The dark side of the moon: structured products from the customer's perspective

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Abstract

This paper develops a framework for the design of optimal structured products (equity- or index-linked notes) allowing us to analyze the maximal utility gain for an investor that can be achieved by introducing structured products. We demonstrate with data from three of the largest markets for structured products (USA, Germany and Switzerland) that most of the successful structured products are not optimal for a perfectly rational investor and we investigate the reasons that make them attractive for behavioral investors.

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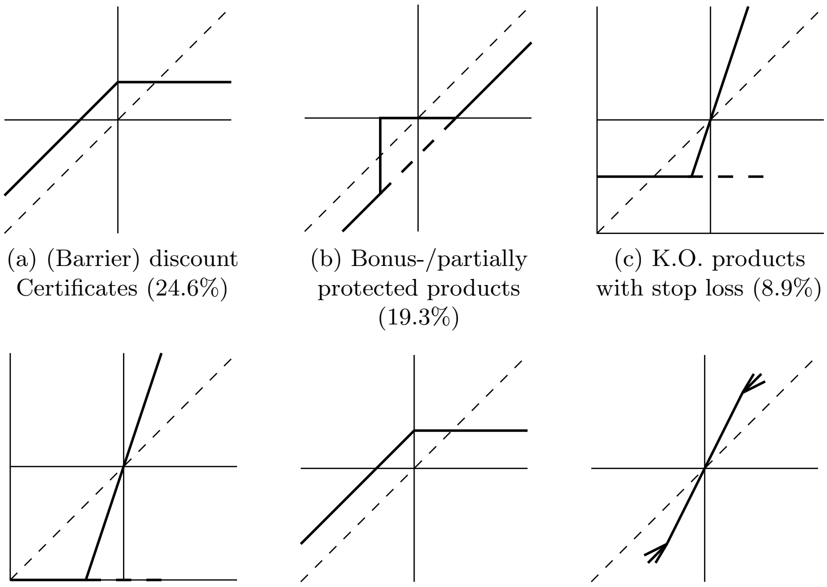
1 Importance of structured financial products

Structured products, SPs, also known as equity- or index-linked notes, combine classical assets (stocks, bonds, indices) with at least one derivative into a bundle that shall have specific interesting features for investors, like capital protection or increased participation. They enable investors with comparatively low budget and knowledge to invest indirectly into derivatives. Banks profit from the restricted participation of these non-institutional investors in the derivatives market, in that they retain a margin profit when issuing structured products. Structured products are immensely popular in Europe: in 2007 in Germany alone the market capitalization with structured products was above 200 billion Euro and 6.8% of invested assets are held in structured products. In Switzerland market capitalization is 340 billion CHF which corresponds to 7-8% of all invested assets. On the other hand, some European states like Norway recently introduced high regulatory burdens on them. Our paper contributes to the understanding of these regulatory issues by analyzing the risks and benefits of structured products for private investors. Structured products are not yet as popular in the USA with the market being approximately half as big as in Germany. Potential reasons for this are regulatory, but also because more private investors in the USA than in Europe hold stocks[1].

Note that the existence of structured products is puzzling since in traditional portfolio models there is no role for SPs. The classical mean-variance model of Markowitz [30], for example, suggests that an investment in the market portfolio and the risk-free asset would be sufficient to construct optimal investments for all degrees of risk-aversion – this is the famous *two-fund separation*. Hence the only role for banks would be to offer the market portfolio at minimal cost, e.g. in the form of an exchange traded fund (ETF).

Not only the widespread existence of structured products is puzzling, but also the large variety of payoff patterns that can be found. Even very complicated payoff patterns are often not as exotic and rare as one would expect: this can be seen in data from the German and Swiss market (the two largest markets for structured products in Europe) and the US market. The Swiss sample consists of $N = 47'362$ products issued from March 2007 to November 2007. The German sample consists of all structured products on the market ($N = 270'254$) as of end of November 2007. The US data is for 2006 and has been composed by JP Morgan (provided by Credit Suisse). Figure 1-3 show typical payoff diagrams for the most popular types in each

[1] Compare [3] for more on the regulatory background for structured products in the US.



(a) (Barrier) discount Certificates (24.6%)

(b) Bonus-/partially protected products (19.3%)

(c) K.O. products with stop loss (8.9%)

(d) K.O. products without stop loss (3.1%)

(e) (Barrier) reverse convertibles (2.3%)

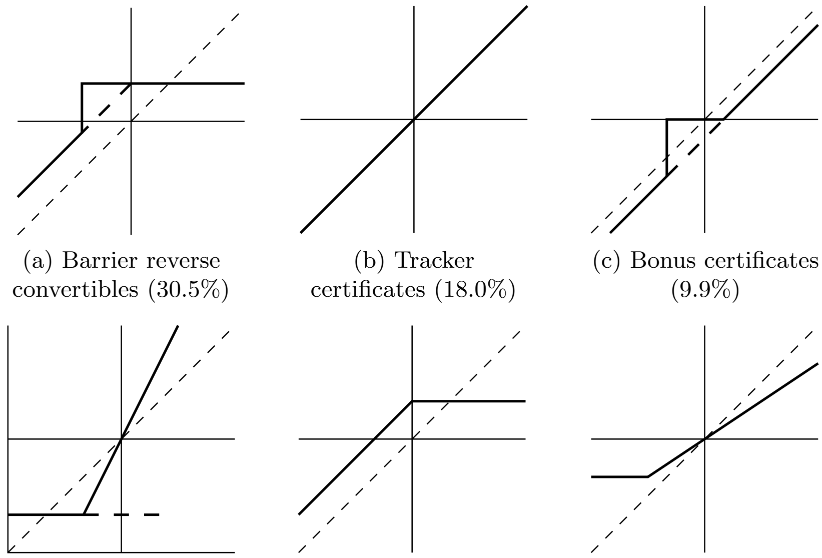
(f) Exotic leverage (2.0%)

Figure 1: The most popular structured products in Germany with their respective shares of all issued products ($N = 270\,254$).

country.[2] We remark that the names for the various types diverge from country to country and even from bank to bank. In the US, e.g., barrier reverse convertibles are usually called reverse convertibles, whereas outperformance products are sold under names like ARES, BARES or TORRERO. We observe that in each country a large variety of products are popular: leverage, capital protection and outperformance can all be found. At the same time there are clear similarities between the countries: three product types (*discount certificates, bonus certificates* and *(barrier) reverse convertibles*) can be found in the top list of both countries.

Structured products have so far been studied in academic research nearly exclusively from the issuer's perspective, mostly in the context of option pricing and hedging. (For an early exception see the article by Shefrin and Statman [37].) Indeed, building on the seminal paper of Black and Scholes [5] a new field of finance, called financial engineering, has emerged in which mathematicians and engineers developed more and more elaborate pricing techniques for ever more complicated structured products. Besides this huge technical literature, recently a few empirical studies on the actual

[2]For clarification, we omit plain vanilla type put and call options that are occasionally also listed as structured products.



(a) Barrier reverse convertibles (30.5%)

(b) Tracker certificates (18.0%)

(c) Bonus certificates (9.9%)

(d) Mini futures (9.2%)

(e) Discount certificates (7.9%)

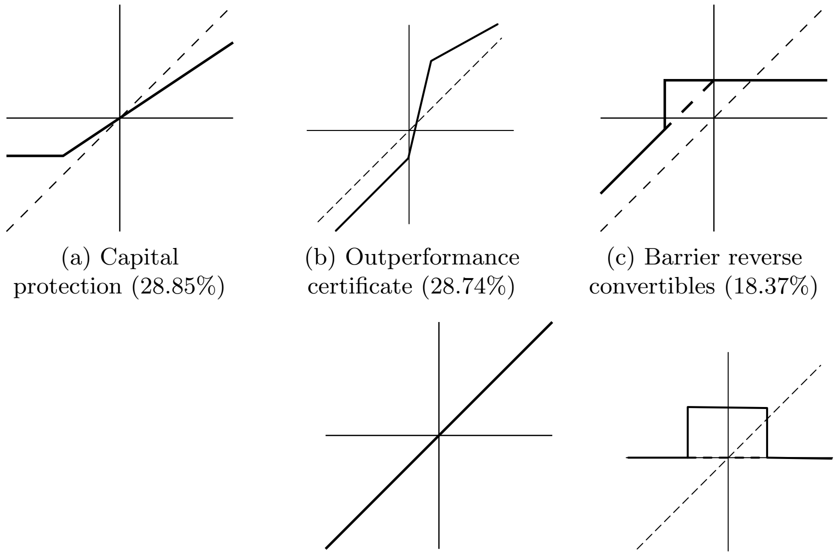
(f) Uncapped capital protection (6.3%)

Figure 2: The same list for Switzerland ($N = 47\,362$).

market prices [42, 19, 41] of SPs can be found. There are interesting first explorative studies on specific puzzles regarding the investors' preferences for certain product classes [38, 20, 7, 8, 2], but all in all the investor's perspective on structured products is still uncharted territory, somehow "the dark side of the moon".

In this article we try to shed some light on this "dark side" and take the investor's perspective as the starting point for our expedition. In this way we ask whether structured products are an appropriate tool to improve investment performance and what types of products are optimal under normative, but also under behavioral models. In particular, we measure how big the potential improvement of a portfolio can be when adding structured products. We also demonstrate that the most popular products derive their popularity not from standard rational, but from behavioral factors like framing, loss aversion and probability mis-estimation.

The paper is organized as follows: In the next section we provide a canonical model that we then use to design optimal structured products. Thereafter we reverse the point of view and ask for any given structured product whether risk-preferences, differences in beliefs or biases could justify its existence.



(a) Capital protection (28.85%)

(b) Outperformance certificate (28.74%)

(c) Barrier reverse convertibles (18.37%)

(d) Mandatory exchangeables (9.89%)

(e) Tracker certificates (5.31%)

(f) Barrier range notes (4.12%)

Figure 3: Most popular structured products in the US (all emissions 2006). Mandatory exchangeables cannot be expressed in a simple payoff diagram.

2 Designing optimal SPs

In the following we introduce a canonical two-period model for a structured product. For simplicity, we assume that the market prices can be described by the CAPM.[3] Using results on the co-monotonicity of optimal investments [34] and extending previous results on optimal investments in the strictly concave expected utility setting [26], we obtain qualitative properties that optimal structured products should satisfy under more and more relaxed conditions on the rationality of the investors. We will use these results to show later that the attractiveness of the currently most popular classes of SPs cannot be understood within a rational decision model.

Some of our analytical results are similar to independent work by Prigent [32] on portfolio optimization and rank-dependent expected utility. In particular, Prigent studies a basic optimization problem in the case of non-concave utility functions, similar to our Lemma 2.

[3] Many of our results could be obtained under much weaker assumptions, e.g. monotonicity of the likelihood ratio, which would include Black-Scholes prices.

2.1 A canonical model

We say that a structured product is *optimal* if its payoff distribution maximizes the given utility of an investor under the constraint that the arbitrage-free price of the product cannot exceed a certain value.
We use a two-period model. This is not a real restriction since this two-period model can indeed be derived from a standard time continuous model. The derivation is summarized in Appendix A.
To obtain intuitive results, we will assume most of the time that the market can be described by the capital asset pricing model. Since we look at an individual decision problem and do not try to explain asset prices, the maximization of non mean-variance utilities is not at odds with the CAPM market pricing assumptions. But even if we were to close the model and needed to derive asset prices as well the validity of the CAPM is not in contradiction to violations of the two-fund separation property. For example heterogeneous beliefs or uncorrelated optimization mistakes give rise to heterogeneous portfolios, while at the market level the security market line property of the CAPM can still hold [9, 17, 6].
Moreover, we assume that the market is complete for the product designer and that the investor does not hold other assets than the structured product (or at least considers them as a separate "mental account").[4]
We assume that the investor maximizes his expected utility, i.e.

$$\max_{y\in\mathbb{R}^S}\sum_{s=1}^{S} p_s u(y_s),$$

where $u\colon \mathbb{R}\to\mathbb{R}$ is the utility function, $s=1,\dots,S$ denote the states that occur with probability $p_s>0$. The payoffs of the structured product at maturity in the states s are denoted by y_s.
Given state prices $\pi=(\pi_1,\dots,\pi_S)\gg 0$, the budget constraint for such an investment is

$$\sum_s \pi_s y_s = B,$$

where $B>0$ is the total budget to be invested. This is the standard two-period model as it can be found, e.g., in the textbooks [29], [28] or [12] or in the work by Leland [27].
It will turn out to be useful to transform this model slightly: normalize π_s to π_s^* with $\sum_s \pi_s^* = 1$, set $B=1$, then we can define the likelihood ratio $\ell_s = \pi_s^*/p_s$. Absence of arbitrage implies $\sum_s \pi_s^* R_s^k = R$, where R_s^k is the return of asset k in state s and R is the risk-free rate. Therefore $\sum_s \pi_s = R$ and we can reformulate the optimization problem using the likelihood ratio

[4] Most of our results carry over if the investor holds additionally risk-free assets.

process

$$\max_{y\in\mathbb{R}^S} \sum_{s=1}^{S} p_s u(y_s),$$
$$\text{subject to } \sum_s p_s \ell_s y_s = R. \tag{1}$$

While studying a discrete state space helps to find an intuition for this problem, it will turn out to be easier if we study a continuous state space with a (non-atomic) market return distribution. One reason for this is the following useful result from [34] that holds under these assumptions:

Lemma 1. *If the likelihood ratio ℓ is a non-increasing function of the market return, then an optimal structured product can be written as a function of the market return (a "payoff function").*

Since ℓ is in most pricing models (in particular in CAPM and Black-Scholes) a decreasing function of the market return[5], we can therefore simplify the optimization problem (1) in the continuous case to the maximization of

$$U(y) := \int_{\mathbb{R}} u(y(x))\, dp(x),$$

over all $y \in L^1_{\ell p}(\mathbb{R})$, i.e. functions measurable with respect to ℓp, subject to the condition

$$\int_{\mathbb{R}} y(x)\ell(x)dp(x) = R. \tag{2}$$

The *likelihood ratio* ℓ can be written in the CAPM as $\ell(x) = a - bx$, with $a = 1 + b\mu$ and $b = (\mu - R)/\sigma^2$, where μ and σ^2 are mean and variance of the market portfolio. We assume, if not otherwise specified, $\mu = 1.09$, $\sigma = 0.20$ and $R = 1.05$. In this case, state prices in the CAPM are positive up to a return of approximately +100%. Negative state prices may arise in the CAPM because of the famous mean-variance paradox, see [22] for details.

2.2 Optimal SPs in the strictly concave case

Let us assume that the investor has rational preferences, i.e. he follows the expected utility approach by von Neumann and Morgenstern [40]. The case where the utility function u is strictly increasing and strictly concave has been studied previously, see [27, 31, 26]. In this case, a straightforward variational approach leads to the optimality condition

$$y(x) = v^{-1}(\lambda\ell(x)), \tag{3}$$

[5]In consumption based models, ℓ is a decreasing function of the market return as a simple consequence of decreasing marginal utility of wealth.

where $v := u'$ and λ is a Lagrange parameter that has to be chosen such that (2) holds.
Note that unless the utility function is quadratic, in the two-period model the two-fund separation property does not hold for any distribution of returns[6].
What conditions on y can we derive from (3) in the CAPM case? In fact, in the most realistic case (when the investor is prudent[7]), y is convex, as the following theorem shows:

Theorem 1. *Let u be a strictly increasing and strictly concave utility function. Assume furthermore that $u''' > 0$ (i.e. the investor is prudent). Then the return of an optimal SP on a CAPM market is strictly convex as a function of the market return. If $u''' = 0$ (i.e. u is quadratic) then the payoff function is linear. If $u''' < 0$ the payoff function is strictly concave.*

Proof. Take the second derivative of $y(x) = v^{-1}(\lambda(a + bx))$ with $v := u'$. This gives:

$$y''(x) = -\frac{\lambda^2 b^2 v''(v^{-1}(\lambda(a + bx)))}{(v'(v^{-1}(\lambda(a + bx))))^3}.$$

The denominator is positive, since $v' = u'' < 0$ (u is strictly concave).
Thus y'' is positive if $u''' = v'' > 0$.[8] □
This result is surprising since prudent investors prefer SPs that look similar to call options which are usually associated with gambling or risk-taking! In particular, the theorem shows that if u is quadratic, the return is a linear function of the market return, i.e. there is no need for a SP at all, which is the classical two-fund separation theorem of the mean-variance portfolio theory.
The following corollary replaces the prudence condition with the standard concept of absolute risk-aversion condition:

Corollary 1. *If the investor on a CAPM market has non-increasing absolute risk-aversion (non-IARA), then the payoff function should be strictly convex.*

For proofs of this and the following statements see the appendix.
Let us now study two prominent examples of specific utility functions, where the payoff function of the optimal structured product can be given explicitly.

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[6] In models with *time continuous* trading, *no transaction costs* and an underlying process that follows a *geometric Brownian motion* the two-fund separation property would hold for any class of utility functions with constant relative risk aversion.

[7] The importance of prudence is well known from the literature in insurance theory, see for example [18]

[8] The result can also be proved using the general results on convexity of optimal payoff profiles by Leland [27].

Power utility

Let $u(x) := \frac{1}{\alpha}x^{\alpha}$ with $\alpha < 1$, $\alpha \neq 0$. Then the optimal SP is given by

$$y(x) = \frac{C}{\ell(x)^{\frac{1}{1-\alpha}}},$$

where C is a positive constant, as a small computation shows (see appendix). We see that y is a decreasing function of ℓ. Thus the structured product is an increasing and convex function of the underlying (as predicted by Corollary 1) if the likelihood ratio $\ell(x)$ is affine in x which is the case in CAPM.

Exponential utility

Let $u(x) := -\frac{1}{\alpha}e^{-\alpha x}$. Then the optimal SP in the CAPM case is

$$y(x) = R - \frac{1}{\alpha}\left(\ln(a - bx) - C\right),$$

where C is a positive constant (see appendix). Again, y is an increasing and strictly convex function of x.

2.3 Optimal SPs in the general case

The assumption of strict concavity for the utility function is classical, but has been much disputed in recent years. Implied utility functions can be computed from stock market data and often show non-concave regions, compare for example [21] and [11]. Moreover, the most popular descriptive theory for decisions under risk, cumulative prospect theory, predicts non-concavity of u in losses [39]. Other descriptive theories also assume risk-seeking behavior at least for small losses.
Given the empirical and experimental evidence, it seems therefore more likely that u depends on a reference point (e.g. the current wealth level) and that it is strictly convex for small losses. What would be an optimal SP for an investor described by such a model?
We first observe that in order to find an optimal SP, it is sufficient to consider the *concavification* of the value function u, i.e. the smallest function larger or equal u which is concave[9]. We state this in the following lemma:

Lemma 2. *Assume that u is concave for large returns and assume that the returns are bounded from below by zero*[10]*. Asssume that ℓ is continuous. Let u_c be the concavification of u, then there is an optimal SP for u_c which is also optimal for u.*

[9] A similar problem has been studied by Prigent [32]

[10] This is always true in applications since there is limited liability for structured products, i.e. the most one can lose is the initial investment.

For a proof see the appendix.
For the concavified utility we can now follow the same computations as before. The only difficulty is that the inverse of the derivative is still not everywhere defined, since it can be constant. This, however, simply corresponds to a jump in y, as can be seen, for instance, by an approximation argument.
This implies particularly that we typically do not have convexity of the optimal structured product in the case of non-concave utility functions. There is, however, one general property that holds in both cases which has been proved (in a more general setting) in [34]:

Theorem 2. *If ℓ is a decreasing function of the market return and the decision model is given by a utility function, then any optimal structured product is a monotonic function (or correspondence) of the market return.*

In other words: the higher the return of the market portfolio at maturity, the higher the return of the optimal structured product.
The intuition for this result is that whenever possible we would like to put large returns on "cheap" states. Since the state price density is decreasing in market returns, both in the case of CAPM and the case of Black-Scholes, the cheapest states are the ones with the largest market returns. This forces us naturally to assign large returns to states with large market return and consequently small returns to states with small market return.
We can see this in an illustrative example (compare Fig. 4): if we have only two equally likely states with payoffs $x_1 < x_2$, then a larger payment y_{large} of the structured products when the market returns x_2 and a smaller payment y_{small} in case of x_1 cannot be optimal: switching the payoffs leaves the return distribution (50% chance for y_{large} and 50% for y_{small}) unchanged, but decreases hedging costs, since $\ell(x_1) > \ell(x_2)$.[11]

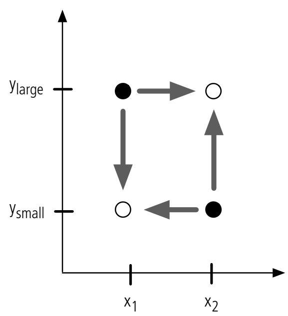

Figure 4: Rearranging a non-monotonic payoff function to optimize a product in the simplest case of two equally likely states.

[11]This example is essentially taken from [13]. For the general proof see [34].

2.4 Numerical results: what drives what?

We now know how optimal structured products can be analytically computed and what properties they satisfy. This framework should enable us to design optimal structured products for investors with a broad range of risk preferences. However, the numerical computation is not as easy as the previous examples suggested: in principle, we just need to evaluate equation (3), however, it is often difficult to compute the Lagrange parameter λ explicitly as we have done in the case of a power and of an exponential utility function. Thus, we use for our numerical computation an iteration method, i.e. we evaluate (3) for a fixed λ, then compute the error of the constraint (2) and correct the λ. We iterate this until the error is sufficiently small.
Let us now give a couple of examples to see how optimal structured products look like.

Power utility

Let us consider the power utility function $u(x) = x^\alpha/\alpha$ (i.e. a typical function with constant relative risk aversion $-x \cdot u''(x)/u'(x) = 1 - \alpha$). The optimal structured products for expected utility investors with this utility function on a CAPM market (here and in the following examples always $R = 1.05$, $\sigma = 0.2$ and $\mu = 1.09$) is shown in Fig. 5. We see that the payoff function

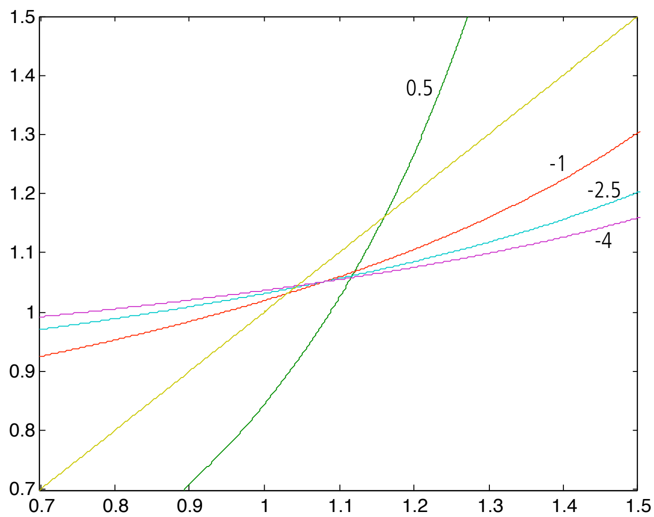

Figure 5: Optimal structured product for CRRA-investors with $\alpha = -4, -2.5, -1, 0.5$.

of the product is strictly convex (which we know already from Theorem 1).

It features a "smooth" capital protection and an increasing participation in gains.

Exponential utility

Let us consider the exponential utility function $u(x) = -e^{-\alpha x}/\alpha$ (i.e. a typical function with constant absolute risk aversion $-u''(x)/u'(x) = \alpha$). The optimal structured product for an expected utility investor with this utility function (for $\alpha = 0.5$) on a CAPM market is shown in Fig. 6. We see that the payoff function of the product is only very slightly convex.

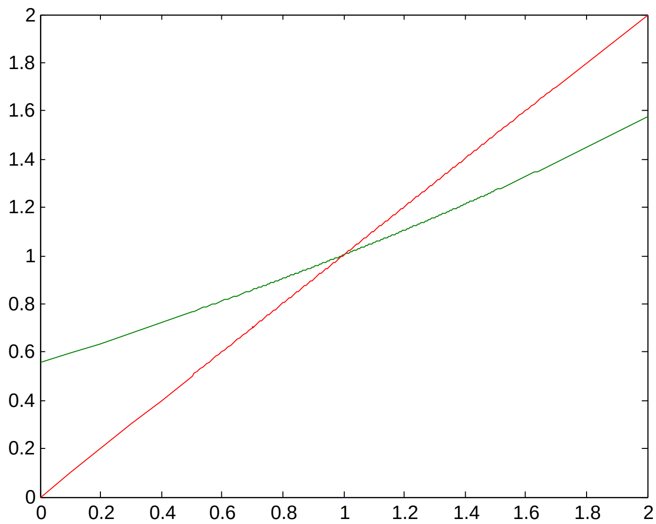

Figure 6: Optimal structured product for an CARA-investor with $\alpha = 0.5$.

Quadratic utility with aspiration level

Let us now consider a non-concave utility function. Non-concave utility functions are a key ingredient of some behavioral decision models, like prospect theory. They can also occur, however, for other reasons: an example would be an investor who plans to buy a house in one year and has saved just about enough money for the installment. (We denote his current wealth level by x_0.) His utility function, when considering a one-year investment, will now necessarily have a jump (maybe slightly smoothed by the uncertainty about the house prices) that will make it locally non-concave.[12]
Let us define the utility function as $v(x) = a(x) + u(x)$ with $a(x) = h(x)u_h$, where u_h is the extra utility gained by the house and $h(x)$ is the probability

[12]This is similar to the aspiration level in the SP/A model, see [36, Chapter 25.5] for an overview and further references.

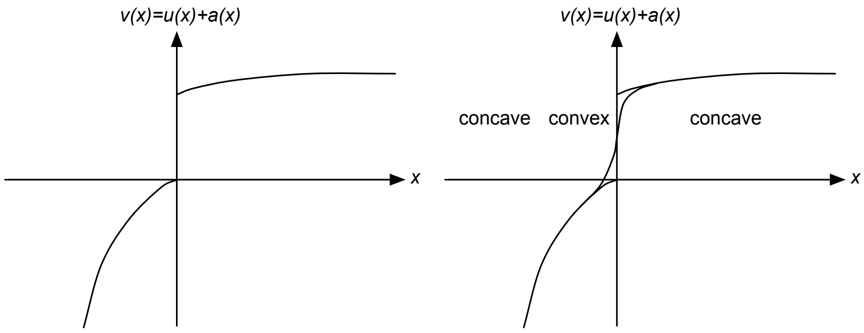

Figure 7: The utility function v of an investor with an aspiration level: his utility increases above a certain threshold, thus making it non-concave with a jump (left). If the precise location of the threshold is uncertain, the effective utility function becomes again continuous, but is still convex around that point (right).

that he is able to afford the house. For simplicity we set $h(x) = 1$ for $x \geq x_0$ and $h(x) = 0$ for $x < x_0$, thus inducing a jump in the utility function at x_0. In reality the probability might be more like a logistic function rather than a precise jump function, thus the overall utility v would look more like a concave–convex–concave function. See Fig. 7 for illustrations.

To keep things simple we choose as u the function $u(x) = x - \alpha x^2$ with $\alpha = 0.2$. (Larger values of α lead to an saturation problem.) Moreover we set $x_0 = 1$ and $u_h = 0.02$. We need to compute the concave hull of

$$v(x) = a(x) + u(x) = \begin{cases} x - \alpha x^2, & \text{for } x < 1, \\ 0.02 + x - \alpha x^2, & \text{for } x \geq 1 \end{cases} .$$

To this end, we compute the derivative of u to find the tangential line on u which crosses the point $(1, v(1))$. A straightforward computation using (3) leads to an explicit solution, depending on the Lagrange parameter λ which is then computed numerically by an iteration scheme as outlined above.

The resulting optimal structured product is piecewise affine (as was to be expected when using a quadratic u in CAPM) and corresponds to a limited capital protection as it can be found in some popular types of structured products (compare Fig. 8).

The model, though certainly simplistic, can in fact be confirmed by empirical evidence: the classical investment goal for private investors is real estate. Whereas in the US the percentage of realty owners in the population is very high, this is not the case in most continental European countries, moreover there houses are usually owned for longer periods of time, thus, buying realty tends to be a very special event. If our previous considerations are correct, we would therefore expect that investors who plan to buy realty in the near future would be more interested in buying capital protected

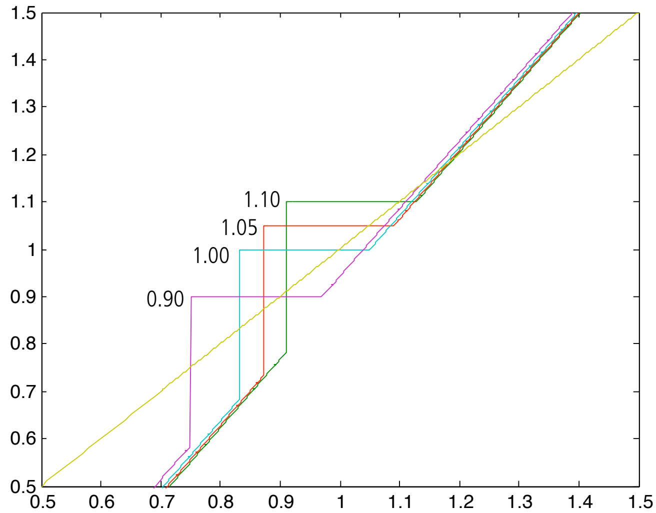

Figure 8: Optimal structured products for investors with quadratic utility ($\alpha = 0.2$) plus an aspiration for $u_h = 0.2$ at position $x_0 = 0.9, 1, 1.05, 1.1$.

SPs. This has been tested in a survey by AZEK [15], where $N = 106$ test subjects have been asked to choose between six structured products on the MSCI World, out of which one provided a full capital protection. The test subjects were mainly bankers and other professionals working in the financial industry, thus their competence in understanding the survey question can be assumed. The subjects were also asked whether they plan to buy a house or an apartment in the next years and whether they already own realty. A simple lottery question was used as proxy for their loss aversion.
According to our model there should be a significant positive relation between the investment into a capital protected product and the plan to buy realty. This correlation could, however, also be triggered by a general affinity to conservative investments, which is just reflected by the plan to buy realty. To test this, we performed a logit regression where we added two independent variables: the loss aversion as estimated from the lottery question and a dummy variable for the owners of realty. It turned out that all of these factors were significant for their decision (see Table 2.4). The effect was stable when removing the dummy variables for realty owners and the loss aversion and was even more pronounced when only considering subjects who did not yet own realty ($N = 85$).

Prospect theory

We can also use prospect theory as underlying preferences, i.e. a utility function which is convex in losses and concave in gains. We use the functional

All subjects ($N = 106$):						
	Model 1		Model 2		Model 3	
	Coeff.	Prob> χ^2	Coeff.	Prob> χ^2	Coeff.	Prob> χ^2
Plan to buy	1.994	0.001***	0.749	0.047**	1.101	0.059*
Owning realty	1.919	0.032**	1.138	0.066*		
Loss aversion	0.819	0.035**				
Only non-owners ($N = 85$):						
	Model 1		Model 2			
	Coeff.	Prob> χ^2	Coeff.	Prob> χ^2		
Plan to buy	1.845	0.002***	1.907	0.001***		
Loss aversion	1.094	0.001***				

Table 1: Factors influencing the decision to invest into a capital protected structured product. $*, **, ***$ =significant on the 10%, 5%, 1% level.

form from [43] that can be seen as a second order approximation of any prospect theory value function:[13]

$$u(x) := \begin{cases} -\lambda(x - \beta x^2), & \text{for } x < 0, \\ x - \alpha x^2, & \text{for } x \geq 0. \end{cases} \tag{4}$$

We can think here of an investor who is (like most investors) afraid of losses, but who does not distinguish much between small and large losses. This leads to risk-seeking behavior in losses. Depending on the amount of loss-aversion λ a smaller or larger amount of capital protection becomes optimal. Figure 9 shows the optimal solution in this case for various values of loss-aversion.

2.5 Utility gain by SPs: much ado about nothing?

We have seen so far that optimal structured products do indeed deliver a payoff structure that is different to classical portfolios including only the market portfolio and a risk-free asset. The effort of computing such an optimal structured product and hedging it can in praxis not be neglected, therefore the natural question arises whether it is worth it; how big is the potential utility improvement?[14]

To answer this question, we compute the expected utility of an optimal structured product and the expected utility of the optimal mix between the market portfolio and the risk-free asset. We translate both values into certainty equivalent interest rates, i.e. the (hypothetical) risk-free asset that

[13]This form is also computationally simpler than the standard specification by Tversky and Kahneman [39] and incorporates mean-variance preferences as a the special case when $\alpha = \beta$ and $\lambda = 1$.

[14][7] has considered a similar question for a selection of typical structured products.

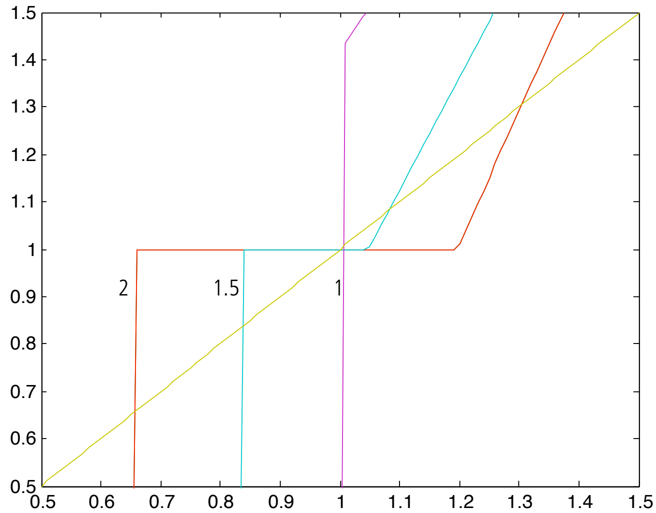

Figure 9: Optimal structured products for investors with a prospect theory utility as in (4) with $\alpha = \beta = 0.2$ and $\lambda = 1, 1.5, 2, 2.5$.

would have the same expected utility as the product. We then consider the difference between both certainty rates and compare it with the gain in certainty rates that can be achieved by a classical portfolio over a risk-free investment. This analysis shows how good the "second order approximation" (the classical two-fund portfolio) is, comparing to the "higher order approximation" (the optimal structured product). Thus we want to answer a couple of questions of practical relevance: is the first (classical) step of improvement big, and the second one (caused by the structured product) negligible? Or are they equally important? Under what circumstances is the additional potential improvement of structured products particularly large?

To keep the analysis simple, we restrict ourselves to the two examples of the previous section: first, we consider a CRRA-utility function of the form $u(x) = x^\alpha/\alpha$. Later we will study the case of a quadratic utility function with aspiration level.

The optimal structured product for CRRA-utility functions has been already computed (see Section 2.2). Thus we can directly compute the expected utility of y, where we assume a normal distribution with mean $\mu = 1.08$ and standard deviation $\sigma = 0.19$ for the return of the market portfolio. Reporting only this value would be useless: we need to compare it with other utilities. Therefore we compute the improvement, as expressed in terms of certainty equivalent interest rate, over the optimal "classical" portfolio, i.e. the optimal combination of the market portfolio and a risk-free investment.

To compute the optimal classical portfolio, we compute the utility of all portfolios with a proportion of θ invested in the market portfolio and $(1 -$

θ) invested into risk-free assets, where $\theta \in \{0, 0.01, \ldots, 0.99, 1\}$. Then we choose the θ yielding the largest expected utility.
The results of this computation are summarized in Table 2.5.

Investment	Certainty equivalent
Fixed interest (4%)	4.00%
Market portfolio (optimal classical)	4.58%
Optimal structured product	4.64%

Table 2: Improvement of a structured product for an investor with a classical CRRA utility with $\alpha = -0.2$ as compared to the improvement by classical portfolios.

We see that the optimal structured product gives an improvement, but it is not as big as the "first step", i.e. the improvement induced by the classical mean-variance portfolio theory. The improvement over the classical portfolio is only 6 basis points.
This is not due to a specific choice of α. Very much to the contrary, Fig. 10 shows that the improvement is in fact small for all natural choices of α.

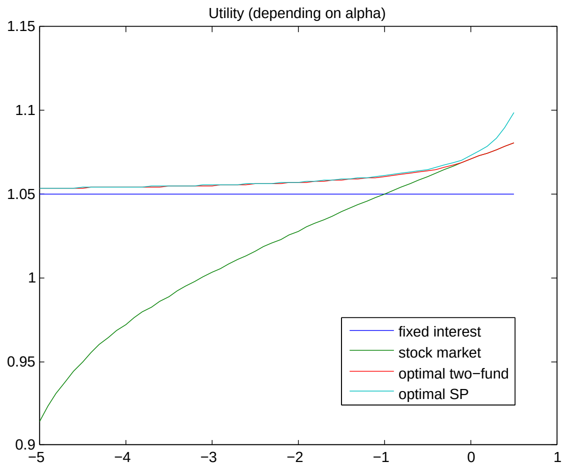

Figure 10: Utility for CRRA (as measured by the certainty equivalent interest rate) of stock, risk-free asset (5% return), optimal two-fund portfolio and optimal structured product. The utility gain by the structured product is small.

The second example, an exponential utility function, can be computed in the same way. Here the improvement is minute (only 0.0004%) and in the precision of the data in Table 2.5 not even visible. This does not come as a big surprise, as the optimal structured product is close to a linear investment

Investment	Certainty equivalent
Fixed interest (4%)	4.00%
Optimal classical portfolio	4.24%
(53% market portfolio, 47% fixed interest)	
Optimal structured product	4.24%

Table 3: Improvement of a structured product for an investor with a classical CARA utility with $\alpha = 0.01$ as compared to the improvement by classical portfolios.

(compare Fig. 6). The result is very similar if we vary the absolute risk aversion.
So far it looks as though the improvements of structured products are somewhere between tiny and small. Structured products – much ado about nothing? Are there no situations in which we can generate a decisive improvement of a portfolio with their help?
Let us consider the third example, the *quadratic utility with aspiration level.* Here, finally, the improvement due to structured products is considerable as we see in Table 2.5. In fact, the improvement is as big (or bigger) than the "first step improvement" done by the classical mean-variance theory! The improvement is similar for various "aspiration levels" as Fig. 11 illustrates.

Investment	Certainty equivalent
Fixed interest (4%)	4.00%
Market portfolio	3.74%
Optimal classical	4.06%
(8% market portfolio, 92% fixed interest)	
Optimal structured product	4.30%

Table 4: Improvement of a structured product for an investor with an aspiration level as compared to the improvement by classical portfolios.

Let us finally consider a prospect theory investor (without probability weighting) with a utility function as in (4) with $\alpha = \beta = 0.2$. His utility improvement for various levels of loss aversion λ is considerable, see Fig. 12.
To summarize our results: it seems that in classical strictly concave settings, the additional amount of risk control due to structured products is not large, and depending on the utility function, sometimes even quite small. As soon as we broaden our horizon and look at situations with partially non-concave utility functions, structured products become much more interesting. In such situations, they can easily improve the portfolio by an amount that is larger than the improvement of the first step from a fixed interest rate to a classical optimal portfolio á la CAPM. However, since banks typically

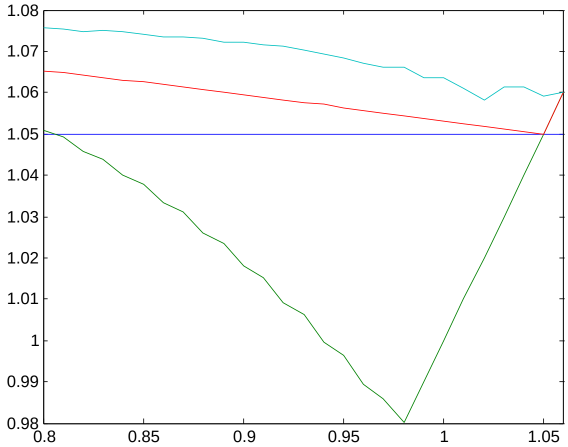

Figure 11: Utility with aspiration level (as measured by the certainty equivalent interest rate) of stock, risk-free asset (5% return), optimal two-fund portfolio and optimal structured product. We have varied the position of the jump in the utility, i.e. the wealth the investor needs to reach his investment goal. The utility gain by the structured product is equivalent to a 1% gain in returns.

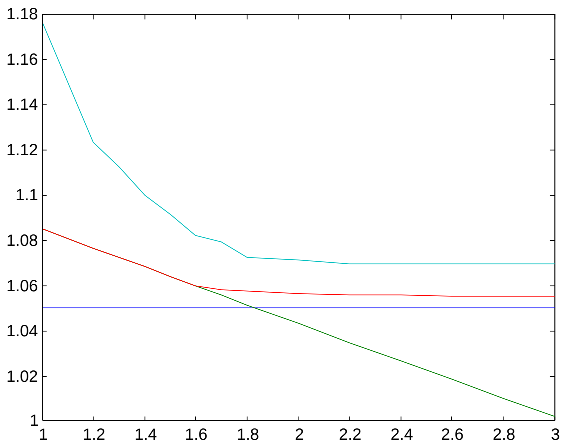

Figure 12: Prospect utility (as measured by the certainty equivalent interest rate) of stock, risk-free asset (5% return), optimal two-fund portfolio and optimal structured product. We have varied the loss aversion λ of the investor. The utility gain by the structured product becomes substantial for low loss aversion; it is equivalent to a 1.5% gain in returns for a normal loss aversion of around 2.

charge at least 1% on SPs the utility improvements we found are often not worth these costs.

3 After all, why do people buy structured products?

So far our aim was to find optimal structured products for given utility functions and characterize the shape of the payoff diagram and the size of the utility gain. In this section we want to find for given structured products (that we observe, e.g., on the market) a utility function and other aspects for which these products are optimal.
As seen in the canonical model an investor with expected utility theory that is strictly increasing, strictly concave and prudent, i.e. a classical rational investor would only invest into SPs with a strictly convex payoff function. This is in striking contrast to the variety of structured products on the market as we have seen them in Fig. 1 and Fig. 2.
In this section we want to investigate other potential reasons that can make structured products attractive for investors, thereby relaxing our assumptions on a classical rational investor step by step.

3.1 Background risk

Let us relax our initial assumptions that an investor only invests into a structured product (or considers the investment in SPs separately from other investments) and allow the investor instead to hold additionally to the SP a portfolio of classical assets – a combination of the market portfolio and risk-free assets. In order to conclude that also in this setting optimal structured products are convex, suppose the contrary, i.e. that the return of the SP is described by a function y which is non-convex. Then the overall portfolio can be described by the function

$$\tilde{y}(x) = \lambda_1 R + \lambda_2 x + (1 - \lambda_1 - \lambda_2) y(x),$$

where $\lambda_1 > 0$ is the proportion of total wealth invested in risk-free assets and $\lambda_2 > 0$ is the proportion of wealth invested in the market portfolio such that $1 - \lambda_1 - \lambda_2 > 0$. According to Theorem 1 the function $\tilde{y}$ is strictly convex, but $\tilde{y}''(x) = y''(x)$, thus we have a contradiction to the assumption that y is non-convex.
We conclude that a non-convex SP for a classical rational investor could only be useful if he already has assets with a strictly convex payoff function in his portfolio. Mixing his assets in this way seems unrealistic, but ignoring this, if we consider such assets as SPs and study the sum of all SPs, the

statements about optimal SPs are still true, but apply to the sum of the SPs in the investor's portfolio.[15]

3.2 Investors with PT-type utility function

There is ample experimental and empirical evidence in favor of non-classical decision theories like prospect theory. In the next step, we will therefore implement one of the key ingredients of prospect theory, namely the convex-concave structure of the utility function with respect to a reference point that is in itself not fixed, but can, for instance, be the initial value of an asset.

We have seen already that in this case loss-aversion can induce a non-convex payoff function of an optimal structured product with a "plateau" at zero return. This mimics a frequent feature of structured products, namely limited capital protection that is valid only up to a certain amount of losses.

More popular structures, however, like the highly popular barrier reverse convertibles can still not be optimal, since their payoff can obviously not be described as a function of the market return at maturity – a requirement that Lemma 1 poses for optimal structured products: all structured products whose return at maturity is not solely defined by the return of the underlying market portfolio at maturity cannot be optimal. Thus the modified utility function of prospect theory can only explain the popularity of some of the structured products offered today. The most popular types cannot be understood within this model.

3.3 Probability weighting

Experimental studies have demonstrated that subjects systematically overweight small probabilities, i.e. small probability events (e.g. large losses when deciding about buying insurance or large gains when deciding about buying a lottery ticket) tend to have more impact on the decision than they ought to [23]. This effect should not be mixed up with probability misestimation: probability weighting even affects decisions where the probabilities are known and do not have to be estimated. In this section we add probability weighting and observe its effects on the qualitative features of optimal structured products.

First, we need to distinguish two models of probability weighting:

1. We consider the return distribution of the structured product and apply a probability weighting to it.

[15] If there are investors with different background risks on the market and market prices are determined by their trades rather than by a general model like CAPM, then differences in the convexity of optimal payoff functions can be explained even if risk attitudes are homogeneous, compare [16].

2. We apply probability weighting to the return distribution of the underlying.

Both approaches are a priori reasonable, but lead to very different results. In particular, in the first case, the monotonicity result of Theorem 2 still holds, thus the payoff function is still monotonic. In the second case, however, this is no longer the case, as the utility becomes in a certain sense state dependent. With this form of probability weighting we can therefore explain, for instance, the attractiveness of constructions that give high payoffs for extreme events (which are overweighted by the probability weighting), in particular straddles.
Nevertheless, the payoff is still (essentially) a function of the underlying as the following theorem demonstrates:

Theorem 3. *Assume that the return of the market portfolio is an absolutely continuous measure with a smooth distribution function p which is nowhere equal to zero. For an investor with a smooth utility function and a probability weighting of the underlying with a smooth probability weighting function w, an optimal structured product has a piecewise smooth payoff function*[16] *of the underlying market portfolio.*

This result implies in particular that y is still a *function* of the market return. This has a strong consequence: some of the most popular classes of products, namely all products with *path-dependent* payoff (in particular barrier reverse convertibles and bonus certificates), still cannot be explained by this model!

3.4 Betting against the market

We have seen from the results in the previous sections that if we want to understand why investors buy structured products like barrier reverse convertibles or basket products, we need to take into account other factors than just risk preferences.
One such factor is disagreement with the market beliefs: so far we have always assumed that investors don't know better than the market and thus the physical return distribution p is also used in their estimate of the utility. If both are different, we label this as "betting against the market" or "speculation". Believing that the market will behave differently than the probability distribution p forecasts might be wise in some circumstances, but it is probably much more frequently a sign of overconfidence – an all too common characteristics of private investors.
First, we notice that the monotonicity result (Theorem 2) does not hold anymore, since it relied on the homogeneity of beliefs. In fact, we have to

[16]This means that the payoff function can have finitely many jumps, but is smooth everywhere else.

change our model. Fortunately, we can use essentially the same idea as with probability weighting: let us denote the estimated probability of the investor by $\tilde{p}$, then his optimization problem becomes

$$\text{Maximize } U(y) := \int_{\mathbb{R}} u(y(x))\tilde{p}(x)\,dx,$$

in $L^1_{\ell p}(\mathbb{R})$ subject to

$$\int_{\mathbb{R}} y(x)\ell(x)p(x)\,dx = R. \tag{5}$$

Defining $w(x) := p(x)/\tilde{p}(x)$ and $\tilde{\ell}(x) := w(x)\ell(x)$ we can transform this problem to

$$\text{Maximize } U(y) := \int_{\mathbb{R}} u(y(x))\tilde{p}(x)\,dx,$$

subject to

$$\int_{\mathbb{R}} y(x)\tilde{\ell}(x)\tilde{p}(x)\,dx = R. \tag{6}$$

This new problem can now be solved in the same way as above. As with probability weighting, we do not necessarily have a monotonic payoff function, but the fact that the optimal SP can be described by a payoff function still holds:

Corollary 2. *Assume that the return of the market portfolio is an absolutely continuous measure with a smooth distribution function p which is nowhere equal to zero. Let $\tilde{p}$ be a smooth return distribution estimated by the investor. Assume that the investor has a smooth utility function. Then an optimal structured product is a piecewise smooth function of the underlying market portfolio.*

Proof. This proof follows the same idea as the proof of Theorem 3. □

Moreover, we can give a condition on w under which the optimal y is monotonic if we assume that ℓ is positive and decreasing (as, e.g., in CAPM for not exceedingly large returns): using the co-monotonicity we know that y is monotonic whenever $\tilde{\ell}$ is a decreasing function. This is the case if $\tilde{\ell}'(x) = \ell'(x)w(x) + \ell(x)w'(x) < 0$. Since $w(x) > 0$, $\ell'(x) < 0$ and $\ell(x) > 0$, this holds particularly when $w'(x) < 0$, i.e.

$$p(x)'\tilde{p}(x) + p(x)\tilde{p}'(x) < 0. \tag{7}$$

Example 1 (Optimism and pessimism)**.** Let p and $\tilde{p}$ be normal distributions. If the investor is optimistic, i.e. $\operatorname{var} p = \operatorname{var} \tilde{p}$, but $\mathbb{E}(p) < \mathbb{E}(\tilde{p})$, then condition (7) is satisfied, and thus the optimal y is an increasing function of the market index. If $\mathbb{E}(p) > \mathbb{E}(\tilde{p})$, the condition is violated, thus y may be non-monotonic.

Example 2 (Over- and underconfidence)**.** Let p and $\tilde{p}$ be normal distributions. If the investor is overconfident in the sense of too narrow probability estimates, i.e. $\operatorname{var} p > \operatorname{var} \tilde{p}$, but $\mathbb{E}(p) = \mathbb{E}(\tilde{p})$, then condition (7) is satisfied for all $x > \mathbb{E}(p)$, thus the optimal y is an increasing function of the market index for $x > \mathbb{E}(p)$. If $\operatorname{var}(p) < \operatorname{var}(\tilde{p})$, this is the case for $x < \mathbb{E}(p)$.

Figure 13 shows examples of optimal SPs for CRRA investors ($\alpha = -2$) with different beliefs.

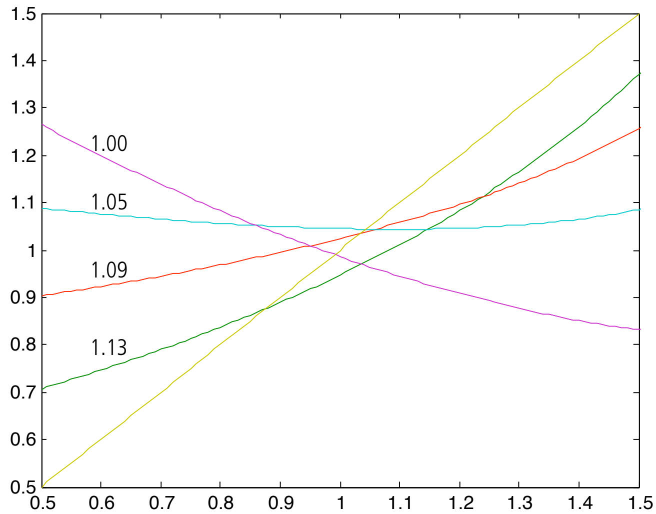

Figure 13: Optimal payoff diagrams for investors with pessimistic to optimistic estimations. $\mathbb{E}(\tilde{p}) = 1.00, 1.05, 1.09, 1.13$ and $\mathbb{E}(p) = 1.09$.

3.5 Probability mis-estimation

Betting against the market can explain the popularity of some investments (like yield enhancing (call option style) or contrarian (put option style) products), but it seems less likely that it can explain the popularity of complicated constructions like barrier reverse convertibles. Here a better explanation is probability mis-estimation. The difference to betting is that probability-misestimation is not caused by a difference in beliefs, but rather a difficulty in translating (potentially accurate) estimates of the market into correct estimates of probabilities for certain events that are important for the payoff of a structured product.

As an example we consider products with a barrier (e.g. barrier reverse convertibles). These are products that have a capital protection that vanishes when the price of the underlying, *at some point before maturity*, falls below a certain threshold ("barrier"), compare Fig. 3c. These products are very popular, as we have seen, although their payoff at maturity is not a function

of the final price of the underlying and thus they are not optimal for any decision model (see above). However, prospect theory utility or an aspiration level cause a similar payoff profile.
Probability mis-estimation can explain this popularity: it has been experimentally demonstrated in [34] that test subjects underestimate the probability that the barrier is hit at *some point in time* before maturity with respect to the probability that the price is below the barrier level *at maturity*. This makes barrier reverse convertibles seemingly more attractive and can thus explain why behavioral preferences alone (without modeling the misestimation of probabilities) cannot explain their popularity.
This is not the only class of popular structured products that conflicts with the rational and behavioral frameworks we have developed. Another important example where probability misestimation seems to play an important role are basket products, e.g. products which return in certain situations the value of the worst performing stock of a predefined basket of assets, for experimental work on this problem see [33].

4 Reverting the point of view: calculating the probability belief or the utility

So far we have computed optimal SPs for given preferences and beliefs. We can revert this computation and (for given preferences) compute the beliefs that make a given SP optimal. From $u'(y(x)) = \lambda\tilde{\ell}(x) = \lambda\ell(x)\frac{p(x)}{\tilde{p}(x)}$ we compute the *relative deviation from rational beliefs* as

$$\frac{\tilde{p}(x)}{p(x)} = \frac{\ell(x)}{u'(y(x))},$$

where we have set $\lambda = 1$[17]. In the CAPM case with CRRA utility this gives

$$\frac{\tilde{p}(x)}{p(x)} = \frac{a - bx}{y(x)^\alpha}.$$

We compute this relative deviation for a typical discount certificate, a capital protected product without cap and a bonus certificate (without barrier feature), where we set for simplicity the coupon to 10%, the barrier at 80% and the participation rates as 100%. The resulting quotient $\tilde{p}(x)/p(x)$ is for some of the products quite peculiar, oscillating in the case of the capital protected product, and in the case of the bonus certificate even having a jump at the barrier level, see Fig. 14.
Given the irregularity of the beliefs necessary to make these typical products optimal, it seems unlikely that beliefs *alone* determine their attractiveness,

[17] Thus we would have to normalize the resulting formula for $\tilde{p}$ to obtain a probability measure

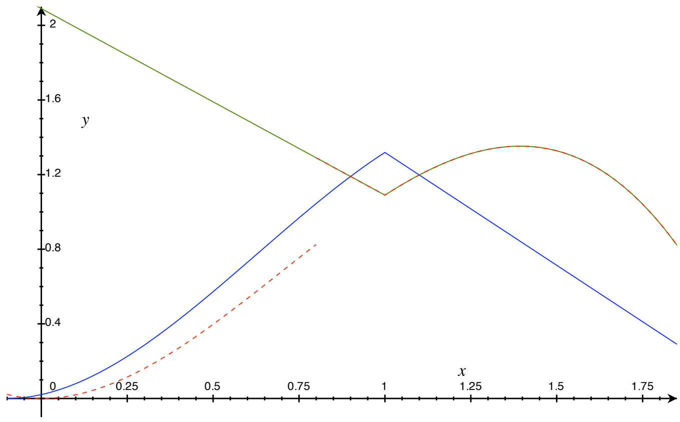

Figure 14: Relative deviation of beliefs for discount certificate (blue line), capital protected product (green line) and bonus certificate (red dots).

particularly in the case of the bonus certificate, but even in the case of the capital protected product, a different explanation seems more likely. In the case of the discount certificate one could accept different beliefs as a motif more easily: in this case the volatility of the returns would be simply underestimated as compared to the market belief.
We can also elicit the utility function of an investor given his probability beliefs and his optimal structured product. Assume for simplicity that his beliefs are correct. Then we start from the derivation of (3) where we had $u'(y(x)) = \lambda\ell(x)$. Since the preferences of expected utility functions are invariant under affine transformation, we can set $\lambda = 1$, thus the concave envelope of the utility function is

$$u'(z) = \ell(y^{-1}(z)).$$

Assuming that $\ell(x)$ is strictly decreasing in x and positive (as in CAPM or Black-Scholes), a jump in y corresponds to a convex part of u (i.e. u' is constant). A capped y (i.e. y is constant above a certain threshold x_0) corresponds to u constant above $y(x_0)$. We use these insights in the next section to give the most likely reasons why investors buy structured products.

5 Summary: a classification of structured products

Our results can be used to classify SPs according to their attractiveness for rational and behavioral investors. We provide a table of the most popular types of structured products, as we had identified them in Fig. 1 and Fig. 2, and give potential reasons for their attractiveness to investors based on our previous analysis.

Here we assumed (in favor of the investors' rationality) that all SPs have the market index as underlying[18].

Ranks GER	CH	USA	Product type (standardized)	Investment motifs (according to our analysis)
1	1	3	Barrier discount cert./ Barrier rev. convert.	IARA plus prob. misestimation (in case of a barrier)
2	3	-	Bonus certificate	non-concave utility plus prob. misestimation
6	-	1	Capital protection	loss-aversion or IARA
-	2	5	Tracker cert.	—
-	-	2	Outperformance cert.	IARA in losses, DARA in gains
3	4	-	Leverage	betting

Table 5: The most popular types of structured products and potential reasons for their popularity.

In this way, we can classify all SPs into several categories, according to the degree of deviation from rational preferences that we need to explain their attractiveness to investors. The overall conclusion is that at least[19] 30.5% of the issued SPs in Germany, 48.6% in Switzerland and 22.49% in the USA have jumps in their payoff diagrams (pointing to non-concave utility functions of investors), 36.2% (resp. 48.6% and 22.49%) cannot be described by a function of the underlying (suggesting systematic probability misestimation as investment motif), and finally 76.6% (resp. 61.6% and 89.97%) have non-convex payoffs (pointing to increasing absolute risk aversion (IARA) in our CAPM model). See Table 6 and 7 (appendix) for details on the German and Swiss data.

We could sum up these examples by saying that the most popular classes of structured products cleverly combine prospect theory-like preferences (in particular loss-aversion and risk-seeking behavior in losses) and probability mis-estimation induced by a complicated payoff structure that leads to a systematic underestimation of the probability for unfavorable outcomes. Hence we have argued that SPs reflect the importance of non-traditional utility functions and of behavioral biases.

[18] In fact, this is a very optimistic assumption, since most have single assets or even worst-of baskets as underlying which makes it even clearer that betting and misestimation play pivotal roles in investment decisions for structured products: for the German market we know, e.g., that as of November 2007 only 25.0% of the structured products used a market index as underlying.

[19] Products with unclear payoff profile could increase these numbers.

6 Conclusions

We have seen that structured products can arise as a solution to enhancing the performance of a portfolio. Depending on the (rational) risk attitudes of an investor it is usually good to use a product that leads to a strictly convex payoff structure for the risky part of the portfolio. This can be done by hedging against losses of different degrees. We estimate the size of the improvement when comparing to a classical Markowitz-style (mean-variance) investment and found that the improvement is typically smaller than the costs of structured products.
Most popular structured products, however, do not follow this rational guideline, but instead use behavioral factors, like loss-aversion or probability mis-estimation to be attractive in the eyes of potential investors. In particular we could show that the currently most popular products clearly cannot be explained even within the framework of prospect theory, but only when taking into account probability mis-estimation. Thus we come to the conclusion that by and large the market for structured products, which is a huge business for banks, offers a utility gain for investors which is most likely only an illusion. Instead of banning structured products completely (as it is currently discussed in some countries), we would suggest to improve the understanding of customers. Also it would be wise to introduce independent rankings that enable specific types of investors to see whether structured products add value to their portfolio or not.

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A Mathematical proofs

Derivation of the two-period model
Let us consider the standard continuous time setting of a financial market (see, e.g., [24, 14]) with a probability space $(\Omega, \mathcal{F}, P)$, a fixed time horizon $T \in (0, \infty)$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ under the usual conditions $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0$ trivial. The stock price process is denoted by S_t. We normalize such that the price of the risk free asset is constant and equals one. A trading strategy $\phi = (\eta, \theta)_t$ consists of an $\mathbb{F}$-adapted η (number od risk free assets) and an $\mathbb{F}$-predictable θ (number of stocks). The cumulative gain process is defined by

$$G_t(\theta) := \int_0^t \theta_\tau \, dS_\tau,$$

where we assume that S is a locally bounded semimartingale and integrable. Then $G(\theta)$ is a stochastic integral of θ with respect to S. We denote the set of all gain processes $G_T(\theta)$ for self-financing and admissible θ by $\mathcal{G}$. Under some regularity assumptions one can show that the market satisfies the no-arbitrage condition if and only if there exists an equivalent martingale measure Q for S (fundamental theorem of asset pricing, compare [10, Theorem 5.10] and [4, Theorem 3, Section 2.1].
We assume that S follows a geometric Brownian motion, i.e.

$$\frac{dS_t}{S_t} = (\mu - r)dt + \sigma_t dW_t, \quad S_0 = s_0 > 0,$$

where μ is the mean return of the stock, σ its volatility and r the risk free return. We define the instantaneous market price of risk by $\lambda = (\mu - r)/\sigma$ and $\hat{Z}_t := \mathcal{E}(-\int \lambda dW)$. One can prove that the measure Q defined by $dQ/dP = \hat{Z}_T$ is an equivalent martingale measure for S, see [24, Theorem 1.6.6].
A utility maximization problem on this financial market for a utility function u can then be described by maximizing $\mathbb{E}(u(R + g))$ over all $g \in \mathcal{G}$, where the initial wealth is normalized. Based on the fundamental theorem of asset pricing, this can be rewritten as maximizing

$$\mathbb{E}(u(X)) \quad \text{subject to } X \in L^0_+, \ \mathbb{E}_P \left[\hat{Z}_T X \right] \leq R.$$

This corresponds to the two-period maximization problem (1) where the likelihood ratio is $\ell := \hat{Z}_T$, thus considering the two-period model is in fact no restricting assumption over the standard time continuous model if we assume that volatility and drift are constant in time.

Proof of Corollary 1.
Consider the Arrow-Pratt risk measure $r(x) := -u''(x)/u'(x)$. Then

$$r'(x) = -\frac{u'''(x)u'(x) - (u''(x))^2}{(u'(x))^2}. \tag{8}$$

Hence, if $r' \geq 0$ (i.e. r is not increasing) then $u'''(x) \geq \frac{(u''(x))^2}{u'(x)}$. Since u is strictly increasing, this is positive. Thus $v'' = u''' > 0$ and we can apply Theorem 1. $\square$

Computation of the optimal SP for CRRA utility.
Let $u(x) := \frac{1}{\alpha}x^\alpha$ with $\alpha < 1$, $\alpha \neq 0$, then $v(x) := u'(x) = x^{\alpha-1}$ and $v^{-1}(z) = z^{\frac{1}{\alpha-1}}$. Therefore, recalling (3), the optimal structured product is given by

$$y(x) = (\lambda\ell(x))^{-\frac{1}{1-\alpha}}.$$

We can compute λ explicitly, if we use the constraint (2):

$$\int (\lambda\ell(x))^{-\frac{1}{1-\alpha}}\ell(x)p(x)\,dx = R,$$

which can be resolved to

$$\lambda = \left(\frac{R}{\int \ell(x)^{\frac{\alpha}{\alpha-1}}p(x)\,dx}\right)^{\alpha-1}.$$

All together we obtain:

$$y(x) = \frac{C}{\ell(x)^{\frac{1}{1-\alpha}}}, \text{ where } C := \frac{R}{\int \ell(x)^{\frac{\alpha}{\alpha-1}}p(x)\,dx}.$$

In the case of the CAPM we obtain

$$y(x) = \frac{C}{(a-bx)^{\frac{1}{1-\alpha}}}.$$

Computation of the optimal SP in the CARA case.
Let $u(x) := -\frac{1}{\alpha}e^{-\alpha x}$. Then $v(x) := u'(x) = e^{-\alpha x}$ and $v^{-1}(z) = -\frac{1}{\alpha}\ln z$. Thus,

$$y(x) = -\frac{1}{\alpha}\ln(\lambda\ell(x)).$$

Again, we can compute λ explicitly, if we use 2, i.e.

$$\int y(x)\ell(x)p(x)\,dx = R.$$

The left hand side can be computed as follows:

$$\begin{aligned}
\int y(x)\ell(x)p(x)\,dx &= -\frac{1}{\alpha}\int \ln(\lambda\ell(x))p(x)\ell(x)\,dx \\
&= -\frac{1}{\alpha}\int (\ln(\lambda) + \ln(\ell(x)))p(x)\ell(x)\,dx \\
&= -\frac{1}{\alpha}\ln(\lambda) - \frac{1}{\alpha}\int \ln(\ell(x))p(x)\ell(x)\,dx.
\end{aligned}$$

Thus we obtain

$$\lambda = e^{-\alpha R - \int p(x)\ell(x)\ln(\ell(x)\,dx}.$$

The optimal structured product is therefore given by

$$y(x) = R - \frac{1}{\alpha}\left(\ln \ell(x) - \int p(x)\ell(x)\ln(\ell(x))\,dx\right).$$

And in the case of the CAPM we obtain

$$y(x) = R - \frac{1}{\alpha}\left(\ln(a - bx) - C\right),$$

where $C := \int p(x)(a - bx)\ln(a - bx)\,dx$.

Proof of Lemma 2.
Let y_0 be a point where $u_c(y_0) > u(y_0)$. We prove for any market return x_0 that an optimal SP for u does not have to yield the value y_0, i.e. $y_0 \notin y(\mathbb{R})$. Suppose the opposite, i.e. $y(x_0) = y_0$ for some x_0. We can find two points y_1, y_2 with $y_1 < y_0 < y_2$ such that $\lambda y_1 + (1 - \lambda)y_2 = y_0$ with $\lambda \in (0, 1)$, $u(y_1) = u_c(y_1)$, $u(y_2) = u_c(y_2)$ and $u_c(y_0) = \lambda u(y_1) + (1 - \lambda)u(y_2)$.
Now we can construct another SP $\tilde{y}$ such that whenever the market return is x_0, $\tilde{y}$ gives a return of y_1 with probability λ and a return of y_2 with probability $1 - \lambda$. Let us do the same construction for all values y_0 where $u_c(y_0) > u(y_0)$. The new product obviously has a utility which is at least as big as before, since $\lambda u(y_1) + (1 - \lambda)u(y_2) = u_c(y_0) > u(y_0)$. Moreover, it satisfies by construction the pricing constraint.
By allowing for lotteries, we have increased the state space. This can be fixed as follows: Let $S \subset [0, \infty)$ be the set on which the modified product takes lotteries in y_1 and y_2. If this is a zero set with respect to p, we can ignore it for the optimization. If not, define $\bar{\lambda} := \frac{1}{p(S)} \int_S \lambda(x)\,dp(x)$. Then decompose S into two disjoint sets S_1 and S_2 with $p(S_1) = \bar{\lambda}p(S)$ and $p(S_2) = (1 - \bar{\lambda})p(S)$ such that $\inf_{x \in S_1} \ell(x) \geq \sup_{x \in S_2} \ell(x)$ (possible since ℓ is continuous). Now modify the product such that it gives the fixed payoffs y_1 on S_1 and y_2 and on S_2. A short computation confirms that this new product yields the same return distribution while being not more expensive than before. □

Proof of Theorem 3.
Let p_* denote the return distribution of the market portfolio after probability weighting has been applied. More precisely, in cumulative prospect theory we define[20]

$$p_*(x) := \frac{d}{dx}\left(w\left(\int_{-\infty}^{x} p(t)\,dt\right)\right).$$

[20] We could also use a decumulative function in gains as in the original formulation in [39], but both is mathematically equivalent if we choose w differently in gains and losses.

Similarly, in prospect theory (see [23, 25, 35]), we define for $\gamma \in (0,1]$:

$$p_*(x) := \frac{p(x)^\gamma}{\int_{\mathbb{R}} p(x)^\gamma \, dx}.$$

Now in both cases it is easy to see that p_* is smooth and nowhere zero. We define $w_*(x) := p_*(x)/p(x)$. Then the optimal structured product maximizes

$$U(p) := \int_{\mathbb{R}} u(y(x)) p_*(x) \, dx,$$

subject to

$$\int_{\mathbb{R}} y(x) \frac{\ell(x)}{w_*(x)} p_*(x) \, dx = R. \tag{9}$$

From the co-monotonicity result [34], we know that the maximizer y of this problem is a monotonic function of $\ell(x)/w_*(x)$ (remember that p is absolutely continuous). Since w_* is smooth, we can differentiate this expression and arrive at

$$\frac{d}{dx}\left(\frac{\ell(x)}{w_*(x)}\right) = \frac{\ell'(x) w_*(x) - \ell(x) w_*'(x)}{w_*(x)^2},$$

which is a smooth function, thus y is piecewise smooth. $\square$

B Number of issued structured products in Germany and Switzerland

Category	Issued
Increasing, convex function of underlying, no jumps (risk optimization, non-increasing ARA)	
Guarantee certificates	1,857
Index certificate	1,998
Outperformance certificates	2,219
Subtotal	3.5%
Same but not convex (risk optimization, increasing ARA)	
Reverse convertible	6,155
Discount certificates	66,511
Subtotal	42.3%
Not necessarily increasing (potentially speculation)	
Exotic leverage	5,319
KO products with stop loss	24,161
Subtotal	17.2%
Several scenarios, in each scenario increasing function of underlying (probability misestimation)	
Basket certificates	1,569
Bonus-/partial proetction cert.	52,321
Subtotal	31.4%
Not classifiable	
KO product w/o stop loss	8,281
Others	1,434
Subtotal	5.7%

Table 6: Approximate distribution of structured products issued in Germany according to normative and behavioral categories.

Category (according to SVSP classification [1].)	Issued
Increasing, convex function of underlying, no jumps (risk optimization, non-increasing ARA)	
Tracker Certificate	8513
Outperformance Certificate	375
Uncapped Capital Protection	2983
Capital Protection with Coupon	170
Subtotal	25.4%
Same but not convex (risk optimization, increasing ARA)	
Airbag Certificate	58
Discount Certificate	3750
Reverse Convertible	1376
Capped Outperformance Certificate	123
Capped Capital Protection	303
Subtotal	11.8%
Not necessarily increasing (potentially speculation)	
Spread Warrant	61
Mini-Future	4335
Subtotal	9.3%
Several scenarios, in each scenario increasing function of underlying (pointing to probability misestimation)	
Knock-Out Warrant	2224
Bonus Certificate	4666
Barrier Discount Certificate	407
Barrier Reverse Convertible	14428
Capped Bonus Certificate	494
Subtotal	46.9%
Partially decreasing function of underlying (speculation)	
Barrier Range Reverse Convertible	476
Twin-Win Certificate	342
Subtotal	1.7%
Not classifiable:	
Diverse Leverage, Diverse Participation, Express Certificate, Diverse Yield Enhancement, Diverse Capital Protection, Diverse Product	
Subtotal	4.8%

Table 7: Approximate distribution of structured products issued in Switzerland according to normative and behavioral categories.