market conditions. This makes margin requirements endogenous and stochastic. Understanding the effect of this stochasticity is the focus of the analysis.

Although the described imperfections yield an exact form of margin requirements, the inverse relation between market conditions and tightness of margin constraints that they produce can be interpreted more broadly. Indeed, the above assumptions capture the idea that the value of the asset is not perfectly known to the financier, who after capturing the collateral has to deal with this uncertainty. Even if the collateral can be sold immediately, the price that the financier gets is highly uncertain and this results in a tight margin constraints.

I consider separately two types of margin requirements. If one financier provides capital for the whole portfolio, the dynamics of portfolio value is taken into account when margin requirements are determined. For instance, if individual asset returns are negatively correlated, then financing of a portfolio is less risky than financing its individual constituents and margins can be set on a lower level.[5] This is a *portfolio collateralization* case.

Although portfolio collateralization is quite natural, some investors prefer to collateralize several parts of their portfolios separately. For example, if a hedge fund wants to keep its arbitrage strategy in a secret from its prime broker it might borrow money from several prime brokerage firms to finance different legs of the strategy. In this case, only a part of portfolio is put as collateral with each financier who imposes individual margin requirements. For expositional purposes, I assume that all individual securities are collateralized separately and margin requirements are imposed on each asset. This is an *individual collateralization* case. In the next two sections, I derive the form of portfolio restrictions for these two types of margin requirements.

[5]E.g., CME group specifies different margin requirements for individual securities and certain strategies combining several securities.

3.3.2 Portfolio collateralization

First, consider a portfolio collateralization case. As explained above, if a borrower does not meet a margin call the portfolio sell-off requires time $\tau > 0$. If there is no need to liquidate the position, the lender at the moment $t+\tau$ will have $B_{it}(1+r\tau)$, where B_{it} is the amount of debt provided to investor i. If the borrower violates the margin requirement, the lender can capture $X'_{it}S_{t+\tau} + X'_{it}D_t\tau$, where the prime denotes matrix transposition. Thus, the lender does suffer a loss if $X'_{it}S_{t+\tau}+X'_{it}D_t\tau < B_{it}(1+r\tau)$ or $X'_{it}\Delta Q_{t+\tau} + W_{it}(1 + r\tau) < 0$. Here $\Delta Q_{t+\tau}$ is excess return from t to $t+\tau$ and W_{it} is wealth of investor i at time t. The lender sets the margin requirement such that the probability of loss is below a certain threshold b:

$$\text{Prob}(X'_{it}\Delta Q_{t+\tau} + W_{it}(1 + r\tau) < 0) \leq b, \tag{6}$$

i.e. the lender uses the value-at-risk approach (Brunnermeier and Pedersen, 2008). If τ is sufficiently small, the distribution of excess return $\Delta Q_{t+\tau}$ is approximately normal and in the first order approximation the condition (6) reduces to

$$\Phi\left(-\frac{W_{it}}{\sqrt{X'_{it}\sigma_Q\sigma'_Q X_{it}}\sqrt{\tau}}\right) \leq b,$$

where $\Phi(\cdot)$ is a cdf of the standard normal distribution. Hence, the margin constraint can be rewritten as

$$\sqrt{X'_{it}\sigma_Q\sigma'_Q X_{it}} \leq \bar{m}W_{it}, \tag{7}$$

where $\bar{m} = 1/(-\Phi^{-1}(b)\sqrt{\tau})$. The constraint is tight ($\bar{m}$ is low) when it takes a long time to sell the asset (τ is large) or the desired probability of loss b is small. Note that in general the margin constraint is not a simple limitation on portfolio leverage; it also takes into account the correlation structure of returns on individual assets. Importantly, margin constraints are dynamic and can be set conditional on the current realization of state variables determining volatility of returns.[6]

Note that if there is only one asset in the economy, Eq. (7) reduces to a direct constraint on the portfolio leverage L_{it}, which is a ratio of the risky asset value over investor's wealth

$$L_{it} = \frac{X_{it}S_t}{W_{it}} \leq \frac{\bar{m}S_t}{\sigma_Q}. \tag{8}$$

Eq. (8) has a natural interpretation: the maximum leverage attainable by professional investors is inverse to the volatility of dollar returns.

3.3.3 Collaterallization of individual assets

Next, consider the case in which each asset is collateralized separately. Investing in K assets, a professional investor borrows money from K lenders. Again, if a professional investor does not meet a margin call then the time τ is needed to liquidate the position. If there is no need to liquidate the position, the lender k at the moment $t + \tau$ will have $B^k_{it}(1 + r\tau)$, where B^k_{it} is the amount of debt provided to investor i by lender k. Following the same logic as before, the lender suffers a loss when $X^k_{it}\Delta Q^k_{t+\tau} + W^k_{it}(1 + r\tau) < 0$, where $W^k_{it} = X^k_{it}S^k_t - B^k_{it}$ is the wealth financing the position in asset k.

[6] Margin constraint in the form of Eq. (7) is effectively a limitation on the volatility of investors' return on wealth.

The lender sets the margin requirement such that the probability of loss is below a certain threshold b:

$$\text{Prob}(X_{it}^k \Delta Q_{t+\tau}^k + W_{it}^k(1+r\tau) < 0) \le b. \tag{9}$$

In the first order approximation in τ the condition (9) is

$$X_{it}^k \sqrt{(\sigma_Q \sigma_Q')_{kk}} \le \bar{m} W_{it}^k, \tag{10}$$

where $\bar{m} = 1/(-\Phi^{-1}(b)\sqrt{\tau})$ and $(\sigma_Q \sigma_Q')_{kk}$ is the k-th diagonal element of the matrix $\sigma_Q \sigma_Q'$. Since each professional investor can decide how to allocate wealth across different assets by adjusting B_{it}^k, the margin requirement he effectively faces is

$$\sum_{k=1}^{K} X_{it}^k \sqrt{(\sigma_Q \sigma_Q')_{kk}} \le \bar{m} W_{it} \tag{11}$$

or in a matrix form

$$X_{it}' \sqrt{diag(\sigma_Q \sigma_Q')} \le \bar{m} W_{it}. \tag{12}$$

Here $\sqrt{diag(\sigma_Q \sigma_Q')}$ is a vector consisting of square roots of diagonal elements $diag(\sigma_Q \sigma_Q')$. Clearly, if there is only one risky asset Eq. (12) reduces to Eq. (8).

3.3.4 Stochastic enforcement of margin constraints

To avoid several technical complications, I introduce another type of randomness in the description of margin requirements. This randomness ensures smoothness of aggregate demand, which in turn makes the endogenous stochastic processes for price and aggregate wealth well defined. I assume that each particular investor at each particular moment of time is bounded by margin constraints with some probability λ only, i.e. margin requirements are enforced stochastically. By the law of large numbers, only a fraction of investors is constrained even if margins are very tight.[7] It is convenient to think about this randomness in the following way. Every moment a professional investor asks a financier for money required to finance the position $\tilde{X}_{it}^k$ in the risky asset k. If the request $\tilde{X}_{it}^k$ does not violate margin requirements, the financier always lends money. However, if margin constraints are not satisfied, the financier enforces them with some probability λ. The outcome can be interpreted as a result of negotiations between professional investors and their financiers. This result depends on many factors such as firm-specific relations, reputation, other assets in the portfolio, etc. The margin requirements of Eqs. (7) and (12) can be interpreted as conservative industry standards pertaining to all investors.

Under both collateralization schemes, the probability of enforcement λ depends on how severely the margin requirement is violated by investor's request $\tilde{X}_{it}^k$. The margin violation ratio M_{it}, which measures the severity is determined by Eqs. (7) and (12). When portfolio is collateralized as a whole, the margin violation ratio is

$$M_{it} = \frac{\sqrt{\tilde{X}_{it}' \sigma_Q \sigma_Q' \tilde{X}_{it}}}{\bar{m} W_{it}}.$$

[7] If margins are enforced deterministically, all professional investors hit the margin constraint simultaneously. This results in a kink in the aggregate demand function and discontinuity in equilibrium expected returns and volatility.

In the case of individual collateralization, the margin violation ratio is

$$M_{it} = \frac{\sum_{k=1}^{K} \tilde{X}_{it}^k \sqrt{(\sigma_Q \sigma_Q')_{kk}}}{\bar{m} W_{it}}.$$

If the request $\tilde{X}_{it}^k$ violates margin requirements ($M_{it} > 1$) and is not acknowledged, the financier allows to borrow up to the margin requirement only and all requests are rescaled accordingly. As a result, a professional investor buys $\hat{X}_{it}^k$ of asset k, where

$$\hat{X}_{it}^k = \frac{\tilde{X}_{it}^k}{M_{it}}. \tag{13}$$

In both cases, the probability of enforcement is $\lambda = \lambda(M_{it})$, where the function $\lambda : R \to [0, 1]$ is continuously differentiable and $\lambda(x) = 0$ for $x \leq 1$. Also, $\lambda(\bar{x}) = 1$ for $x \geq \bar{x} > 1$ where $\bar{x}$ is the maximum permissible deviation from the margin requirement. As an example, I use the following specification of $\lambda(x)$:

$$\lambda(x) = \begin{cases} 0, & x < 1; \\ \bar{\lambda}(x-1)^2, & 1 \leq x < 1 + \frac{1}{\sqrt{2\bar{\lambda}}}; \\ 1 - \bar{\lambda}\left(1 + \sqrt{\frac{2}{\bar{\lambda}}} - x\right)^2, & 1 + \frac{1}{\sqrt{2\bar{\lambda}}} \leq x < 1 + \frac{2}{\sqrt{2\bar{\lambda}}}; \\ 1, & x \geq 1 + \frac{2}{\sqrt{2\bar{\lambda}}}. \end{cases} \tag{14}$$

The function $\lambda(x)$ contains one parameter $\bar{\lambda}$, which admits a clear interpretation. Indeed, small $\bar{\lambda}$ means that financiers are quite tolerant to violation of margin requirements, which are enforced with a low probability. In the limit $\bar{\lambda} \to 0$ margin constraints are not enforced at all and become irrelevant. On the contrary, large $\bar{\lambda}$ corresponds to the case in which financiers allow only small violation of margin requirements. In the limit $\bar{\lambda} \to \infty$ they are perfectly enforced and $\lambda(x)$ converges to a step function. A typical form of the function $\lambda(x)$ with $\bar{\lambda} = 50$ is depicted in Panel A of Figure 1.

3.4 Portfolio problem of professional investors

Now consider the portfolio problem of professional investors with stochastic margin constraints. For a moment assume that the dynamics of the model can be described by a one-dimensional state variable z_t. This assumption will be justified later and it will be shown that z_t corresponds to the aggregate wealth of professional investors. The stochastic process for z_t is

$$dz_t = \mu_z(z_t)dt + \sigma_z(z_t)dB_t, \tag{15}$$

and functions $\mu_z(z_t)$ and $\sigma_z(z_t)$ are determined by equilibrium conditions. Similarly, the process for excess return is

$$dQ_t = \mu_Q(z_t)dt + \sigma_Q(z_t)dB_t,$$

where $\mu_Q(z_t)$, and $\sigma_Q(z_t)$ are also determined in the equilibrium. The problem of a professional investor is to maximize a recursive utility function specified by Eqs. (2) and (3) subject to the stochastic margin constraint (7) or (12) and the budget constraint

$$dW_t^i = \begin{cases} \hat{X}_{it}'(\mu_Q(z_t)dt + \sigma_Q(z_t)dB_t) + (rW_{it} - C_{it})dt, & \text{if margin binds;} \\ \tilde{X}_{it}'(\mu_Q(z_t)dt + \sigma_Q(z_t)dB_t) + (rW_{it} - C_{it})dt, & \text{if margin does not bind.} \end{cases} \tag{16}$$

Recall that $\tilde{X}_{it}$ is a request submitted to a financier and acknowledged with probability $\lambda(M)$ and $\hat{X}_{it} = \tilde{X}_{it}/M_{it}$ is actual portfolio if margin requirements are strictly enforced.

First, consider the case in which portfolio is collateralized as a whole. The solution to the portfolio problem is given by the following proposition, where time subscripts and individual investor indexes are omitted for simplification of notations.

Proposition 1 *The optimal consumption C^* and the optimal requested position $\tilde{X}^*$ solving the optimization problem of a professional investor with Epstein-Zin utility function (2), (3) subject to the budget constraint (16) and a stochastic margin constraint (7) with the probability function (14) are*

$$C^* = \beta W, \tag{17}$$

$$\tilde{X}^* = \frac{2}{\gamma}\frac{y(A)}{A}\left[\sigma_Q \sigma_Q'\right]^{-1}\left(\mu_Q + H'\sigma_Q\sigma_z'\right)W, \tag{18}$$

where

$$A = \frac{2}{\gamma\bar{m}}\sqrt{\left(\mu_Q + H'\sigma_Q\sigma_z'\right)'\left[\sigma_Q\sigma_Q'\right]^{-1}\left(\mu_Q + H'\sigma_Q\sigma_z'\right)}. \tag{19}$$

The function $y(x)$ is implicitly determined by an algebraic equation

$$x\left(1 - \lambda(y) + \lambda'(y) - \lambda'(y)y\right) - \left(2(1-\lambda(y))y + \lambda'(y) - \lambda'(y)y^2\right) = 0, \tag{20}$$

where the unique root on the interval $[0, 1 + 1/\sqrt{3\bar{\lambda}}]$ is chosen. The function H solves the following differential equation:

$$\beta(1-\gamma)(\log\beta - 1) - \beta H + H'\mu_z + \frac{1}{2}(H'' + (H')^2)\sigma_z\sigma_z' + r(1-\gamma) \tag{21}$$

$$+(1-\gamma)\frac{\gamma\bar{m}^2}{2}\left[\lambda(A-1) + (1-\lambda)\,y(A)(A - y(A))\right] = 0,$$

where $\lambda = \lambda(y(A))$. The value function is

$$V(z, W) = \frac{1}{1-\gamma}W^{1-\gamma}\exp(H(z)).$$

Proof. See Appendix B.

Note that the state of the world for professional investors is determined by the state variable z_t only (it determines expected returns and volatility as well as the tightness of margin constraints). Whether the margin constraint binds or not at the particular moment does not affect the value function since this knowledge does not help to predict whether the margin requirement binds in future.

The function $y(x)$ is closely related to $\lambda(x)$ and it is not difficult to show that it is well-defined for all $x \in R^+$. Its typical graph with $\bar{\lambda} = 50$ is depicted in Panel B of Figure 1. Note that in the range $x \leq 2$ it has a very simple form: $y(x) = x/2$. This is exactly the case when the investor submits a financing request that will be deterministically fulfilled. Hence, the requested position $\tilde{X}$ coincides with the optimal unconstrained demand

$$\tilde{X}^* = \frac{1}{\gamma}\left[\sigma_Q\sigma_Q'\right]^{-1}\left(\mu_Q + H'\sigma_Q\sigma_z'\right)W. \tag{22}$$

In the interval $x > 2$ the investor starts taking into account the possibility of being constrained immediately since the optimal unconstrained demand violates margin requirements. As a result, the requested position is reduced appropriately. The function $y(x)$ determines the margin violation ratio: $M = y(A)$. Eq. (18) together with the form of $y(x)$ indicates that the margin violation ratio never exceeds $1 + 1/\sqrt{3\bar{\lambda}}$. In the one asset case it means that professional investors never request leverage exceeding the contemporaneous margin constraint by more than $1 + 1/\sqrt{3\bar{\lambda}}$.

The optimal strategy $\tilde{X}^*$ from Proposition 1 is a complicated vector function. The optimal portfolio takes a significantly simpler form when all assets are identical. The result is stated by Corollary 1.

Corollary 1 *Let ι be a $K \times 1$ vector of ones. If all assets are identical, i.e. $\mu_Q = \bar{\mu}_Q \iota$, $\sigma_Q = \bar{\sigma}_Q I + \bar{\bar{\sigma}}_Q(\iota\iota' - I)$ and all innovations affect the state variable z symmetrically $\sigma_z = \bar{\sigma}_z \iota'$ then demands for assets are also identical: $\tilde{X}^* = \tilde{x}^* \iota$. If portfolio is collateralized as a whole,*

$$\tilde{x}^* = \frac{\bar{m} y(A)}{\sqrt{K}(\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1))} W, \qquad A = \frac{2\sqrt{K}}{\gamma \bar{m}} \left[\frac{\bar{\mu}_Q}{\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1)} + H' \bar{\sigma}_z \right]$$

and the function $H(z)$ solves

$$\beta(1-\gamma)(\log \beta - 1) - \beta H + H' \mu_z(z) + \frac{K}{2}(H'' + (H')^2)\bar{\sigma}_z^2 + r(1-\gamma) \\ + (1-\gamma)\frac{\gamma \bar{m}^2}{2}\left[\lambda(A-1) + (1-\lambda)\, y(A)(A - y(A))\right] = 0, \quad (23)$$

where $\lambda = \lambda(y(A))$.

Note that Corollary 1 describes the solution to the portfolio problem that is faced by professional investors in the symmetric equilibrium analyzed below.

Next, consider the portfolio problem of professional investors when individual assets are collateralized separately. Unfortunately, in this case if processes of asset returns are not identical or the state variable z_t is asymmetrically affected by innovations dB_t^k the optimal policy $\tilde{X}$ does not admit an analytical solution analogous to Eq. (18). However, if all assets are identical the solution to the optimal portfolio problem is relatively simple and provided by Proposition 2.

Proposition 2 *If all assets are identical, i.e. $\mu_Q = \bar{\mu}_Q \iota$, $\sigma_Q = \bar{\sigma}_Q I + \bar{\bar{\sigma}}_Q(\iota\iota' - I)$ and all innovations affect the state variable z symmetrically $\sigma_z = \bar{\sigma}_z \iota'$ then the optimal consumption C^* and the optimal requested position $\tilde{X}^*$ solving the optimization problem of a professional investor with Epstein-Zin utility function (2), (3) subject to the budget constraint (16) and a stochastic margin constraint (12) with the probability function (14) are*

$$C^* = \beta W, \qquad \tilde{X}^* = \tilde{x}^* \iota \quad (24)$$

with

$$\tilde{x}^* = \frac{\bar{m} y(A)}{K\sqrt{\bar{\sigma}_Q^2 + (K-1)\bar{\bar{\sigma}}_Q^2}} W, \quad (25)$$

where

$$A = \frac{2K}{\gamma \bar{m}} \frac{\left[\bar{\mu}_Q + H'(\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1))\bar{\sigma}_z\right] \sqrt{\bar{\sigma}_Q^2 + (K-1)\bar{\bar{\sigma}}_Q^2}}{(\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1))^2} \quad (26)$$

and the function $y(x)$ is defined in Section 3.3.4. The function H solves the following differential equation:

$$\beta(1-\gamma)(\log\beta-1)-\beta H+H'\mu_z+\frac{K}{2}(H''+(H')^2)\bar{\sigma}_z^2+r(1-\gamma) \\ +(1-\gamma)\frac{\gamma\bar{m}^2}{2K}\frac{(\bar{\sigma}_Q+\bar{\bar{\sigma}}_Q(K-1))^2}{\bar{\sigma}_Q^2+(K-1)\bar{\bar{\sigma}}_Q^2}\left[\lambda(A-1)+(1-\lambda)\,y(A)(A-y(A))\right]=0, \quad (27)$$

where $\lambda=\lambda(y(A))$. The value function is

$$V(z,W)=\frac{1}{1-\gamma}W^{1-\gamma}\exp(H(z)).$$

Proof. See Appendix B.

Note that Proposition 2 is an analog to Corollary 1 and will serve as a building block for symmetric equilibrium of Section 3.5.

The structure of the solution to the individual portfolio problem (18) and the stochastic nature of margin constraints make it possible to find aggregate optimal demand of professional investors in a closed form. Since professional investors are identical, they follow the same strategy $\tilde{X}_i=\tilde{X}(z,W_i)$ and have the same function $H(z)$. Due to linearity of the optimal strategy with respect to individual wealth W_i they all have the same ratio $\tilde{X}_i/W_i$, which can be inferred from Propositions 1 and 2. Since under both collateralization schemes the margin violation ratio depends on ratios $\tilde{X}_i/W_i$, margin requirements bind all investors with the same probability. Thus, using the law of large numbers the aggregate demand X can be written as

$$X=\int_0^1 X_i di=\int_0^1\left[\lambda\hat{X}_i+(1-\lambda)\,\tilde{X}_i\right]di \quad (28)$$

$$=\int_0^1\left[\frac{\lambda}{M}+(1-\lambda)\right]\tilde{X}_i di=\left[\lambda+(1-\lambda)\,y(A)\right]\frac{\tilde{X}_i}{W_i}\frac{W}{y(A)},$$

where W is the aggregate wealth of professional investors. In the symmetric equilibrium studied below, all assets have identical aggregate demands x, i.e. $X=x\iota$. Using Corollary 1 in the portfolio collatarization case we get

$$x=\frac{\bar{m}\left[\lambda+(1-\lambda)\,y(A)\right]}{\sqrt{K}(\bar{\sigma}_Q+\bar{\bar{\sigma}}_Q(K-1))}W, \quad (29)$$

whereas in the case with individual collateralization

$$x=\frac{\bar{m}\left[\lambda+(1-\lambda)\,y(A)\right]}{K\sqrt{\bar{\sigma}_Q^2+(K-1)\bar{\bar{\sigma}}_Q^2}}W. \quad (30)$$

Note that the aggregate demand explicitly depends on the aggregate wealth of professional investors. Hence, if X/W is not constant W should be a state variable. Since the aggregation of consumption policies (17) is trivial and immediately yields $C=\beta W$, we arrive at the following dynamics for W:

$$dW_t=X'(\mu_Q dt+\sigma_Q dB_t)+(r-\beta)W_t dt. \quad (31)$$

Thus, if W is the only state variable, Eq. (31) coincides with Eq. (15) and $\mu_z=X'\mu_Q+(r-\beta)W$, $\sigma_z=X'\sigma_Q$.

3.5 Equilibrium

The equilibrium in the model is effectively determined by interaction between professional investors and long-term investors. The following formal definition provides the details.

Definition. The equilibrium of the model is a set of matrix functions $\tilde{X}_{it} = \tilde{X}_{it}(F_t, W_t, W_{it})$, $\mu_Q(F_t, W_t)$, $\sigma_Q(F_t, W_t)$, and $S_t = S(F_t, W_t)$ such that

1. $\tilde{X}_{it}$ solves the portfolio problem of professional investors with Epstein-Zin utility function (2), (3) subject to the budget constraint (16) and a stochastic margin constraint (7) or (12) with the probability function (14);

2. Market clearing condition holds:

$$X_t + X_t^L = \bar{X}\iota, \tag{32}$$

where X_t^L is the demand of long term investors (4) and X_t is the aggregate demand of professional investors given by Eq. (28);

3. Aggregate demand of professional investors satisfies Eq. (28).

The market clearing condition immediately relates the price function and the aggregate demand of professional investors:

$$S_t = F_t - \bar{\gamma}(\bar{X}\iota - X_t). \tag{33}$$

Eq. (33) implies that the price can be easily found if we know the aggregate demand of professional investors X. Moreover, Eq. (33) is quite intuitive. Long-term investors require a premium for keeping risky assets. As a result, if professional investors hold only a fraction of risky assets the price is below the fundamental value and the gap is determined by the parameter $\bar{\gamma}$.

The above definition is quite general and allows for the existence of multiple very complex equilibria. For the sake of tractability, I put additional restrictions on equilibrium functions, i.e. I am looking for certain types of equilibria. Specifically, I impose two conditions. First, aggregate demand X, expected returns μ_Q and volatility σ_Q are functions of only one state variable, which is the aggregate wealth of professional investors W_t. Note that the existence of such an equilibrium essentially depends on the form of margin constraints. In general, the level of fundamentals F_t is also a state variable, what significantly complicates the analysis. Second, given that asset fundamentals are identical, I focus on symmetric equilibria, i.e. equilibria with identical demands for all risky assets.

The price function (33) and the equation for fundamentals (5) fix the functional form of $\mu_Q(W)$ and $\sigma_Q(W)$, which are completely determined by the aggregate demand function $X(W)$. The following Proposition states the result.

Proposition 3 *If the consumption-wealth ratio of professional investors is β and all assets have identical demands $X = x\iota$, then expected returns are also identical with $\mu_Q = \bar{\mu}_Q\iota$, where*

$$\bar{\mu}_Q = \frac{\bar{\gamma}(r-\beta)x_W W + \frac{K}{2\bar{\gamma}}\frac{x_{WW}}{x_W^2}\bar{\bar{\sigma}}_Q^2 + r\bar{\gamma}(\bar{X}-x)}{1-\bar{\gamma}Kx_W x} \tag{34}$$

and the covariance matrix of returns can be rewritten as $\sigma_Q = \bar{\sigma}_Q I + \bar{\bar{\sigma}}_Q(\iota\iota' - I)$, where

$$\bar{\sigma}_Q = \frac{\sigma_D}{r+\phi}\frac{1-(K-1)\bar{\gamma}x_W x}{1-\bar{\gamma}Kx_W x}, \qquad \bar{\bar{\sigma}}_Q = \frac{\sigma_D}{r+\phi}\frac{\bar{\gamma}x_W x}{1-\bar{\gamma}Kx_W x}. \tag{35}$$

Both μ_Q and σ_Q are determined by the aggregate demand of professional investors X and subscripts on x denote derivatives.

Proof. See Appendix B.

Eqs. (34) and (35) show that if X is a function of W only, then σ_Q and μ_Q also depend only on W. Proposition 3 together with the solution to the portfolio problem from Corollary 1 or Proposition 2 and the aggregation equation (28) deliver a system of differential equations determining equilibrium demands, expected returns, and volatility. Combining Eqs. (34), (35) with Eqs. (23) and (29) in the portfolio collateralization case or with Eqs. (27) and (30) in the individual asset collateralization case we get equations for functions $x(W)$ and $H(W)$. The key object here is $x(W)$, which determines equilibrium expected returns and their volatility as functions of the state variable W.

Note that the function $x(W)$ must satisfy natural boundary conditions. When aggregate wealth of professional investors is zero, they cannot hold the risky asset: $x(W) = 0$. In the opposite extreme, when aggregate wealth tends to infinity professional investors compete for the risky asset and keep its total supply $\bar{X}$. Thus, $x(\infty) = \bar{X}$.

An interesting characteristic of the equilibrium at each moment of time is liquidity of risky assets, which is defined as market depth. Specifically, if any of professional investors have an urgent need to dump one of their securities on the market, the price should change appropriately to induce the rest of investors to accommodate the excess supply. The depth of the market (liquidity) is defined as $LQ_t = dS_t'/d\bar{X}$, where it is taken into account that the variation in $\bar{X}$ affects the equilibrium functions directly (they depend on $\bar{X}$ as a parameter) and indirectly through instantaneous changes in the aggregate wealth. It is not difficult to show that in the equilibrium considered above

$$LQ = -\bar{\gamma}\left(I - X_{\bar{X}} - SX_W'\right)\left(I - \bar{\gamma}XX_W' - SX_W'\right)^{-1}. \tag{36}$$

Note that liquidity depends not only on the current value of W, but also on the price S (or, equivalently, on the current level of fundamentals). As a conservative estimate, for computation of LQ I fix $F = \bar{\gamma}$. This choice guarantees that even if $W = 0$ the price of the asset does not become negative. Increase in F amplifies LQ and for some levels of fundamentals Eq. (36) becomes ill-defined.

4 Analysis: single asset case

4.1 Equilibrium without margin constraints

I start the analysis with a single asset economy ($K = 1$). As a benchmark case, consider a setup without margin constraints in which professional investors can borrow unlimitedly. This case can be obtained either when $\bar{\lambda} \to 0$ (the probability of enforcement is negligibly small), or $\bar{m} \to \infty$ (the margin requirement is infinitely loose). Without margin constraints, the model resembles the setup of Xiong (2001) with several important differences. First, in my model there are no noise traders. Second, the preferences of professional traders are different: in Xiong (2001) speculators maximize the logarithmic utility function giving rise to a myopic demand whereas in my model professional investors are allowed to have arbitrary risk aversion. Taking into account that $\sigma_z = X'\sigma_Q$, Eq. (18) together with Eq. (28) yields

$$X = \frac{\mu_Q}{\gamma \sigma_Q^2} \frac{\gamma W}{\gamma - H'W}. \tag{37}$$

Note that X is a product of two terms. The first one is $\mu_Q/(\gamma\sigma_Q^2)$ and corresponds to myopic demand. The second term $\gamma W/(\gamma - H'W)$ is a correction for time variation in investment opportunities. Expected returns μ_Q and volatility σ_Q are given by Eqs. (34) and (35), respectively. The function H solves the following equation:

$$\beta(1-\gamma)(\log\beta - 1) - \beta H + H'(X\mu_Q + (r-\beta)W) + \frac{1}{2}(H'' + (H')^2)X^2\sigma_Q^2 + r(1-\gamma)$$

$$+\frac{1-\gamma}{2\gamma}\left(\frac{\mu_Q}{\sigma_Q} + H'X\sigma_Q\right)^2 = 0 \tag{38}$$

with the boundary conditions $X(0) = 0$ and $X(\infty) = \bar{X}$.

Unfortunately, the system of non-linear differential equations (35), (34), (37), and (38) is rather complex and does not admit an analytical solution. Thus, we have to rely on numerical analysis. To find an approximate solution, a projection method is used, which approximates the functions $X(W)$ and $H(W)$ by linear combinations of Chebyshev polynomials. One of advantages this method has is that the obtained solutions are automatically twice continuously differentiable for all $W \in (0, \infty)$. Other details on the implementation of the projection method are collected in Appendix C.

Although most of my results are valid for a broad range of model parameters, for illustrative purposes I set $\gamma = 4$, $\bar{\gamma} = 6$, $\beta = 0.05$. The volatility of fundamentals is $\sigma_F = 0.2$. The choice $r = 0.03$ guarantees the stationarity of aggregate wealth. The aggregate amount of assets is normalized to one: $\bar{X} = 1$.

To visualize the structure of the equilibrium, several statistics of interest are plotted as a function of the state variable w, which is a rescaled aggregate wealth of professional investors W: $w = (W - 1)/(W + 1)$. Clearly, the point $w = -1$ corresponds to $W = 0$ and $w = 1$ corresponds to $W = \infty$. This rescaling is quite natural when the solution is approximated by Chebyshev polynomials (see Appendix C). The results are presented in Figure 2, where solid lines correspond to the equilibrium under consideration.

The upper left panel of Figure 2 reports the aggregate demand of professional investors X. Consistent with boundary conditions, when $W = 0$ no one holds the asset and $X = 0$. When $W \to \infty$ all assets are held by professional investors: $X = 1$. For intermediate levels of wealth the asset is split between professional investors and long-term investors.

The upper right panel shows the equilibrium Sharpe ratio. When professional investors have a lot of wealth they fiercely compete with each other and drive the expected excess return to zero. Since the volatility of the risky asset is not eliminated, the Sharpe ratio also tends to zero. In the opposite extreme when the wealth of professional investors is close to zero their risk bearing capacity is small and they require high premium even for keeping a small amount of the risky asset. Correspondingly, the Sharpe ratio is also high.

The bottom left panel of Figure 2 presents the volatility of returns, which has a hump-shaped form. In the extremes $W = 0$ and $W = \infty$ it coincides with the fundamental volatility σ_F: in both cases the price is driven by the fundamental value F_t only. However, at intermediate levels of wealth the volatility is amplified by the wealth effect (Xiong, 2001; Kyle and Xiong, 2001). A negative shock to fundamentals decreases the price of the asset and increases investors' leverage. Trying to maintain optimal portfolio weights professional investors start selling the asset to long-term investors pushing the price down even further. This decline in price erodes investors equity even more and triggers additional sell-offs. As a result, the impact of the fundamental shock on price is amplified and volatility increases.

The most interesting graph is given by the bottom right panel: it represents the liquidity of the market LQ. For all W the liquidity LQ is negative: positive shock to $\bar{X}$ leads to a decrease in the price. In the limit $W = \infty$ the market is perfectly liquid since competing professional investors would be eager to absorb an extra amount of assets. When $W = 0$ and professional agents are absent in the market the liquidity is completely determined by long-term investors and is equal to $-\bar{\gamma}$. Interestingly, for intermediate levels of the aggregate wealth the liquidity can be even lower than $-\bar{\gamma}$. Intuitively, at these levels an unexpected increase in the asset supply causes a decrease in prices, which in turn stimulates additional sell-off by professional investors due to the wealth effect. This produces an additional pressure on long-term investors and diminishes the depth of the market.

To facilitate understanding of the impact of the hedging demand, Figure 2 also plots equilibrium statistics in a counterfactual economy with semi-rational professional investors (dashed lines). Instead of the optimal demand (37), semi-rational professional investors have a myopic demand $X = \mu_Q/(\gamma\sigma_Q^2)W$ and ignore hedging needs. The upper left panel indicates that in the absence of margin constraints the hedging demand is positive. It is quite intuitive, since negative shock in wealth increases the Sharpe ratio. The risky asset provides a natural hedge to investors and this explains the positivity of the hedging demand. Since aggregate demand is higher in the non-myopic case, a lower share of the asset is kept by long-term investors and this explains why the Sharpe ratio is also lower. It is interesting that the hedging demand increases volatility and reduces liquidity relative to the myopic equilibrium. Intuitively, in the middle range of wealth non-myopic investors who have larger investment in the risky asset relative to myopic investors are more sensitive to the wealth effect and at some point liquidate their positions more aggressively than myopic investors in a response to an adverse fundamental shock. As a result, the price becomes more sensitive to such shocks, i.e. returns become more volatile and the market less liquid.

4.2 Equilibrium with margin constraints

To demonstrate the impact of margin constraints on the equilibrium, consider the setting from Section 4.1 in which now professional investors are subject to margin requirements. The margin parameter $\bar{m}$ is chosen such that the constraint is binding when the aggregate wealth of professional investors is below some threshold, and this happens sufficiently often. In the numerical example $\bar{m} = 0.1$. The parameter of the function $\lambda(x)$ in Eq. (14) is set to $\bar{\lambda} = 50$, which makes stochastic deviations from the margin m_t relatively small.

For comparison with the no margins case, Figure 3 depicts the difference between statistics in the economy with margin constraints and the economy without margins. Vertical lines indicate the aggregate wealth of professional investors at which the margin constraint becomes binding. The upper left panel depicting the aggregate demand of professional investors shows that the presence of margin constraints uniformly reduces holdings of professional investors. This reduction mostly results from the hedging of margin constraints: professional agents underinvest in the risky asset because its price is negatively correlated with the tightness of margin constraints when margins do not bind. This hedging demand is fully examined in Appendix A.

Upper right panel demonstrates that when the margin constraint binds, the Sharpe ratio is higher than in the unconstrained case. Intuitively, when professional investors have a limited ability to invest in the risky asset this burden falls on long-term investors who require higher compensation for holding the asset. As in the unconstrained case, when $W = 0$ the Sharpe ratio approaches $r\bar{\gamma}\bar{X}/\sigma_F$.

The bottom left panel shows the volatility of returns. On the one hand, when the aggregate wealth is sufficiently high the volatility is higher than in the unconstrained case and peaks approximately when the margin starts binding. On the other hand, when W is low and professional investors are constrained the volatility is lower than in the benchmark case. Although the overall effect of margin constraints on volatility is ambiguous, the model shows that negative shock in good times when wealth is high and expected returns are low leads to increase in volatility. This is consistent with empirical evidence. In addition, the decrease in volatility when margins bind is in line with Brunnermeier and Pedersen (2008), who claim that "stabilizing margins are ... hard to escape in a theoretical model" and resembles the result of Saha (2007).

The bottom right panel shows the difference in liquidity LQ. Clearly, the presence of time-varying margin requirements substantially influences the reaction of the price to unanticipated sell-offs. When the wealth W is relatively high, margin constraints drastically reduce liquidity, which becomes especially low near the threshold separating the binding and non-binding regions. Intuitively, an additional amount of equity on the market reduces the price of the asset, and consequently, leads to decline of the aggregate wealth W. However, in addition to the wealth effect, which reduces liquidity even without margin constraints, now the reduction in wealth also increases the probability of hitting the margin constraint in future. Because of that, the selling pressure created by professional investors becomes stronger making the market significantly less liquid.

Interestingly, deep in the region with binding constraints market can become more liquid than in the unconstrained case. Indeed, because of constraints professional investors leave the market earlier and adding an extra amount of assets generates less rebalancing, leading to smaller decline in the asset price.

Note that in the equilibrium model professional investors solving their portfolio problem need to hedge two types of risks. First, expected returns and volatility change over time because of time variation in the aggregate wealth (this is the wealth effect as explained in Section 4.1). The resulting time variation in the Sharpe ratio generates the standard hedging demand. Second, margin requirements also vary in the equilibrium and their variation must be hedged as well. In the presence of fluctuating margins, it is not easy to separate these two hedging demands. Indeed, the hedging of the second type changes the aggregate demand of professional investors, which in turn determines the equilibrium volatility and risk premium. As a result, individual investors hedging the time varying Sharpe ratio partially respond to evolving margins.

The possibility that under some circumstances time varying margin constraints can play an amplification role was also pointed out by Brunnermeier and Pedersen (2008). However, there are several substantial differences between this paper and Brunnermeier and Pedersen (2008). First, in Brunnermeier and Pedersen (2008) the security payoffs are exogenously heteroscedastic, and this assumption is important for their results. In this paper fundamental volatility is constant and heteroscedasticity in returns is generated endogenously. Second, the model of Brunnermeier and Pedersen (2008) features risk-neutral speculators who exploit temporary imbalance in market orders and provide liquidity, whereas in this paper professional agents investing in a risky asset demand liquidity when their aggregate wealth changes. Third, to arrive at destabilizing margins Brunnermeier and Pedersen (2008) (at least in the main analysis of period $t = 1$) need asymmetry of information between lenders and borrowers. This is quite natural, since the investment opportunity is of an arbitrage type. In my model, all agents have the same information and margins are destabilizing because of their dependence on volatility, which in turn can be altered by the wealth effect. Fourth, as opposed to Brunnermeier and Pedersen (2008) hedging motives play an important role in my analysis and results.

To illustrate the last point and assess the role of the hedging demand, I compare the impact of margin requirements in two separate cases. In the first one, investors are fully rational, and this is exactly the equilibrium discussed above. In the second case, investors ignore all hedging needs and have a purely myopic demand $X = \mu_Q/(\gamma\sigma_Q^2)W$. The differences of several conditional variables in economies with and without margin constraints for myopic (dashed line) and non-myopic (solid line) equilibria are presented in Figure 4. The upper right panel demonstrates that the decline in the demand X caused by margin constraints is more pronounced when investors are fully rational. This is very intuitive: they correctly anticipate that the margin can bind in future and try to avoid it. Note that the desire to hedge margins is stronger than the desire to use the asset as a natural hedge (cf. Figure 2).

Having a greater decline in X caused by margin constraints in the fully rational equilibrium, it is not surprising to see that the Sharpe ratio is more strongly affected by margins (upper right panel of Figure 4). The logic is familiar: there is a larger burden on long-term investors and they require a higher risk premium.

The hedging component also changes the impact of margin constraints on volatility and liquidity. Bottom panels of Figure 4 indicate that when the demand of agents has a hedging component at intermediate levels of wealth the introduction of margin constraints significantly increases volatility and reduces the liquidity of the market relative to the myopic case. Partially this difference can be explained by the fact that in the myopic equilibrium agents behave more aggressively and do not sell assets in advance. Thus, the margin constraint becomes binding at higher levels of aggregate wealth (this is a vertical dashed line) than in the equilibrium with fully rational investors (this point is indicated by a vertical solid line). As a result, myopic agents sell assets more smoothly and this reduces the sensitivity of the price to fundamental shocks or changes in the aggregate amount of wealth.

Finally, consider how the effect of margin constraints depends on their average tightness characterized by $\bar{m}$. Figure 5 plots the equilibrium statistics for $\bar{m} = 0.01$, $\bar{m} = 0.1$, and $\bar{m} = 0.2$. Clearly, if margins are tight ($\bar{m}$ is small) they start binding at higher levels of aggregate wealth w as indicated by vertical lines. The reduction in the position X and increase in the Sharpe ratio are monotonic in $\bar{m}$: when margins are very restrictive professional agents cannot hold many assets and long-term investors require higher risk premium for keeping the rest of them. Interestingly, the destabilizing impact of margins on σ_Q and LQ has a hump-shaped form: it is stronger for intermediate values of $\bar{m}$.

Indeed, if margin requirements are loose they do not have a substantial effect on the economy since most of the time they do not bind and can be almost ignored. However, if financiers are very sensitive to potential losses margin constraints on average are very tight and start binding at relatively high levels of professional investors' wealth. As a result, professional investors typically do not hold many risky assets and, hence, their rebalancing has only a limited effect, which translates into only a weak amplification of volatility and reduction of liquidity.

5 Analysis: multiple asset case

5.1 Equilibrium without margin constraints

In this section, I analyze the general model with the number of assets $K > 1$. This extension makes it possible to answer several additional questions. First, it can shed light on the impact of margin constraints on correlation between returns and the contribution of time varying margins into contagion. Second, it provides a framework for analysis of how the impact of margins depends on the number of assets available for investing. Third, given several assets it is possible to study how different collateralization schemes (portfolio collateralization vs. collateralization of individual securities) affect asset returns.

Similar to the analysis of a single asset model, I start with the setup without margin constraints. As mentioned earlier, this is a special case of the general model obtained in the limit $\bar{m} \to \infty$ or $\bar{\lambda} \to 0$. This simplified model serves as a benchmark for the fully-fledged model with margin constraints. Following the theoretical analysis of Section 4, I focus on a symmetric equilibrium in which all risky assets have identical processes for returns and professional investors split their wealth equally across all assets. To facilitate comparison of results with the previous section, it is assumed that model parameters are exactly the same as before: $r = 0.03$, $\beta = 0.05$, $\gamma = 4$, $\bar{\gamma} = 6$, $\sigma_F = 0.2$. Similar to Section 4, the model is solved numerically and interesting statistics are reported as functions of rescaled aggregate wealth of professional investors. The results are presented in Figure 6.

The top left panel of Figure 6 depicts the optimal number of shares that a professional investor puts in his portfolio. Clearly, as the number of assets available for investment increases, professional investors with the same amount of aggregate wealth hold smaller amount of each asset leaving a higher share of each asset to long-term investors. Effectively, the aggregate risk-bearing capacity of professional investors becomes smaller relative to the total amount of risk available in the economy. This observation is confirmed by the top right panel of Figure 6, which shows Sharpe ratios of individual securities. Since all assets are identical, Sharpe ratios are

$$SR = \frac{\bar{\mu}_Q}{\sqrt{\bar{\sigma}_Q^2 + (K-1)\bar{\bar{\sigma}}_Q^2}}. \tag{39}$$

Sharpe ratios unambiguously increase with the number of assets since professional investors require additional compensation for holding extra amount of risk brought by additional assets at each level of wealth.

Bottom left panel of Figure 6 reports total conditional volatility of individual securities for different levels of rescaled aggregate wealth w. Strikingly, the highest volatility achievable for each K increases with K. The intuition is as follows. A negative shock to fundamentals of asset k gives rise to wealth effect: rebalancing triggers new sell-offs, which drive the price even further from fundamentals and increase sensitivity of the price to adverse shocks. However, with multiple stocks rebalancing caused by fundamentals of one asset leads to sell-off in others. These sell-offs drive prices of other stocks down and generate new losses which instigate further rebalancing and sell-offs. This is contagion described by Kyle and Xiong (2001). However, these authors consider only two assets. This paper generalizes their analysis to a multiple asset setting and shows that volatility amplification caused by contagion can become stronger as the number of assets increases.

Another measure related to contagion is the correlation between returns. For K identical assets, a correlation coefficient between each two of them is

$$\rho_Q = \frac{cov(dQ^k, dQ^l)}{var(dQ^k)} = \frac{\bar{\bar{\sigma}}_Q(2(\bar{\sigma}_Q - \bar{\bar{\sigma}}_Q) + K\bar{\bar{\sigma}}_Q)}{\bar{\sigma}_Q^2 + (K-1)\bar{\bar{\sigma}}_Q^2}. \tag{40}$$

This correlation is shown in the bottom right panel of Figure 6. Note that despite independence of assets fundamentals, returns appear to be highly correlated for intermediate levels of wealth. Only if the aggregate wealth tends to zero or infinity the correlation of returns becomes closer to the correlation of fundamentals. However, as opposed to volatility, the range of attainable correlations appears to be significantly less sensitive to changes in the number of assets K.

Under certain circumstances, the inference drawn from bottom panels of Figure 6 can be misleading. Indeed, if the distribution of aggregate wealth has a relatively narrow support, only a certain range of wealth might be relevant for analysis and extreme values of volatility and correlation might be unattainable. To mitigate this concern, Table 1 reports 5% and 95% quantiles of total volatility of returns and their pairwise correlations. Consistent with the previous discussion, 95% quantile for volatility consistently increases and almost coincides with the top points of graphs depicted in the left bottom panel of Figure 6. This indicates that volatility increase is not spurious and in certain states of the world in the model with seven assets total volatility indeed can be higher than in any state in the the model with, for example, one asset. Table 1 also confirms the conclusions on correlations between assets: although the 95% quantile increases with the number of assets, the effect is substantially smaller than for volatility.

5.2 Equilibrium with margin constraints

To visualize the impact of endogenously varying margin requirements, Figure 7 reports differences in several statistics in economies with and without margin constraints. For illustrative purposes it is assumed that there are three identical assets in each economy. To make the results comparable, in both economies all other parameters are set as in the previous sections except for margin requirements. Following the analysis of Section 3, I examine two types of margin requirements associated with different collateralization schemes. In Figure 7 the dashed line corresponds to the case of individual collateralization whereas the solid line corresponds to the portfolio collateralization case. Figure 7 reports the aggregate demand of professional investors X, the Sharpe ratio of each asset SR, the volatility of returns σ_Q^{tot}, and the correlation between returns ρ_Q. In consistency with one asset analysis of Section 4, time varying margin constraints decrease optimal holdings of professional investors and increase volatility of returns. Also, when margins bind the Sharpe ratio of assets becomes higher.

A new result relative to the one asset case of Section 4 is that margin requirements can also increase correlation between returns. As shown in the right bottom panel of Figure 7, the correlation ρ_Q increases relative to no margins case as the aggregate wealth approaches the interval in which margin constraint binds. The intuition is as follows. A negative shock to fundamentals of one asset induces price change of other assets and increases their volatilities through the wealth effect described in the previous section. However, in the presence of margin requirements the hedging demand in each asset is responsible for additional sales of *all* assets. Hence, prices of all assets go down and this further reduces wealth of professional investors leading to tighter margin constraints affecting all assets. Thus, not only the wealth effect makes individual innovations in asset fundamentals partially systematic, but also hedging of time variation in margin requirements contributes to the transformation of idiosyncratic shocks into systematic and makes returns more correlated.

Comparing dashed lines with solid lines we conclude that margin requirements imposed on individual assets have a stronger impact than margins applied to the portfolio as a whole. This finding is quite intuitive since margin requirements imposed on individual assets are effectively more restrictive than those imposed on the portfolio level even if the VaR rule followed by financiers is exactly the same in both cases leading to identical $\bar{m}$. Indeed, multiple stocks provide diversification opportunities and if financiers care about portfolio volatility the perceived volatility of collateral value is lower. As a result, margin constraints are less tight and the probability that they will bind is lower. Because of that, investors are more risk tolerant and at each level of aggregate wealth allocate their capital more aggressively maintaining higher positions in each asset (see left top panel of Figure 7). Moreover, in the range of aggregate wealth where margins do not bind lower risk aversion produces weaker hedging demand, which makes volatility and correlation between returns lower (see bottom panels of Figure 7). This leads to further relaxation of margin constraints. Thus, the tendency to use different financiers for different parts of the overall portfolio might have an important aggregate effect manifesting itself in an even higher volatility of returns and their mutual correlations.

Finally, consider how the number of assets in the economy affects the impact of margin constraints on volatility of returns and their correlations. Figure 8 reports differences between several statistics in the economy with margin constraints and the economy without margins for different K. In the economy with margins a portfolio collateralization scheme is assumed. Strikingly, the impact of margins is almost insensitive to the number of assets. This is a manifestation of two counteracting effects. On the one hand, the increase in the number of assets makes the portfolio more diversified. Since the portfolio is collateralized as a whole, diversification reduces its volatility making margin constraints less tight. As a result, volatility of returns and their correlation decrease. On the other hand, an extra asset brings a new systematic component since because of the wealth effect and hedging needs all returns would react to the new shock. This new systematic component increases volatility of returns and their correlation. Which effect dominates depends on the parameters of the model, but to a large extent they cancel each other.

6 Conclusion

The main conclusion of this paper is that time variation of margin requirements can have an economically significant impact on volatility of returns, their correlation, and liquidity. Trying to hedge changes in margin constraints, risk-averse agents underinvest in risky assets and start selling them when the probability that margin constraints become binding increases. In equilibrium, the hedging behavior of agents substantially changes volatility and liquidity of the market. In particular, negative shocks reducing the wealth of relatively risk tolerant investors in the states when margins do not bind can be significantly amplified by hedging.

Theoretical understanding of the role of margin constraints is important from a practical point of view and might have significant policy implications. There is a long-lasting discussion in the empirical literature about the relation between margin requirements mandated by the Federal Reserve and stock market volatility. Many authors claim that there is no robust evidence in favor of margin requirements as an effective policy instrument (e.g., Moore, 1966; Schwert, 1989; Hsieh and Miller, 1990). This point of view is challenged by Hardouvelis (1990), who argues that historically higher margin requirements are associated with lower stock market volatility. This conclusion is partially reaffirmed by Hardouvelis and Theodossiou (2002) who demonstrate that there is a negative causal relation between tightness of margins and volatility in bull markets and no relation in bear markets. Similarly, using data from the Tokyo Stock Exchange Hardouvelis and Peristiani (1992) show that an increase in margin requirements is followed by decline in volatility of daily returns.

An interesting twist in this discussion is the opinion that not margins per se, but *changes* in margin requirements can be an effective policy tool. Shiller (2000) claims that the Federal Reserve should return to its pre-1974 policy of actively changing margin requirements in response to stock market speculation. In particular, it argues that the announcement effect of increased margin requirements might help to stabilize an exuberant stock market. This idea is in line with this paper, which suggests that the volatility of markets can be reduced by making the tightness of margins procyclical.

The general equilibrium model considered in the paper can be generalized in numerous ways. First, in this paper it is assumed that margins are solely determined by the impossibility to sell the collateral immediately and lock in the current price. However, there might be other reasons why lenders impose margin requirements. In particular, if the amount of assets the lender needs to sell is not negligible, price impact might become an important issue for consideration. In this case, in addition to volatility the financier starts worrying about liquidity of the market, and it becomes another determinant of margin requirements.

Second, the ability of financiers to change margin requirements might open a door for their strategic interaction. In particular, they might inefficiently coordinate on volatility or get involved in predation against their customers in some states of the world. Understanding the strategic behavior of financiers is likely to be the most fruitful direction for future research.

Appendix A. Portfolio problem with constant Sharpe ratio and dynamic margin constraints

A.1 Setup

To develop some intuition on the impact of dynamic margin constraints consider a portfolio choice problem of a risk-averse investor who is subject to time-varying margin requirements. Although all results would be similar for investors with the CRRA utility function, to make the analysis consistent with the equilibrium model presented in Section 3 I assume that investors have Epstein-Zin preferences (Epstein and Zin, 1989; Weil, 1989) with unit EIS over an infinite stream of consumption C_t:

$$J_t = E_t \left[\int_t^{\infty} f(C_s, J_s) ds \right], \tag{41}$$

where

$$f(C_s, J_s) = \beta(1-\gamma) J_s \left(\log C_s - \frac{1}{1-\gamma} \log((1-\gamma) J_s) \right). \tag{42}$$

There are two assets the investor can invest in, a risk-free asset with a constant rate of return r and a risky asset, whose price S_t follows the process

$$dS_t = (r + \mu_S) S_t dt + \sigma_S S_t dB_t^S + Q_S S_{t-} (dN_t - \lambda dt).$$

Innovations in the price can be of two types. The diffusion part is represented by standard Brownian motion B_t^S and corresponds to continuous changes in the price. Besides that, there can be infrequent jumps arriving according to a Poisson process N_t with constant intensity λ. The size of jumps is also assumed to be fixed and is equal to Q_S. As usually, S_{t-} denotes the price of the asset just before the jump arrival.

To delineate the impact of margin constraints, I deliberately assume that expected excess returns on the asset μ_S and volatility σ_S are constant. It is not the case in the equilibrium studied in Section 3 where the hedging of margin constraints is inextricably intertwined with the hedging of time-varying Sharpe ratio. To ensure the stationarity of the wealth process I assume that $\beta > r$.

The investor is subject to margin constraints, which restrict the leverage of the portfolio: the proportion of wealth ω_t allocated to the risky asset cannot exceed a prespecified threshold m_t: $\omega_t \leq m_t$. In general, m_t is time-varying with the dynamics specified as

$$dm_t = \phi_m(\bar{m} - m_t) dt + \bar{\sigma}_m \sqrt{m_t - 1} dB_t^m + Q_m(m_{t-} - 1)(dN_t - \lambda dt). \tag{43}$$

Similar to the price of the risky asset, innovations in margins have diffusion and jump parts. The diffusive innovations dB_t^m correspond to gradual changes in margin requirements and it is convenient to think about them as an outcome of continuous negotiations between investors and their financiers. I assume that $cov(dB_t^m, dB_t^S) = \rho_{mS} dt$ and $\rho_{mS} > 0$. Although the sign of the correlation depends on how margins are set, it is quite natural to think that margins become less tight when asset price increases.[8]

Jumps in the margin requirements are controlled by the same Poisson process N_t that is responsible for jumps in the price. Hence, jumps can be interpreted as rare events which simultaneously shift the price and margin requirements.[9] Consistent with empirical evidence that returns more abruptly go down than up ("up by stairs, down in elevator"), I assume that $-1 < Q_m < 0$ and $-1 < Q_S < 0$.

Excluding jumps, the process (43) coincides with a Cox-Ingersoll-Ross process. This implies that the margin fluctuates around $\bar{m}$ and the square root in the diffusion term guarantees that m_t is always greater than one. The latter condition means that the portfolio constraint can be binding only if the investor borrows money and there are no limitations on investments of his own wealth. To ensure that the boundary $m = 1$ is inaccessible,

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[8]In the equilibrium model in Section 3 innovations in returns are perfectly positively correlated with the state variable determining margin constraints.

[9]A good example of this sort of events is provided by stock market crash of October 19, 1987. At that day S&P 500 index lost about 20%. Simultaneously, all investors who traded on a margin experienced a substantial increase in margin requirements. For example, margins for S&P futures jumped from 4% to more than 12%. A similar jump, although of a smaller size, occurred during the LTCM crisis in 1998.

the standard Feller condition is imposed: $2\phi_m \bar{m} \geq \bar{\sigma}_m^2$. The factor $m_{t-} - 1$ in the jump term is also introduced to guarantee that jumps do not push the margin below one.[10]

The problem of investor is to choose consumption and asset allocation maximizing the Epstein-Zin utility function (41), (42) subject to the budget constraint

$$dW_t = \omega_t W_t((\mu_S - \lambda Q_S)dt + \sigma_S dB_t^S) + (rW_t - C_t)dt + W_{t-}\omega_{t-}Q_S dN_t \tag{44}$$

and the margin requirement $\omega_t \leq m_t$. The solution to this problem is summarized in Proposition 4.

Proposition 4 *The solution to the portfolio problem of an investor with Epstein-Zin utility function (41), (42) subject to the budget constraint (44) and the margin requirement (43) is*

$$C^* = \beta W, \qquad \omega^* = \min(m, \bar{\omega}(m)), \tag{45}$$

where $\bar{\omega}(m)$ solves the following equation

$$\bar{\omega}(m) = \frac{\mu_S - \lambda Q_S}{\gamma \sigma_S^2} + \frac{\bar{\sigma}_m \sqrt{m-1}\rho_{mS}}{\gamma \sigma_S} H'(m) + \frac{\lambda Q_S}{\gamma \sigma_S^2}(1 + \bar{\omega}(m)Q_S)^{-\gamma} \exp(H(m + (m-1)Q_m) - H(m)). \tag{46}$$

A twice continuously differentiable function $H(m)$ solves the following ODE

$$\begin{aligned}\beta(1-\gamma)(\log\beta - 1) + r(1-\gamma) - \beta H + H'\left(\phi_m(\bar{m} - m) - Q_m\lambda(m-1)\right) \\ + \frac{1}{2}(H'' + (H')^2)\bar{\sigma}_m^2(m-1) + (1-\gamma)\left(\omega^*(\mu_S - \lambda Q_S) - \frac{\gamma}{2}(\omega^*)^2\sigma_S^2 + H'\omega^*\sigma_S\bar{\sigma}_m\sqrt{m-1}\rho_{mS}\right) \\ + \lambda\left[(1 + \omega^* Q_S)^{1-\gamma}\exp(H(m + (m-1)Q_m) - H(m)) - 1\right] = 0\end{aligned} \tag{47}$$

and the value function is

$$V(m, W) = \frac{1}{1-\gamma} W^{1-\gamma} \exp(H(m)).$$

Proof. The value function of an investor maximizing stochastic differential utility (41), (42) satisfies the following Bellman equation (Duffie and Epstein, 1992a)

$$\max_{\{C, \omega \leq m\}} [f(C, J) + DJ] = 0. \tag{48}$$

An aggregator $f(C, J)$ is given by Eq. (42) and

$$DJ = J_W(\omega W(\mu_S - \lambda Q_S) + rW - C) + \frac{1}{2} J_{WW} W^2 \omega^2 \sigma_S^2 + J_m \phi_m(\bar{m} - m) - J_m Q_m \lambda(m-1)$$

$$+ \frac{1}{2} J_{mm} \bar{\sigma}_m^2 (m-1) + J_{Wm} W \omega \sigma_S \bar{\sigma}_m \sqrt{m-1}\rho_{mS} + \lambda \Delta J,$$

where

$$\Delta J = J(W(1 + \omega Q_S), m + (m-1)Q_m) - J(W, m).$$

Assuming $(1-\gamma)J > 0$, maximization with respect to C yields

$$C = \arg\max_{\{C\}} [\beta(1-\gamma) J \log C - J_W C] = \beta(1-\gamma)\frac{J}{J_W}. \tag{49}$$

[10] Clearly, the suggested model can be generalized in numerous ways. For example, it is possible to introduce jumps with random sizes, consider stochastic intensity of jump arrival, etc. Although such modifications may enrich model implications, their consideration is beyond the scope of this paper.

Due to homotheticity of preferences it is natural to look for the value function in the following standard form:

$$J(m, W) = \frac{1}{1-\gamma} W^{1-\gamma} \exp(H(m)), \tag{50}$$

where the function $H(m)$ is assumed to be twice continuously differentiable for all m achievable by the process of Eq. (43). Eq. (50) satisfies the condition $(1-\gamma)J > 0$ and immediately yields $C^* = \beta W$. Substituting optimal consumption back into (48) we reduce it to

$$\max_{\{\omega \leq m\}} [G(\omega, m)] = 0, \tag{51}$$

where

$$G(\omega, m) = \beta(\log \beta - 1) + r - \frac{\beta}{1-\gamma} H + \frac{1}{1-\gamma} H' \left(\phi_m(\bar{m} - m) - Q_m \lambda(m-1)\right) + \\ \frac{1}{2(1-\gamma)}(H'' + (H')^2)\bar{\sigma}_m^2 (m-1) + \omega(\mu_S - \lambda Q_S) - \frac{\gamma}{2}\omega^2 \sigma_S^2 + H' \omega \sigma_S \bar{\sigma}_m \sqrt{m-1}\rho_{mS} + \\ \frac{\lambda}{1-\gamma}\left[(1+\omega Q_S)^{1-\gamma} \exp(H(m + (m-1)Q_m) - H(m)) - 1\right] = 0. \tag{52}$$

The solution to the maximization problem ω^* is either an internal point satisfying the first order condition $G_\omega(\omega^*, m) = 0$ or a point on the boundary $\omega^* = m$ if $G_\omega(\omega, m) > 0$ for all $\omega \leq m$. The first order condition yields Eq. (46) and its solutions (if they exist) are denoted as $\bar{\omega}$. Since the optimal wealth can never become negative (this will lead to infinitely negative utility), it is sufficient to consider only those portfolio weights for which $1 + \omega Q_S > 0$ (see Liu, Longstaff, and Pan (2003) for a similar result). Note that if $G(\omega, m)$ has at most one local maximum in $\{\omega : \omega < m\}$, then $\omega^* = \min(m, \bar{\omega})$. The latter condition is guaranteed by $G_{\omega\omega} < 0$ for all admissible ω and m, where

$$G_{\omega\omega} = -\gamma \left(\sigma_S^2 + \lambda Q_S^2 (1 + \omega Q_S)^{-1-\gamma} \exp(H(m + (m-1)Q_m) - H(m))\right). \tag{53}$$

The verification of $G_{\omega\omega} < 0$ is equivalent to checking the "verification theorem" (e.g. Fleming and Soner, 2006) and completes the proof.

Proposition 4 indicates that the solution to the portfolio problem has a form of a free boundary problem: choosing the portfolio policy the investor decides at which point m^* the margin constraint becomes binding. This point is determined by the equation $m = \bar{\omega}(m)$ and the optimal portfolio policy (45) has a two-region structure. If the margin constraint is sufficiently tight ($m \leq m^*$) the investor prefers to invest up to this limit and $\omega^* = m$. However, if $m > m^*$ the optimal portfolio is $\omega^* = \bar{\omega}$, where $\bar{\omega}$ solves Eq. (46).

The unconstrained demand $\bar{\omega}$ has a very intuitive structure. The first term is a standard myopic demand corrected for jumps. The second term is a hedging demand caused by continuous changes in the marginal requirement. From the structure of the value function, if $\gamma > 1$ the increase of $V(m, W)$ with m implies $H'(m) < 0$. Hence, when $\rho_{mS} > 0$ the hedging demand is negative and anticipating the possibility of binding margin constraints investors should diminish their positions in the risky asset. On the contrary, if $0 < \gamma < 1$ the increase in indirect utility with relaxation of margin constraints implies $H'(m) > 0$ and positive hedging demand when returns are positively correlated with innovations in m_t. Note that although this hedging demand is similar to the standard hedging demand of Merton (1971) and can be interpreted as investor's response to time-varying investment opportunities, its origin is different. Merton's hedging demand is produced by time variation in the Sharpe ratio, whereas the hedging demand in Eq. (46) results from variation in margin constraints. This difference manifests itself in the equation for the function $H(m)$. The third term in Eq. (46) corresponds to jump-related component and typically appears when price follows a jump-diffusion process (Liu, Longstaff, and Pan, 2003; Wu, 2003).

A.2 Analysis

To illustrate the implications of Proposition 4 consider sequentially several special cases. To avoid unnecessary complications, there are no jumps in Sections A.2.1, A.2.2, and A.2.3.

A.2.1 No margin requirements

This is the simplest benchmark with no restrictions on portfolio weights. The solution to the portfolio problem of Epstein-Zin investor facing constant investment opportunities is well-known (Svensson, 1989; Obstfeld, 1994). Similar to a CRRA investor analyzed in Merton (1969), investors with Epstein-Zin preferences also have constant myopic demand

$$\omega^* = \frac{\mu_S}{\gamma \sigma_S^2}, \tag{54}$$

which is completely determined by expected returns and volatility. The weight (54) coincides with the optimal leverage of an unconstrained investor, and this leverage will be considered a benchmark in the subsequent analysis.

A.2.2 Constant margin requirements

The next special case corresponds to a constant margin constraint: $m_t = \bar{m} = const$. It is a special case of borrowing constraints studied in the literature (Cvitanić and Karatzas, 1992; Teplá, 2000; Schroder and Skiadas, 2003). There are two possible outcomes here. If the myopic demand (54) does not violate the margin constraint, it solves the portfolio problem and the margin constraint appears to be irrelevant. When the myopic portfolio is not feasible, it is optimal for the investor to choose the highest possible weight of the risky asset $\omega = \bar{m}$ and the margin constraint binds all the time.[11] Thus, the optimal portfolio is

$$\omega^* = \min\left(\bar{m}, \frac{\mu_S}{\gamma \sigma_S^2}\right). \tag{55}$$

A.2.3 Time varying margin requirements without jumps

Now consider a more interesting situation in which the investor is exposed to time-varying margin constraints. To separate the effect of continuous changes in margin requirements from the effect of jumps, I first focus on the case with $\lambda = 0$. Without jumps, Eq. (46) becomes trivial and the optimal portfolio takes the form

$$\omega^* = \min\left(m, \frac{\mu_S}{\gamma \sigma_S^2} + \frac{\bar{\sigma}_m \sqrt{m-1}\rho_{mS}}{\gamma \sigma_S} H'(m)\right), \tag{56}$$

where the function $H(m)$ solves Eq. (47).

It is worth to compare the obtained hedging behavior of the investor with the standard dynamic hedging first examined by Merton (1971). In the latter case, hedging needs are due to time-varying expected excess returns or volatility of assets causing variation in the Sharpe ratio, which is associated with *instantaneous* investment opportunities. In the portfolio problem under consideration expected returns and volatility are fixed

[11] Since there is only one risky asset in the economy, the solution to the portfolio problem with a constant margin constraint is quite trivial. In the multiple asset case, however, the optimal portfolio is less obvious. As demonstrated by Cvitanić and Karatzas (1992) and Teplá (2000), when the myopic portfolio is not feasible and the margin constraint limits the aggregate position in risky assets the CRRA investors act as if unconstrained, but facing a higher interest rate. Similar to the single asset economy, the optimal portfolio sits on the boundary in the portfolio space.

and *instantaneous* investment opportunities do not vary over time. The hedging demand arises because of a distant possibility of binding margin constraints and has a long-run nature.

To illustrate the impact of dynamic margin constraints on the optimal portfolio policy, I consider several numerical examples. In these examples the rate of time preference β is chosen to be 5%, and the risk free rate is set at 3%. To make examples comparable, instead of fixing the parameters of the price process I keep constant the Sharpe ratio $SR = \mu_S/\sigma_S$ and the optimal leverage of unconstrained investors $L = \mu_S/(\gamma\sigma_S^2)$. In the benchmark specification $SR = 1$ and $L = 4$. Note that the return on the risky asset can be interpreted as a return on any risky strategy. Hence, the parameters μ_S and σ_S can significantly deviate from the values typical for the stock market.

It is natural to assume that margin requirements are quite persistent: long periods of high margins are typically followed by long periods of low margins. To reflect this fact, I assume that $\phi_m = 0.05$. The unconditional mean of margin is $\bar{m} = 4$, i.e. on average 75% of a position in the risky asset is financed by borrowed money. The volatility of shocks to the margin constraint σ_m is set at 0.5. It means that the unconditional volatility of m is 2.74. I also assume that innovations to asset prices are positively correlated with innovations to the margin constraints: $\rho_{mS} = 1$. This positive correlation is consistent with the intuition that positive returns are associated with declining volatility and, hence, more aggressive lending by financiers.

The optimal portfolio is determined by the solution to Eq. (47), which is a complicated ODE admitting only numerical solutions. To solve for $H(m)$, I use a projection method and approximate $H(m)$ by appropriately scaled Chebyshev polynomials. This approach guarantees that $H(m)$ is twice continuously differentiable for all $m \in (1, \infty)$. Further details about the numerical procedure are relegated to Appendix C.

Optimal demands of investors with various risk preferences are depicted in Figure 9. Similar to the standard hedging demand caused by variations in the Sharpe ratio, the sign of hedging demand generated by time-varying margin constraints depends on whether $\gamma > 1$ or $\gamma < 1$. When $\gamma = 1$ the hedging demand disappears and the investor maintains the optimal myopic leverage from Eq. (54) unless the margin constraint hits. If the allowed leverage is below myopic demand an investor with logarithmic preferences borrows up to the limit. However, a risk-averse investor with $\gamma > 1$ underinvests in comparison with the no margin requirements case and even when risk aversion is moderate hedging needs can significantly influence the optimal portfolio. For instance, even if the myopic optimal leverage is 4 an investor with $\gamma = 4$ keeps his leverage at a level below 3.5 and decreases it further as the margin gets tighter. For sufficiently low m an investor stops hedging and uses the whole borrowing capacity (sits on the margin constraint). Note that an anticipated constraint changes both the level and the slope of the demand function. Thus, not only an investor reduces his investments into the risky asset, but also he becomes sensitive to variation in margins. The latter feature will be especially important in the general equilibrium framework.

If $\gamma < 1$ the hedging demand has an opposite sign: constrained investors invest more in the risky asset than their unconstrained peers. Although quantitatively the effect is less pronounced than in the $\gamma > 1$ case, qualitatively the optimal demand resembles a strategy followed by some arbitrageurs: when prices decrease, they buy more assets until hitting the margin constraint and at this point they start selling. Hence, hedging time variation in margin constraints can at least partially rationalize this strategy.

To quantify the hedging demand, I consider average adjustments to the optimal portfolio policy caused by dynamic margin constraints. Specifically, I simulate (discretized) paths of prices and margin constraints and estimate an average hedging demand as

$$HD = E\left[\frac{\omega^* - \omega^*_{myopic}}{\omega^*_{myopic}}\right],$$

where ω^* is the optimal constrained demand and ω^*_{myopic} is myopic (unconstrained) demand. To examine the relation between the size of the hedging demand, investors' risk preferences, and characteristics of the risky asset I consider several parameter sets. The results are reported in Panel A of Table 2.

It is not surprising that the size of hedging demand is positively related to risk aversion for any considered values of the Sharpe ratio and myopic leverage. What is more interesting, hedging demand caused by margin requirements might have a substantial size even for moderate coefficients of risk aversion. Thus, when $\gamma = 4$, $SR = 2$, and $L^* = 4$ hedging reduces optimal demand by 17%. This is a clear indication that accounting for time varying margin requirements might be very important for portfolio managers.

Table 2 shows that the hedging demand is the largest when optimal leverage is equal to the mean value of the margin constraint, which is four in the considered examples. Indeed, if L^* is too low (e.g. $L^* = 2$), the margin constraint rarely binds and its existence is less relevant. As a result, hedging needs are small and the solution in close to the myopic demand. On the other hand, when optimal leverage is high (e.g. $L^* = 6$) the margin constraint binds quite often both in myopic and non-myopic cases. Hence, the impact of hedging demand is again reduced.

Another general conclusion from Table 2 is that the size of hedging demand grows with the Sharpe ratio SR. Since $SR = \gamma\sigma_S L$, keeping the coefficient of risk aversion and the optimal leverage fixed, an increase in the Sharpe ratio implies an increase in the volatility of returns. This makes the asset a better hedging instrument and increases the size of the hedging demand. This also indicates that hedge funds with higher Sharpe ratios should be more attentive to time variation of margin requirements.

Panel B of Table 2 reports the boundary values of margin requirements below which an investor finds it optimal to use the whole borrowing capacity. When $\gamma > 1$ this threshold is below optimal leverage indicating that the hedging demand is negative. Also, the difference $L^* - m^*$ is more pronounced for higher γ, higher SR, and higher L^* indicating an increasing size of hedging demand. When $\gamma < 1$ an investor might prefer to be on the constraint even when myopic allocation is feasible and $L^* < m^*$. This is consistent with Figure 9.

It is worth to note that hedging of time-varying margin requirements might take different forms depending on the exact form of portfolio constraints and financially constrained arbitrageurs might underinvest as well as overinvest in a risky asset. In Grossman and Vila (1992) investors with CRRA preferences are bounded by a borrowing constraint which is non-homogeneous in wealth. In contrast to this paper, the leverage constraint binds at *high* levels of wealth making indirect utility function more concave. Investors with high risk aversion (with $\gamma > 1$) overinvest in a risky asset at low levels of wealth (when margin does not bind) to be able to postpone the effect of binding margins in future. When $\gamma < 1$ investors underinvest in the risky asset. Thus, the effect of Grossman and Vila (1992) which is solely driven by an intercept in the margin constraint works in the opposite direction to the hedging demand of this paper. Shleifer and Vishny (1997) argue that arbitragers invest cautiously in underpriced assets since observing bad performance outside investors can withdraw capital exactly in the time when arbitrage opportunities are especially profitable. In Liu and Longstaff (2004) a risk-averse arbitrageur investing in an arbitrage opportunity represented by a Brownian bridge process faces a constant collateral constraint expressed in terms of "units of arbitrage", but not the dollar value of the position. In this framework, the arbitrageur finds it optimal to invest in the arbitrage less than the constraint permits to be able to exploit the opportunities if the arbitrage gap widens. In Attari and Mello (2006) risk-neutral arbitrageurs take a smaller position because a forced liquidation will drive the price downward exacerbating financial loss. The price impact is also the cause of underinvestment in Basak and Croitoru (2006).

A.2.4 Margin requirements with jumps

Next, consider the effect of pure jumps in the margin constraint on the optimal portfolio policy. Specifically, it is assumed that $\bar{\sigma}_m = 0$ and margins are updated irregularly when the price also abruptly changes. Note that even without margin constraints the equation for an optimal weight (46) is non-linear and in general cannot be solved analytically (Liu, Longstaff, and Pan, 2003; Wu, 2003). Moreover, in contrast to the pure diffusion case some combinations of risk aversion γ, the Sharpe ratio SR, the optimal leverage L, and the probability of jumps are unattainable. It is not difficult to show that the following inequality must be satisfied:

$$SR^2 + 4L\lambda\gamma Q_S \left[(1 + LQ_S)^{-\gamma}\right] > 0. \tag{57}$$

Thus, as opposed to the diffusion case, it is not straightforward to compare hedging demands for various leverages, risk aversion parameters, and the Sharpe ratios.

Consider the effect of risk aversion on the portfolio keeping other parameters as follows: the Sharpe ratio is $SR = 2$, and the optimal leverage without constraints is $L = 2$. As in the previous section, an average margin constraint coincides with the optimal leverage: $\bar{m} = 2$ and $\phi_m = 0.05$. Also, I set $\beta = 0.05$, $r = 0.03$ and maintain the desired optimal leverage and the Sharpe ratio by adjusting expected returns and volatility of the asset.

It is assumed that the jump probability parameter is $\lambda = 0.05$. In such event, the asset price drops by 20% and the margin constraint becomes tighter by 50%, i.e. $Q_S = -0.2$, $Q_m = -0.5$. Given the facts collected in Section 2, such intensity and sizes of jumps look quite reasonable.

Optimal portfolio policies for different γ are plotted in Figure 10. Qualitatively, the effect of jumps in margins is similar to that of continuous changes. Risk averse investors with $\gamma > 1$ prefer to underinvest in the asset if they anticipate even small probability of price drop accompanied by tightening of margins. For example, if $\gamma = 4$ instead of maintaining the optimal leverage $L = 2$ investors prefer to have a leverage slightly above 1.7. On the contrary, if $\gamma < 1$ it is optimal to overinvest in the risky asset, but quantitatively the effect is quite small.

This result indicates that even if investors do not need to renegotiate their margin constraints with lenders on a regular basis, the risk of sudden price drops accompanied by abrupt changes in margin requirements should be taken into consideration. Specifically, such risk should make a rational risk-averse investor with $\gamma > 1$ follow a more conservative strategy even if the jump risk does not affect instantaneous investment opportunities measured by the Sharpe ratio.

Appendix B.

This Appendix collects the proofs of several propositions stated in the main part of the paper.

Proof of Proposition 1 and 2.

Omitting arguments of the value function J as well as time and investor subscripts we get the following Bellman equation:

$$\max_{\tilde{X},C} \left(f(C, J) + \frac{1}{dt} E_t[dJ] \right) = 0,$$

where

$$\frac{1}{dt}E_t[dJ] = J_z\mu_z + \frac{1}{2}J_{zz}\sigma_z\sigma_z' + J_W\mu_Q'\left(\lambda(M)\hat{X} + (1-\lambda(M))\,\tilde{X}\right) + J_W(rW - C)$$
$$+\frac{1}{2}J_{WW}\left(\lambda(M)\hat{X}'\sigma_Q\sigma_Q'\hat{X} + (1-\lambda(M))\,\tilde{X}'\sigma_Q\sigma_Q'\tilde{X}\right) + J_{zW}\left(\lambda(M)\hat{X} + (1-\lambda(M))\,\tilde{X}\right)'\sigma_Q\sigma_z'. \quad (58)$$

Eq. (58) uses Ito's formula and the individual wealth dynamics (16). Recall that if the bid of a professional investor violates the margin constraint, the margin binds with a probability $\lambda(M)$, where

$$M = \frac{\sqrt{\tilde{X}'\sigma_Q\sigma_Q'\tilde{X}}}{\bar{m}W}$$

if portfolio is collateralized as a whole and

$$M = \frac{\tilde{X}'\sqrt{diag(\sigma_Q\sigma_Q')}}{\bar{m}W}$$

if individual assets are collateralized separately. Maximization with respect to C is quite standard and yields

$$C = \beta(1-\gamma)\frac{J}{J_W}.$$

Using $\hat{X} = \tilde{X}/M$ the optimal portfolio problem reduces to

$$\max_{\tilde{X}}\left(\tilde{X}'\left(\frac{\lambda(M)}{M} + 1 - \lambda(M)\right)(J_W\mu_Q + J_{zW}\sigma_Q\sigma_z') + \frac{1}{2}J_{WW}\left(\frac{\lambda(M)}{M^2} + 1 - \lambda(M)\right)\tilde{X}'\sigma_Q\sigma_Q'\tilde{X}\right) = 0.$$

Hence, maximization with respect to $\tilde{X}$ leads to the following first order condition:

$$(J_W\mu_Q + J_{zW}\sigma_Q\sigma_z')\left(\frac{\lambda}{M} + 1 - \lambda + \tilde{X}'M_{\tilde{X}}\left(\frac{\lambda'}{M} - \frac{\lambda}{M^2} - \lambda'\right)\right)$$
$$+ J_{WW}\left(\frac{\lambda(M)}{M^2} + 1 - \lambda(M)\right)\sigma_Q\sigma_Q'\tilde{X} + \frac{1}{2}J_{WW}(\tilde{X}'\sigma_Q\sigma_Q'\tilde{X})M_{\tilde{X}}\left(\frac{\lambda'}{M^2} - \frac{2\lambda}{M^3} - \lambda'\right) = 0, \quad (59)$$

where $M_{\tilde{X}}$ is a vector derivative of M with respect to $\tilde{X}$. Assuming the value function in the form

$$J(z, W) = \frac{1}{1-\gamma}W^{1-\gamma}\exp(H(z))$$

we get $C = \beta W$ and the following equation for $\tilde{X}$:

$$(\mu_Q + H'\sigma_Q\sigma_z')\left(\frac{\lambda}{M} + 1 - \lambda + \frac{\tilde{X}'}{W}(M_{\tilde{X}}W)\left(\frac{\lambda'}{M} - \frac{\lambda}{M^2} - \lambda'\right)\right)$$
$$-\gamma\left(\frac{\lambda}{M^2} + 1 - \lambda\right)\sigma_Q\sigma_Q'\frac{\tilde{X}}{W} - \frac{1}{2}\gamma\left(\frac{\tilde{X}'}{W}\sigma_Q\sigma_Q'\frac{\tilde{X}}{W}\right)(M_{\tilde{X}}W)\left(\frac{\lambda'}{M^2} - \frac{2\lambda}{M^3} - \lambda'\right) = 0. \quad (60)$$

Equation for $H(z)$ comes from the Bellman equation

$$\beta(1-\gamma)(\log\beta - 1) - \beta H + H'\mu_z(z) + \frac{1}{2}(H'' + (H')^2)\sigma_z\sigma_z' + r(1-\gamma)$$
$$+(1-\gamma)(\mu_Q + H'\sigma_Q\sigma_z')'\frac{\tilde{X}}{W}\left(\frac{\lambda}{M} + (1-\lambda)\right) - \frac{\gamma}{2}(1-\gamma)\left(\frac{\tilde{X}'}{W}\sigma_Q\sigma_Q'\frac{\tilde{X}}{W}\right)\left(\frac{\lambda}{M^2} + (1-\lambda)\right) = 0. \quad (61)$$

Eqs. (60) and (61) are quite complicated and, in general, cannot be simplified further without specification of the margin violation ratio M. Hence, I consider the cases of portfolio collateralization and collateralization of individual securities separately.

Portfolio collateralization. When portfolio is collateralized as a whole, the margin violation ratio and its derivatives are

$$M = \frac{\sqrt{\tilde{X}'\sigma_Q\sigma_Q'\tilde{X}}}{\bar{m}W}, \qquad M_{\tilde{X}}W = \frac{1}{\bar{m}^2 M}\sigma_Q\sigma_Q'\frac{\tilde{X}}{W}.$$

Hence, Eq. (60) yields

$$\left(\mu_Q + H'\sigma_Q\sigma_z'\right)\left(1-\lambda+\lambda'-\lambda' M\right) - \gamma\sigma_Q\sigma_Q'\frac{\tilde{X}}{W}\left(1-\lambda+\frac{\lambda'}{2M}-\frac{1}{2}\lambda' M\right) = 0.$$

Assuming invertibility of σ_Q, we get

$$\frac{1}{\bar{m}}\sigma_Q'\frac{\tilde{X}}{W}\left(2(1-\lambda)+\frac{\lambda'}{M}-\lambda' M\right) = \frac{2}{\gamma\bar{m}}\sigma_Q^{-1}\left(\mu_Q + H'\sigma_Q\sigma_z'\right)\left(1-\lambda+\lambda'-\lambda' M\right). \tag{62}$$

Lemma 2 *The solution to Eq. (62) is*

$$\frac{\tilde{X}}{W} = \frac{2}{\gamma}\frac{y(A)}{A}\left[\sigma_Q\sigma_Q'\right]^{-1}\left(\mu_Q + H'\sigma_Q\sigma_z'\right), \tag{63}$$

where

$$A = \frac{2}{\gamma\bar{m}}\sqrt{\left(\mu_Q + H'\sigma_Q\sigma_z'\right)'\left[\sigma_Q\sigma_Q'\right]^{-1}\left(\mu_Q + H'\sigma_Q\sigma_z'\right)}. \tag{64}$$

and the function $y(x)$ is defined in Section 3.3.4.

Proof. Multiplying Eq. (62) by its own transposition and using the difinition of M we arrive at

$$M\left(2(1-\lambda)+\frac{\lambda'}{M}-\lambda' M\right) = \frac{2}{\gamma\bar{m}}\sqrt{\left(\mu_Q + H'\sigma_Q\sigma_z'\right)'\left[\sigma_Q\sigma_Q'\right]^{-1}\left(\mu_Q + H'\sigma_Q\sigma_z'\right)}\left(1-\lambda+\lambda'-\lambda' M\right). \tag{65}$$

Using Eq. (64) and noticing that Eq. (65) coincides with Eq. (20) we get

$$M = y(A).$$

Combining Eq. (62) with (65) we arrive at

$$A\frac{1}{\bar{m}}\sigma_Q'\frac{\tilde{X}}{W} = \frac{2}{\gamma\bar{m}}\sigma_Q^{-1}\left(\mu_Q + H'\sigma_Q\sigma_z'\right)M, \tag{66}$$

which immediately yields Eq. (63). ■

Given the solution for $\tilde{X}/W$, the Bellman equation (61) can be significantly simplified. Indeed,

$$(\mu_Q + H'\sigma_Q\sigma_z')'\frac{\tilde{X}}{W} = \frac{\gamma\bar{m}^2}{2}y(A)A, \qquad \frac{\tilde{X}'}{W}\sigma_Q\sigma_Q'\frac{\tilde{X}}{W} = \bar{m}^2 y(A)^2.$$

Hence, the equation for $H(z)$ is

$$\beta(1-\gamma)(\log\beta - 1) - \beta H + H'\mu_z(z) + \frac{1}{2}(H'' + (H')^2)\sigma_z\sigma_z' + r(1-\gamma)$$
$$+(1-\gamma)\frac{\gamma\bar{m}^2}{2}A\left(\lambda + (1-\lambda)\,y(A)\right) - \frac{\gamma\bar{m}^2}{2}(1-\gamma)\left(\lambda + (1-\lambda)\,y(A)^2\right) = 0$$

or

$$\beta(1-\gamma)(\log\beta - 1) - \beta H + H'\mu_z(z) + \frac{1}{2}(H'' + (H')^2)\sigma_z\sigma_z' + r(1-\gamma)$$
$$+(1-\gamma)\frac{\gamma\bar{m}^2}{2}\left[\lambda(A-1) + (1-\lambda)\,y(A)(A - y(A))\right] = 0.$$

In the symmetric equilibrium further simplification is attainable. Indeed,

$$\frac{\tilde{X}}{W} = \frac{\tilde{x}}{W}\iota, \quad \mu_Q = \bar{\mu}_Q\iota, \quad \sigma_z = \bar{\sigma}_z\iota', \, \sigma_Q = \bar{\sigma}_Q I + \bar{\bar{\sigma}}_Q(\iota\iota' - I)$$

where $\tilde{x}$, $\bar{\mu}_Q$, $\bar{\sigma}_z$, $\bar{\sigma}_Q$, and $\bar{\bar{\sigma}}_Q$ are scalars. I is a K-dimensional unit matrix and ι is a $K \times 1$ vector consisting of ones. Simple calculation yields

$$\mu_Q + H'\sigma_Q\sigma_z' = \left[\bar{\mu}_Q + H'(\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1))\bar{\sigma}_z\right]\iota,$$

$$\sigma_Q\sigma_Q' = (\bar{\sigma}_Q - \bar{\bar{\sigma}}_Q)^2 I + \left[2(\bar{\sigma}_Q - \bar{\bar{\sigma}}_Q) + K\bar{\bar{\sigma}}_Q\right]\bar{\bar{\sigma}}_Q\iota\iota',$$

$$\left[\sigma_Q\sigma_Q'\right]^{-1}\iota = \frac{1}{(\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1))^2}\iota.$$

Thus, the optimal strategy of Eq. (63) reduces to

$$\frac{\tilde{x}}{W} = \frac{\bar{m}y(A)}{\sqrt{K}(\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1))},$$

where

$$A = \frac{2\sqrt{K}}{\gamma\bar{m}}\left[\frac{\bar{\mu}_Q}{\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1)} + H'\bar{\sigma}_z\right]. \tag{67}$$

Collateralization of individual assets. When individual assets are collateralized separately, the margin violation ratio and its derivatives are

$$M = \frac{\tilde{X}'\sqrt{diag(\sigma_Q\sigma_Q')}}{\bar{m}W}, \qquad M_{\tilde{X}}W = \frac{\sqrt{diag(\sigma_Q\sigma_Q')}}{\bar{m}}.$$

The equation for the optimal policy (60) gives

$$\begin{aligned}&\left(\mu_Q + H'\sigma_Q\sigma_z'\right)\left(1 - \lambda + \lambda' - \lambda' M\right)\\ &\quad - \gamma\left(\frac{\lambda}{M^2} + 1 - \lambda\right)\sigma_Q\sigma_Q'\frac{\tilde{X}}{W} - \frac{1}{2}\gamma\left(\frac{\tilde{X}'}{W}\sigma_Q\sigma_Q'\frac{\tilde{X}}{W}\right)\frac{\sqrt{diag(\sigma_Q\sigma_Q')}}{\bar{m}}\left(\frac{\lambda'}{M^2} - \frac{2\lambda}{M^3} - \lambda'\right) = 0.\end{aligned} \tag{68}$$

This equation cannot be simplified further without using that all assets are identical. Similar to the case with portfolio collateralization, it is convenient to introduce $\tilde{x}$, $\bar{\mu}_Q$, $\bar{\sigma}_z$, $\bar{\sigma}_Q$, and $\bar{\bar{\sigma}}_Q$ such that

$$\frac{\tilde{X}}{W} = \frac{\tilde{x}}{W}\iota, \quad \mu_Q = \bar{\mu}_Q\iota, \quad \sigma_z = \bar{\sigma}_z\iota', \, \sigma_Q = \bar{\sigma}_Q I + \bar{\bar{\sigma}}_Q(\iota\iota' - I).$$

Then, the margin violation ratio is

$$M = \frac{\tilde{x}}{W}\frac{K\sqrt{(\sigma_Q\sigma_Q')_{11}}}{\bar{m}}, \quad (\sigma_Q\sigma_Q')_{11} = \bar{\sigma}_Q^2 + (K-1)\bar{\bar{\sigma}}_Q^2.$$

Multiplying Eq. (68) by $\tilde{X}/W$ we get

$$\frac{\tilde{X}'}{W}\left(\mu_Q + H'\sigma_Q\sigma_z'\right)\left(1 - \lambda + \lambda' - \lambda' M\right) - \gamma\frac{\tilde{X}'}{W}\sigma_Q\sigma_Q'\frac{\tilde{X}}{W}\left(1 - \lambda + \frac{\lambda'}{2M} - \frac{1}{2}\lambda' M\right) = 0.$$

The substitution $\tilde{X}/W = \tilde{x}/W\iota$ yields

$$\iota'\left(\mu_Q + H'\sigma_Q\sigma_z'\right)\left(1-\lambda+\lambda'-\lambda' M\right) - \gamma\left(\frac{\tilde{x}}{W}\right)\iota'\sigma_Q\sigma_Q'\iota\left(1-\lambda+\frac{\lambda'}{2M}-\frac{1}{2}\lambda' M\right)=0$$

or

$$\iota'\left(\mu_Q + H'\sigma_Q\sigma_z'\right)\left(1-\lambda+\lambda'-\lambda' M\right) - \frac{\bar{m}\gamma}{K\sqrt{(\sigma_Q\sigma_Q')_{11}}}\iota'\sigma_Q\sigma_Q'\iota\left((1-\lambda)M+\frac{\lambda'}{2}-\frac{1}{2}\lambda' M^2\right)=0$$

or

$$\frac{2\iota'\left(\mu_Q + H'\sigma_Q\sigma_z'\right)K\sqrt{(\sigma_Q\sigma_Q')_{11}}}{\gamma\bar{m}\iota'\sigma_Q\sigma_Q'\iota}\left(1-\lambda+\lambda'-\lambda' M\right) = 2(1-\lambda)M+\lambda'-\lambda' M^2. \tag{69}$$

Remarkably, Eq. (69) has exactly the same structure as Eq. (20). Denoting

$$A = \frac{2\iota'\left(\mu_Q + H'\sigma_Q\sigma_z'\right)K\sqrt{(\sigma_Q\sigma_Q')_{11}}}{\gamma\bar{m}\iota'\sigma_Q\sigma_Q'\iota}$$

we get

$$M = y\left(A\right).$$

The optimal strategy is

$$\frac{\tilde{x}}{W} = \frac{\bar{m}}{K\sqrt{(\sigma_Q\sigma_Q')_{11}}}y\left(A\right).$$

or, noting that

$$(\sigma_Q\sigma_Q')_{11} = \bar{\sigma}_Q^2 + (K-1)\bar{\bar{\sigma}}_Q^2, \qquad \iota'\sigma_Q\sigma_Q'\iota = K(\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1))^2,$$

we get

$$\frac{\tilde{x}}{W} = \frac{\bar{m}}{K\sqrt{\bar{\sigma}_Q^2 + (K-1)\bar{\bar{\sigma}}_Q^2}}y\left(\frac{2K\left[\bar{\mu}_Q + H'(\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1))\bar{\sigma}_z\right]\sqrt{\bar{\sigma}_Q^2 + (K-1)\bar{\bar{\sigma}}_Q^2}}{\gamma\bar{m}(\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1))^2}\right).$$

Given the solution for $\tilde{X}/W$ the Bellman equation (61) is

$$\beta(1-\gamma)(\log\beta - 1) - \beta H + H'\mu_z(z) + \frac{1}{2}(H'' + (H')^2)\sigma_z\sigma_z' + r(1-\gamma)$$

$$+(1-\gamma)\bar{m}\frac{\bar{\mu}_Q + H'(\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1))\bar{\sigma}_z}{\sqrt{\bar{\sigma}_Q^2 + (K-1)\bar{\bar{\sigma}}_Q^2}}\left(\lambda + (1-\lambda)\,y(A)\right) - \frac{\gamma\bar{m}^2}{2K}(1-\gamma)\frac{(\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1))^2}{\bar{\sigma}_Q^2 + (K-1)\bar{\bar{\sigma}}_Q^2}\left(\lambda + (1-\lambda)\,y^2(A)\right) = 0,$$

$$\beta(1-\gamma)(\log\beta - 1) - \beta H + H'\mu_z(z) + \frac{K}{2}(H'' + (H')^2)\bar{\sigma}_z^2 + r(1-\gamma)$$

$$+(1-\gamma)\frac{\gamma\bar{m}^2}{2K}\frac{(\bar{\sigma}_Q + \bar{\bar{\sigma}}_Q(K-1))^2}{\bar{\sigma}_Q^2 + (K-1)\bar{\bar{\sigma}}_Q^2}\left[\lambda(A-1) + (1-\lambda)\,y(A)(A - y(A))\right] = 0.$$

This completes the proof.

Proof of Proposition 3.

To simplify notations, subscripts are omitted. Using the definition of excess return $dQ = dS + Ddt - rSdt$ together with the equilibrium price function (33), the definition of fundamentals (5), and the dividend process (1) we get

$$dQ = \bar{\gamma}dX + r\bar{\gamma}(\bar{X}\iota - X)dt + \frac{\sigma_D dB}{r+\phi}.$$

In the equilibrium X is a function of W only. Hence, applying Ito's formula to $X(W)$ and using the dynamics of aggregate wealth

$$dW = X'(\mu_Q dt + \sigma_Q dB) + (r-\beta)W dt \tag{70}$$

together with the linear relation between aggregate consumption and aggregate wealth $C = \beta W$ we get:

$$dX = X_W(X'(\mu_Q dt + \sigma_Q dB) + (r-\beta)W dt) + \frac{1}{2}X_{WW}(X'\sigma_Q\sigma_Q' X)dt.$$

Hence,

$$dQ = \bar{\gamma}X_W(X'(\mu_Q dt + \sigma_Q dB) + (r-\beta)W dt) + \frac{1}{2}\bar{\gamma}X_{WW}(X'\sigma_Q\sigma_Q' X)dt + r\bar{\gamma}(\bar{X}\iota - X)dt + \frac{\sigma_D dB}{r+\phi}$$

and using the definition $dQ = \mu_Q dt + \sigma_Q dB$ we arrive at matrix equations for μ_Q and σ_Q

$$\mu_Q = \bar{\gamma}X_W(X'\mu_Q + (r-\beta)W) + \frac{1}{2}\bar{\gamma}X_{WW}(X'\sigma_Q\sigma_Q' X) + r\bar{\gamma}(\bar{X} - X), \tag{71}$$

$$\sigma_Q = \bar{\gamma}X_W(X'\sigma_Q) + \frac{\sigma_D I}{r+\phi}, \tag{72}$$

where I is a K-dimensional unit matrix. Solving Eqs. (71) and (72) for μ_Q and σ_Q we get

$$\mu_Q = \frac{\bar{\gamma}(r-\beta)X_W W + \frac{1}{2}\bar{\gamma}X_{WW}(X'\sigma_Q\sigma_Q' X) + r\bar{\gamma}(\bar{X}\iota - X)}{1 - \bar{\gamma}X_W' X} \tag{73}$$

$$+\frac{\bar{\gamma}^2 r\left[X'(\bar{X}\iota - X)X_W - (X'X_W)(\bar{X}\iota - X)\right] + \frac{1}{2}\bar{\gamma}^2(X'\sigma_Q\sigma_Q' X)\left[(X'X_{WW})X_W - (X'X_W)X_{WW}\right]}{1 - \bar{\gamma}X_W' X},$$

$$\sigma_Q = \frac{\sigma_D}{r+\phi}\frac{I + \bar{\gamma}(X_W X' - X'X_W I)}{1 - \bar{\gamma}X_W' X}. \tag{74}$$

If all assets are identical, i.e. $X(W) = x(W)\iota$ then Eqs. (73) and (74) can be represented as $\mu_Q = \bar{\mu}_Q\iota$ and $\sigma_Q = \bar{\sigma}_Q I + \bar{\bar{\sigma}}_Q(\iota\iota' - I)$, where $\bar{\mu}_Q$, $\bar{\sigma}_Q$, $\bar{\bar{\sigma}}_Q$ are the following scalars:

$$\bar{\mu}_Q = \frac{\bar{\gamma}(r-\beta)x_W W + \frac{K}{2\bar{\gamma}}\frac{x_{WW}}{x_W^2}\bar{\bar{\sigma}}_Q^2 + r\bar{\gamma}(\bar{X} - x)}{1 - \bar{\gamma}Kx_W x}$$

$$\bar{\sigma}_Q = \frac{\sigma_D}{r+\phi}\frac{1 - (K-1)\bar{\gamma}x_W x}{1 - \bar{\gamma}Kx_W x}, \qquad \bar{\bar{\sigma}}_Q = \frac{\sigma_D}{r+\phi}\frac{\bar{\gamma}x_W x}{1 - \bar{\gamma}Kx_W x}.$$

This completes the proof.

Appendix C.

This Appendix reports some details on numerical methods used to solve non-linear differential equations appearing in Section 3 and Appendix A. The key idea is to use the projection method, which is essentially equivalent to looking for an approximate solution in the form of a linear combination of orthogonal polynomials (Judd, 1998). As an orthogonal basis Chebyshev polynomials of the first kind has been used. Since Chebyshev polynomials form an orthogonal basis in $L^2([-1,1])$, the independent variables are rescaled to fit this interval. Denote the state variable as Z: it is the aggregate wealth of professional investors W in Section 3 and exogenous margin requirement $m-1$ in Appendix A. In both cases the range of Z is $(0,\infty)$. The rescaled variable is

$$z = \frac{Z-1}{Z+1} \tag{75}$$

and the range of z is $(-1,1)$. Thus, the transformation (75) monotonically and smoothly maps an interval $(0,\infty)$ into an interval $(-1,1)$: the boundary $Z=0$ corresponds to $z=-1$ and when Z tends to infinity z approaches 1. Besides being a natural domain for Chebyshev polynomials, the interval $(-1,1)$ also facilitates the reporting of results. The inverse map to (75) is

$$Z = \frac{1+z}{1-z}.$$

The change of variables (75) also affects derivatives appearing in the differential equations. It is easy to show that derivatives with respect to Z and z are related as

$$\frac{\partial}{\partial Z} = \frac{(1-z)^2}{2}\frac{\partial}{\partial z}, \qquad \frac{\partial^2}{\partial Z^2} = \frac{(1-z)^4}{4}\frac{\partial^2}{\partial z^2} - \frac{(1-z)^3}{2}\frac{\partial}{\partial z}.$$

More specifically, denote the solution to the system of differential equations as $\{Y^i(z)\}$, $i = 1, ..., L$. For example, in Section 3 $L = 2$ and $\{Y^1(z), Y^2(z)\} \equiv \{H(z), X(z)\}$, whereas in Appendix A $L = 1$ and $Y^1(z) \equiv H(z)$. Then, the projection method prescribes to look for a solution in the following form

$$Y^i(z) = \sum_{j=0}^{N} a_j^i T_j(z),$$

where $\{T_j(z), j = 0...N\}$ are Chebyshev polynomials of the first kind and N denotes the highest order. Thus, the problem is to find the coefficients a_j^i such that $Y^i(z)$ minimizes the deviation of the left hand side of differential equations from zero and satisfies the boundary conditions. Since $T_j(-1) = (-1)^j$ and $T_j(1) = 1$ the boundary conditions at $z = -1$ and $z = 1$ put linear restrictions on the unknown coefficients a_j^i

$$Y^i(-1) = \sum_{j=0}^{N} a_j^i (-1)^j, \qquad Y^i(1) = \sum_{j=0}^{N} a_j^i.$$

To set the objective function, I use the overidentified collocation method, which prescribes to minimize the sum of squared errors computed at points $\{z_m,\ m = 0, ..., M\}$ and $M > N$. For better approximation, instead of a uniform grid I use a Chebyshev array, which is a set of points where $T_M(z_m) = 0$ for all $\{z_m,\ m = 0, ..., M\}$. Since the optimization problem is overidentified, it is possible to ensure the existence of the solution if the value of the objective function at the optimal point is close to zero. In the practical implementation of the projection method I use $M = 100$ and $N = 30$. For these parameters, the error of approximation is typically of order 10^{-7}. Moreover, the results are stable with respect to variation in the degree of Chebyshev polynomials and the number of points, and this stability provides additional evidence that the numerical approximation indeed converges to the exact solution.

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Table I
Quantiles of σ_Q^{tot} and ρ_Q

This table reports 5% and 95% quantiles q_α of simulated distributions of σ_Q^{tot} and ρ_Q for different number of assets N in the economy without margin constraints. The model parameters are as follows: $r = 0.03$, $\beta = 0.05$, $\gamma = 4$, $\bar{\gamma} = 6$, $\sigma_F = 0.2$.

	$N = 1$	$N = 3$	$N = 5$	$N = 7$
$q_{0.05}(\sigma_Q^{tot})$	0.2003	0.2003	0.2004	0.2004
$q_{0.95}(\sigma_Q^{tot})$	0.3755	0.4180	0.4256	0.4270
$q_{0.05}(\rho_Q)$	—	0.0032	0.0038	0.0043
$q_{0.95}(\rho_Q)$	—	0.7710	0.7792	0.7806

Table II
Relative size of hedging demand under dynamic margin requirements

This table reports the percentage reduction HD of the optimal portfolio weight produced by hedging time varying margin constraints (Panel A) and the boundary m^* between constrained and unconstrained regions (Panel B). The variable HD is defined as $HD = E[(\omega^* - \omega^*_{myopic})/\omega^*_{myopic}]$, where ω^* is optimal portfolio weight and ω^*_{myopic} is myopic demand. SR is the Sharpe ratio, γ is the coefficient of risk aversion, and L^* is the optimal leverage of an unconstrained investor. Other parameters are as follows: $r = 0.03$, $\beta = 0.05$, $\phi_m = 0.05$, $\bar{m} = 4$, $\sigma_m = 0.5$, $\rho_{mS} = 1$.

	$SR = 1$			$SR = 2$		
	$\gamma = 0.5$	$\gamma = 2$	$\gamma = 4$	$\gamma = 0.5$	$\gamma = 2$	$\gamma = 4$
Panel A: Contribution of hedging demand HD						
$L^* = 2$	$< 10^{-2}$	-3.08	-4.96	$< 10^{-2}$	-6.69	-9.07
$L^* = 4$	3.24×10^{-1}	-6.54	-9.87	$< 10^{-2}$	-13.0	-17.2
$L^* = 6$	$< 10^{-2}$	-6.01×10^{-2}	-2.41×10^{-1}	$< 10^{-2}$	-5.30×10^{-1}	-1.51
Panel B: Boundary m^* between constrained and unconstrained regions						
$L^* = 2$	2.02	1.89	1.85	2.01	1.81	1.76
$L^* = 4$	4.06	3.56	3.39	4.01	3.25	3.06
$L^* = 6$	6.08	5.24	4.96	6.02	4.74	4.45

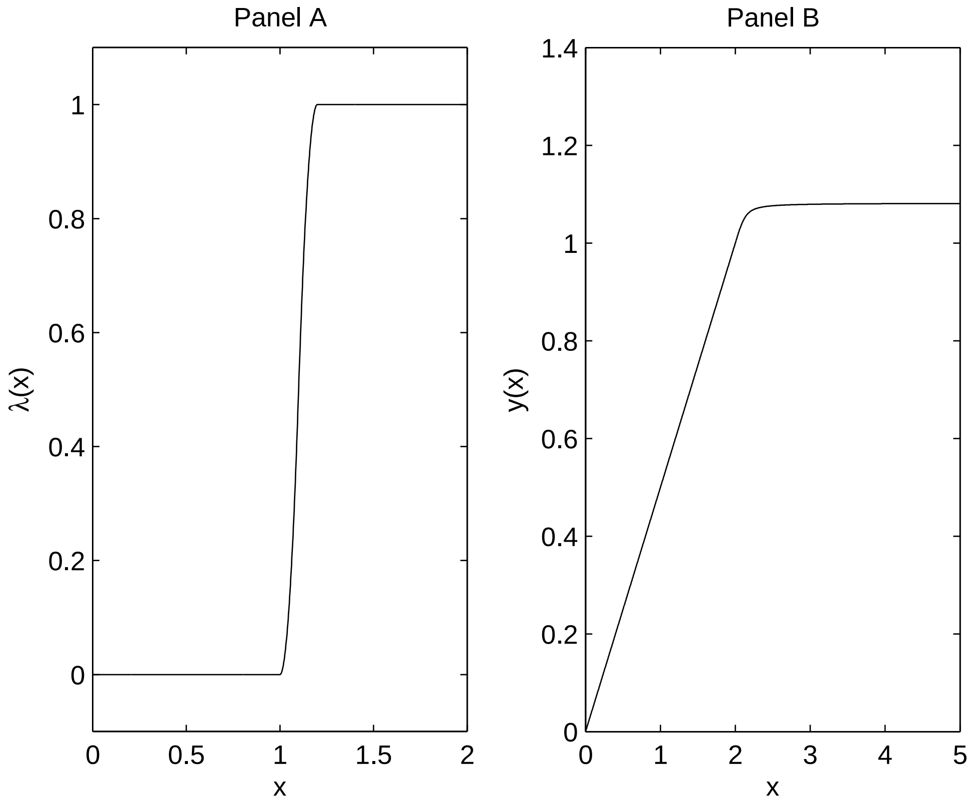

Figure 1. Functions $\lambda(x)$ and $y(x)$. Panel A plots the probability function $\lambda(x)$ defined by Eq. (14). Panel B depicts the function $y(x)$ defined in Proposition 1. The parameter $\bar{\lambda}$ is 50.

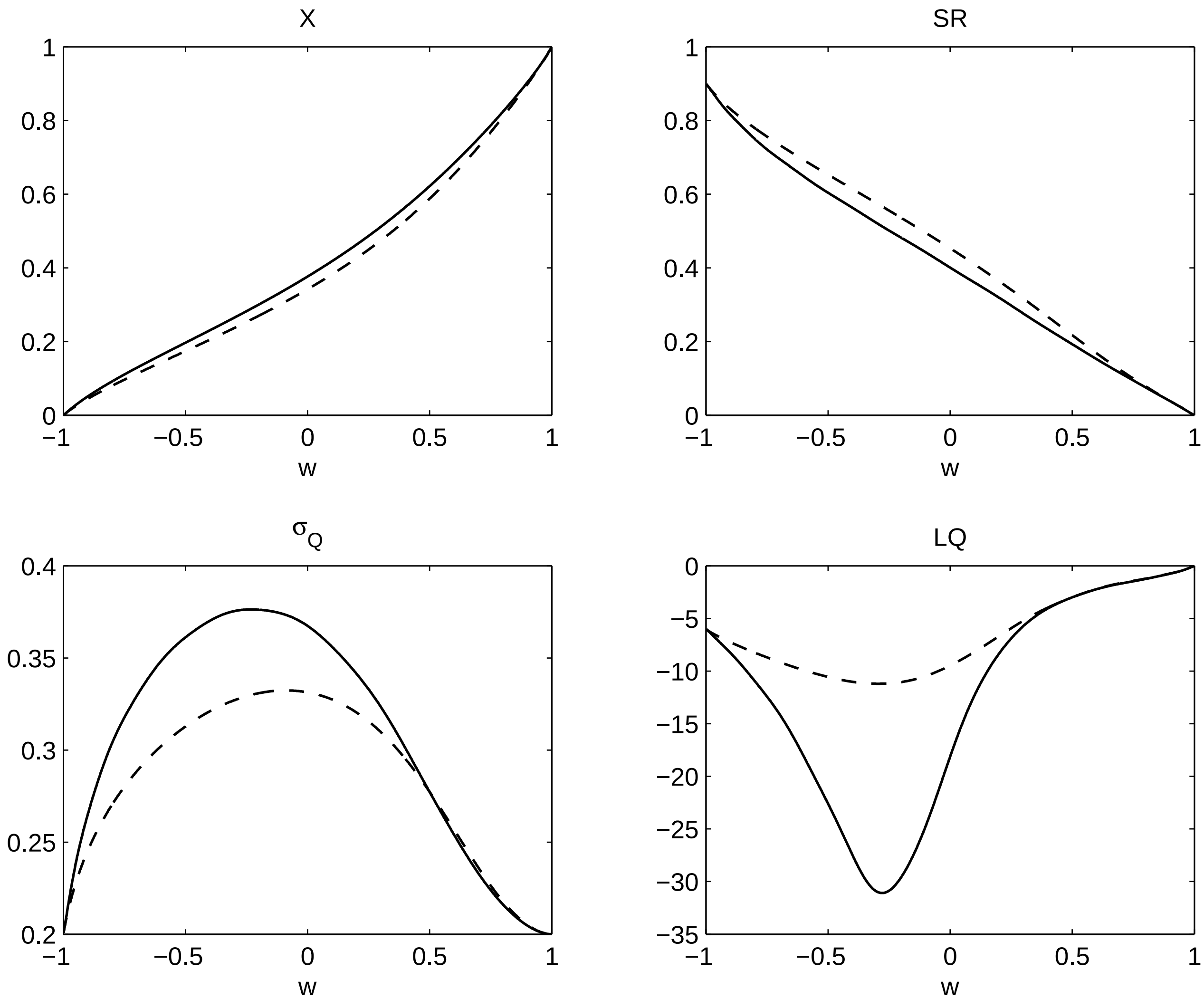

Figure 2. The equilibrium without margin constraints in the single asset setup. This figure presents the aggregate demand of professional investors X, the Sharpe ratio $SR = \mu_Q/\sigma_Q$, the volatility of returns σ_Q, and the liquidity of the market LQ in the absence of margin constraints. All statistics are functions of rescaled wealth of professional investors $w = (W-1)/(W+1)$. The model parameters are as follows: $r = 0.03$, $\beta = 0.05$, $\gamma = 4$, $\bar{\gamma} = 6$, $\sigma_F = 0.2$. The dashed line corresponds to the equilibrium with myopic investors ($H' = 0$), the solid line corresponds to the equilibrium with fully rational investors.

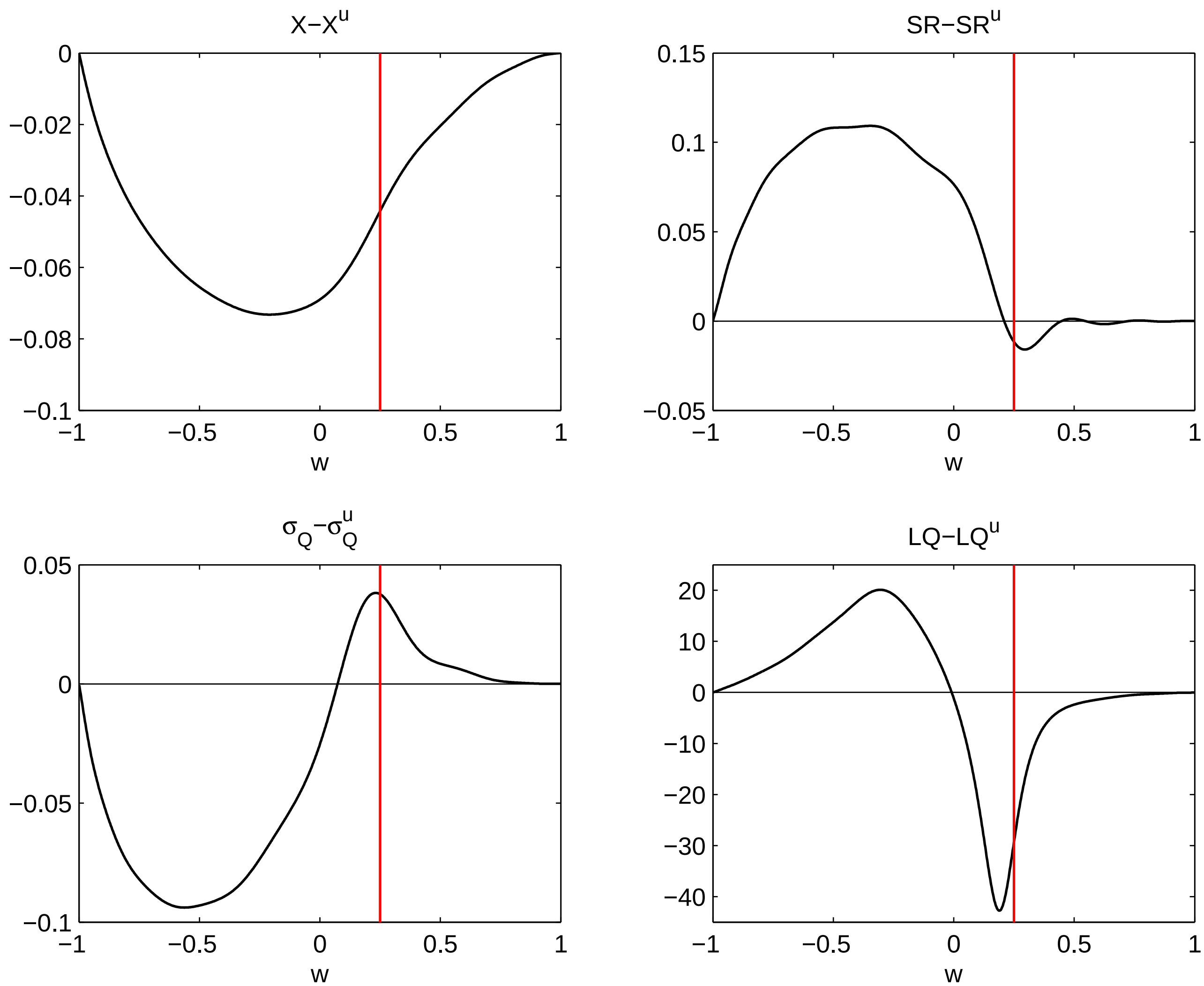

Figure 3. The impact of margin constraints on equilibrium statistics in the single asset setup. This figure presents the difference between several statistics in the economy with and without margin constraints. X is the aggregate demand of professional investors, $SR = \mu_Q/\sigma_Q$ is the Sharpe ratio, σ_Q is the volatility of returns, and LQ is the liquidity of the market. All statistics are functions of rescaled wealth of professional investors $w = (W-1)/(W+1)$. The model parameters are as follows: $r = 0.03$, $\beta = 0.05$, $\gamma = 4$, $\bar{\gamma} = 6$, $\sigma_F = 0.2$, $\bar{m} = 0.1$, $\bar{\lambda} = 50$. The vertical solid line indicates the wealth of professional investors at which the margin constraint starts binding.

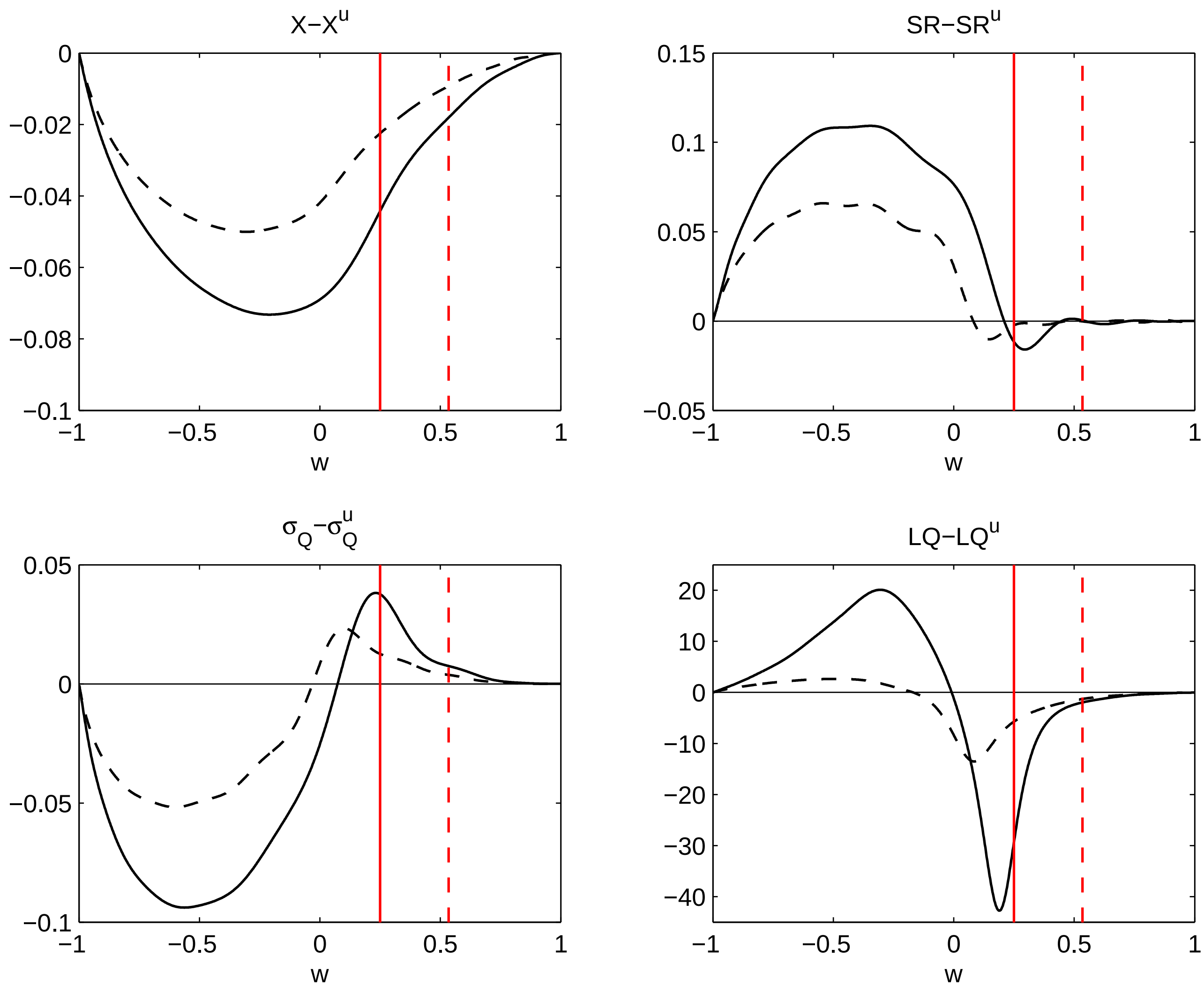

Figure 4. The effect of margin constraints with and without hedging demand in the single asset setup. This figure presents the difference between several statistics in the economy with and without margin constraints. X is the aggregate demand of professional investors, $SR = \mu_Q/\sigma_Q$ is the Sharpe ratio, σ_Q is the volatility of returns, and LQ is the liquidity of the market. All statistics are functions of rescaled wealth of professional investors $w = (W-5)/(W+5)$. The model parameters are as follows: $r = 0.03$, $\beta = 0.05$, $\gamma = 4$, $\bar{\gamma} = 6$, $\sigma_F = 0.2$, $\bar{m} = 0.1$, $\bar{\lambda} = 50$. The dashed line corresponds to the equilibrium with myopic investors, the solid line corresponds to the equilibrium with fully rational investors. The vertical lines indicate wealth at which the margin constraint starts binding in the myopic (dashed line) and fully rational (solid line) equilibria.

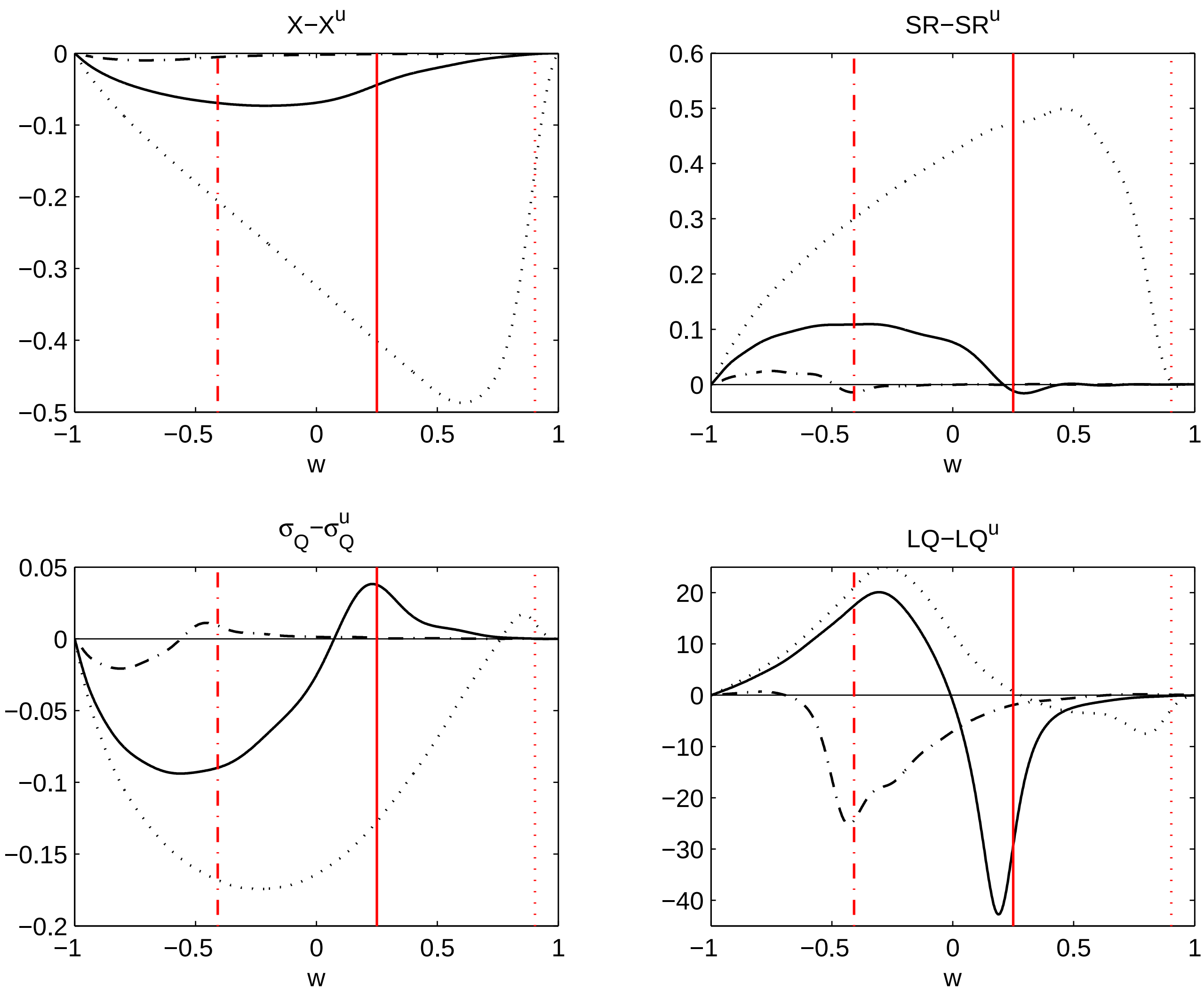

Figure 5. **Comparative statics with respect to $\bar{m}$.** This figure presents the difference between several statistics in the economy with and without margin constraints. X is the aggregate demand of professional investors, $SR = \mu_Q/\sigma_Q$ is the Sharpe ratio, σ_Q is the volatility of returns, and LQ is the liquidity of the market. All statistics are functions of rescaled wealth of professional investors $w = (W-1)/(W+1)$. The model parameters are as follows: $r = 0.03$, $\beta = 0.05$, $\gamma = 4$, $\bar{\gamma} = 6$, $\sigma_F = 0.2$, $\bar{\lambda} = 50$. Dotted line corresponds to $\bar{m} = 0.01$, solid line corresponds to $\bar{m} = 0.1$, dashed-dot line corresponds to $\bar{m} = 0.2$. The vertical solid line indicates the wealth of professional investors at which the margin constraint starts binding.

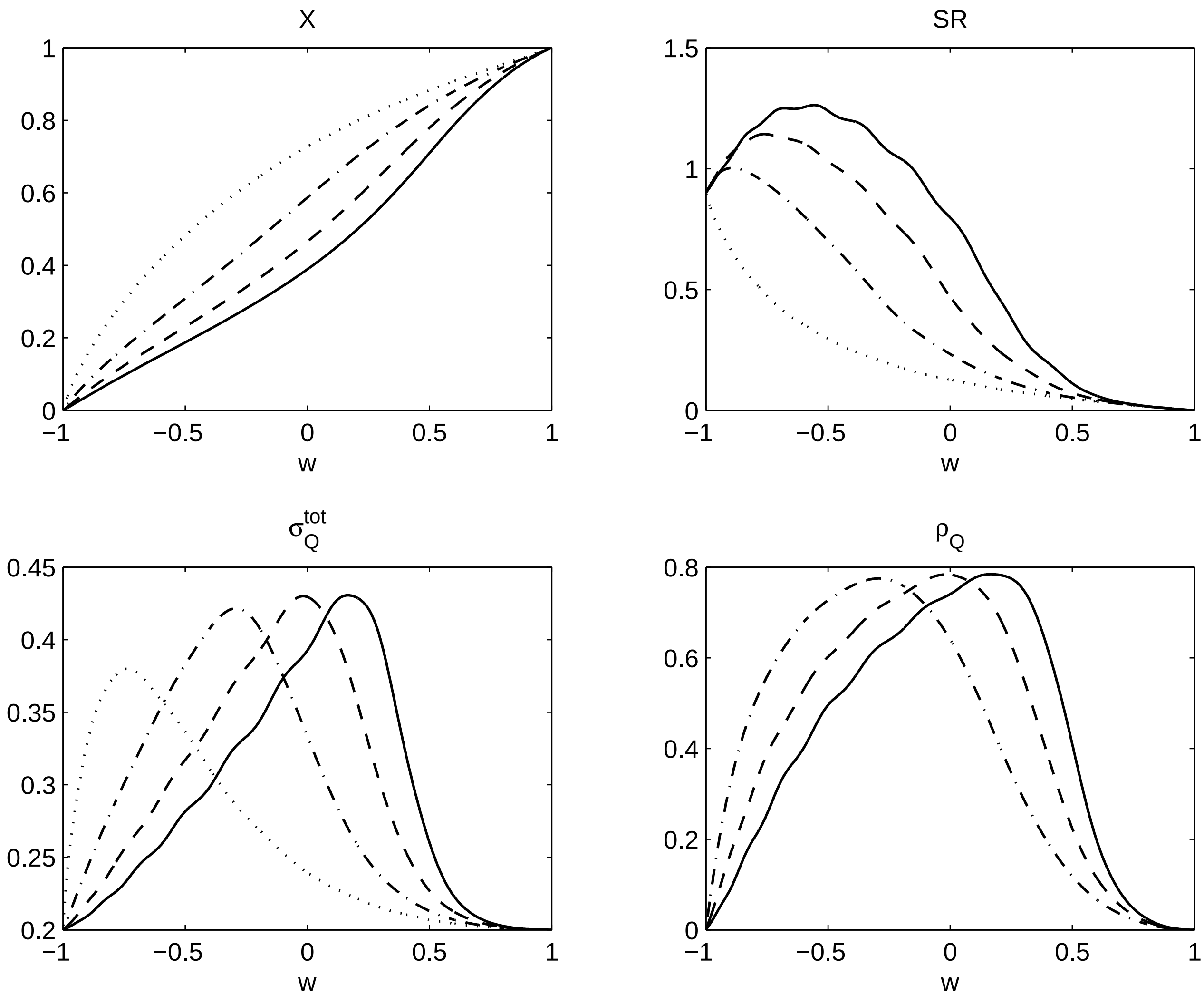

Figure 6. Equilibrium statistics in a multiple asset case without margin constraints. This figure presents the aggregate demand of professional investors X, the Sharpe ratio SR defined by Eq. (39), the total volatility of returns σ_Q^{tot}, and the correlation of returns ρ_Q defined by Eq. (40). All statistics are functions of the rescaled wealth of professional investors $w = (W-1)/(W+1)$ and computed for different number of assets K. Dotted line corresponds to $K = 1$, dash-dotted line corresponds to $K = 3$, dashed line corresponds to $K = 5$, and solid line corresponds to $K = 7$. The model parameters are as follows: $r = 0.03$, $\beta = 0.05$, $\gamma = 4$, $\bar{\gamma} = 6$, $\sigma_F = 0.2$.

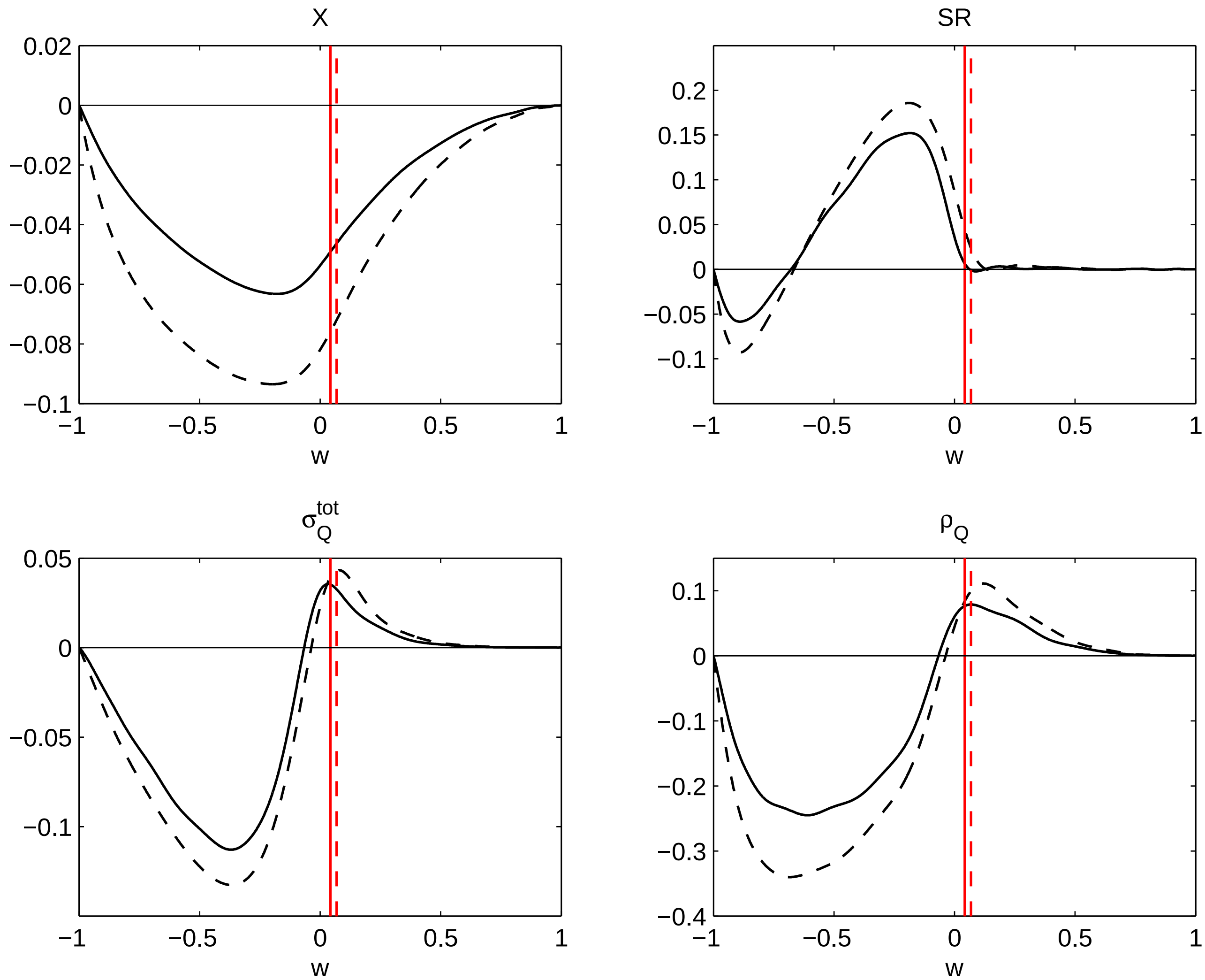

Figure 7. Equilibrium statistics in a multiple asset case with margin constraints. This figure presents the difference between several statistics in the economy with margin constraints and the economy without margins. X is the aggregate demand of professional investors; the Sharpe ratio SR is defined by Eq. (39); σ_Q^{tot} is the total volatility of returns; ρ_Q is the correlation of returns defined by Eq. (40). All statistics are functions of the rescaled wealth of professional investors $w = (W - 10)/(W + 10)$ and computed for the number of assets $K = 3$. Dashed line corresponds to the case when individual securities are collateralized separately and solid line corresponds to the case of portfolio collateralization. The model parameters are as follows: $r = 0.03$, $\beta = 0.05$, $\gamma = 4$, $\bar{\gamma} = 6$, $\sigma_F = 0.2$, $\bar{m} = 0.1$. The vertical lines indicate wealth at which the margin constraint becomes binding under individual collateralization scheme (dashed line) and portfolio collateralization scheme (solid line).

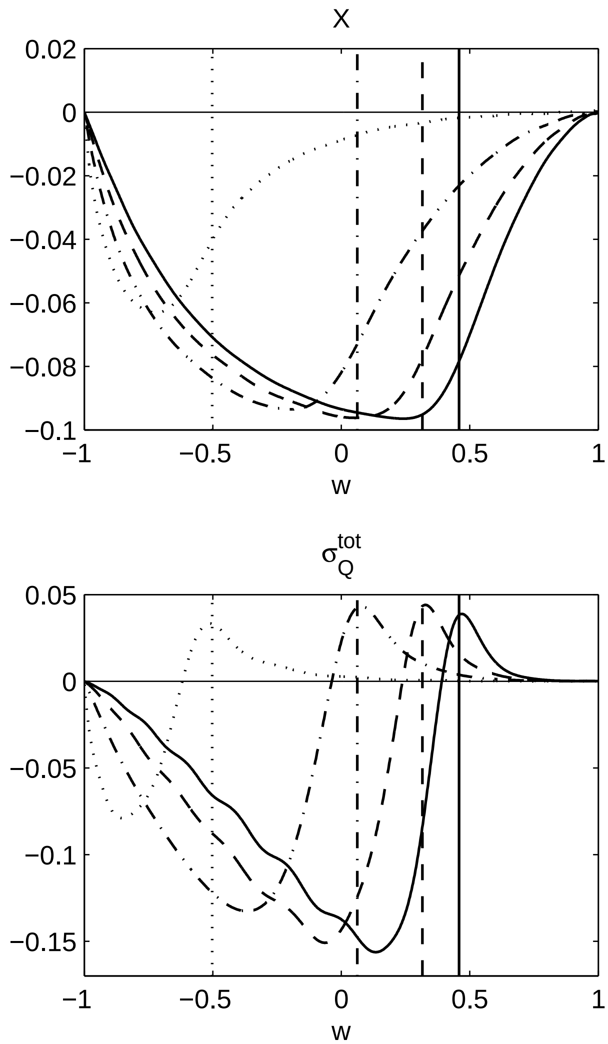

Figure 8. The effect of margin constraints in models with different number of assets. This figure presents the difference between several statistics in the economy with margin constraints and the economy without margins. X is the aggregate demand of professional investors; the Sharpe ratio SR is defined by Eq. (39); σ_Q^{tot} is the total volatility of returns; ρ_Q is the correlation of returns defined by Eq. (40). All statistics are functions of the rescaled wealth of professional investors $w = (W - 5)/(W + 5)$ and computed for different number of assets K. Dotted line corresponds to $K = 1$, dash-dotted line corresponds to $K = 3$, dashed line corresponds to $K = 5$, and solid line corresponds to $K = 7$. The vertical lines indicate wealth at which the margin constraint starts binding in models with different K. The model parameters are as follows: $r = 0.03$, $\beta = 0.05$, $\gamma = 4$, $\bar{\gamma} = 6$, $\sigma_F = 0.2$, $\bar{m} = 0.1$.

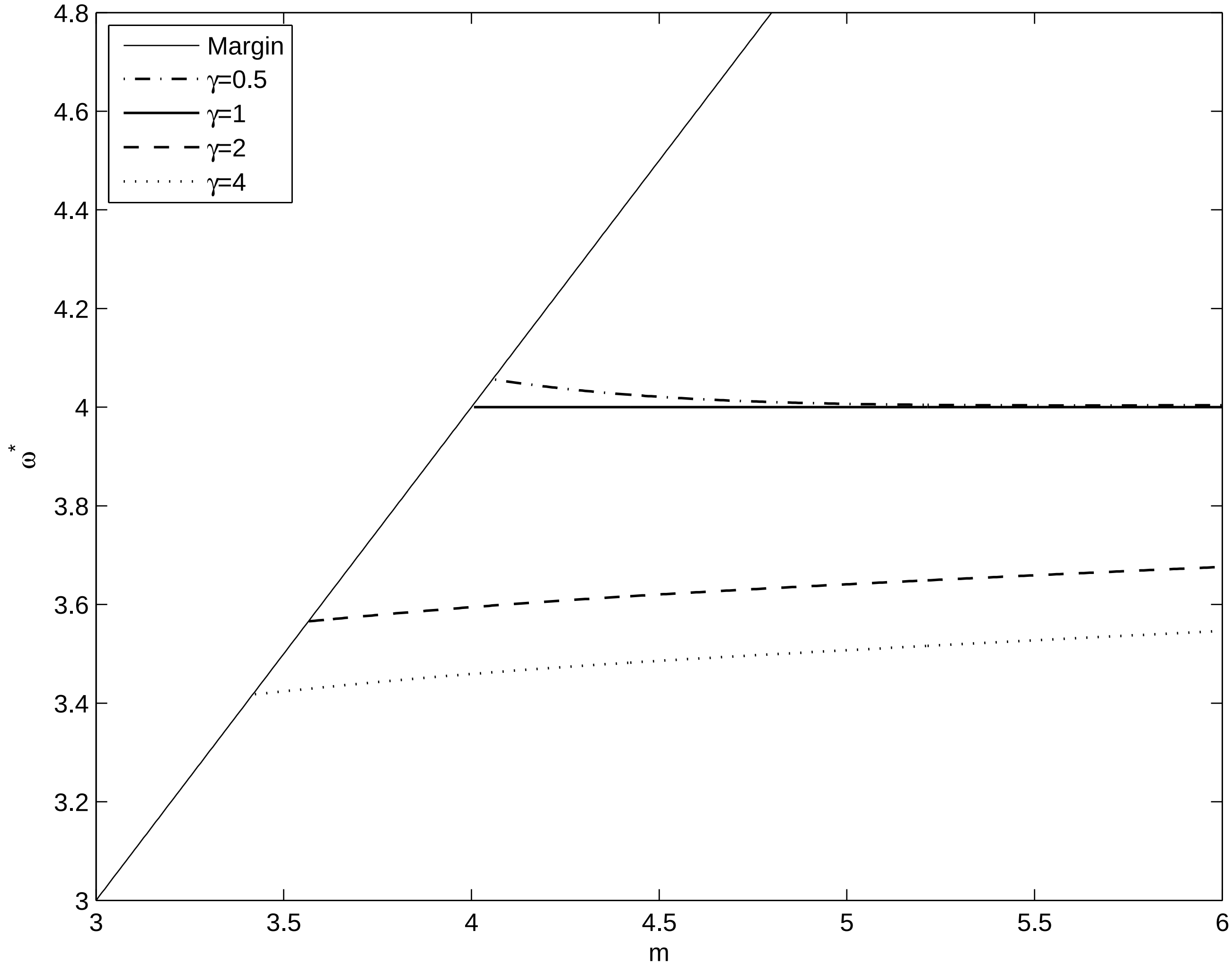

Figure 9. Optimal portfolios with margin requirements following a diffusion process. This figure presents optimal portfolio policy $\omega^*(m)$ as a function of the current level of margin constraints m for investors with various risk aversion γ. The model parameters are as follows: $r = 0.03$, $\beta = 0.05$, $\rho_{mS} = 1$, $\sigma_m = 0.5$, $\phi_m = 0.05$, $\bar{m} = 4$. Expected returns μ_S and volatility σ_S are set separately for each γ such that the Sharpe ratio is one and the optimal myopic leverage is four.

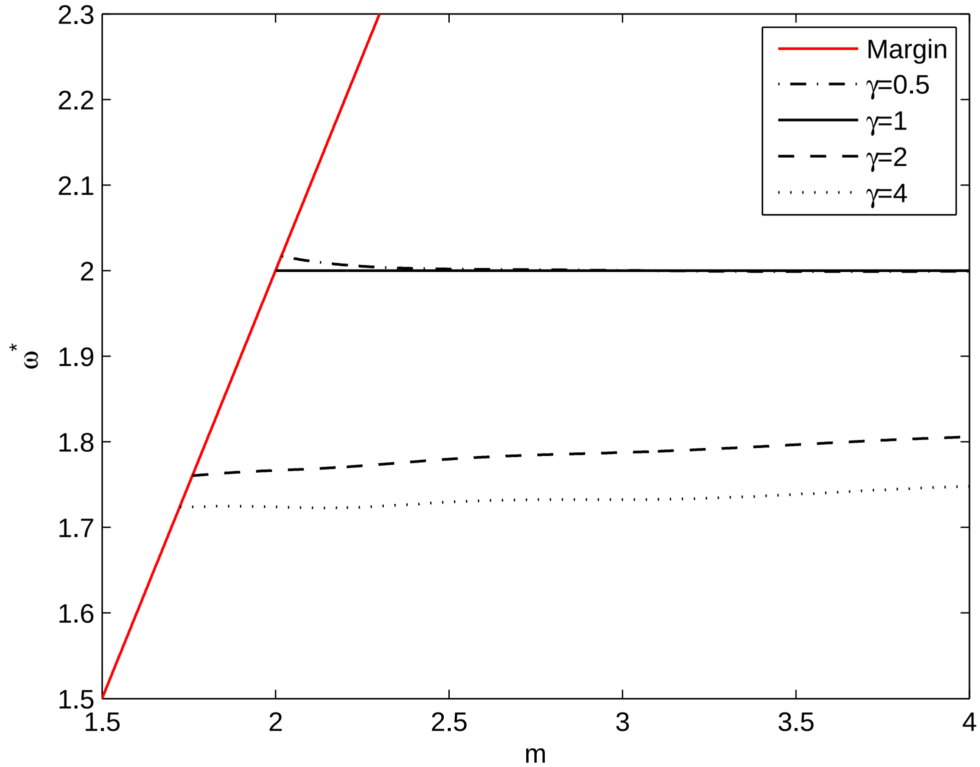

Figure 10. Optimal portfolios with jumps in margin requirements. This figure presents optimal portfolio policy $\omega^*(m)$ as a function of the current level of margin constraints m for investors with various risk aversion γ. The model parameters are as follows: $r = 0.03$, $\beta = 0.05$, $\phi_m = 0.05$, $\bar{m} = 2$, $\lambda = 0.05$, $Q_S = -0.2$, $Q_m = -0.5$. Expected returns μ_S and volatility σ_S are set separately for each γ such that the Sharpe ratio $SR = 2$ and the optimal myopic leverage $L = 2$.