Asset Market Games of Survival*

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Abstract

The paper examines a game-theoretic model of a financial market in which asset prices are determined endogenously in terms of short-run equilibrium. Investors use general, adaptive strategies depending on the exogenous states of the world and the observed history of the game. The main goal is to identify strategies, allowing an investor to "survive," i.e. to possess a positive, bounded away from zero, share of market wealth over the infinite time horizon. This work links recent studies on evolutionary finance to the classical topic of games of survival pioneered by Milnor and Shapley in the 1950s.

Keywords: evolutionary finance, dynamic games, stochastic games, games of survival.

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1 Introduction

In this paper we consider a discrete-time, game-theoretic model of a financial market with endogenous asset prices determined by short run equilibrium of supply and demand, given agents' monetary bids. Uncertainty on asset payoffs at each period is modeled via a general, exogenous stochastic process governing the evolution of the states of the world. The states of the world are meant to capture various macroeconomic and business cycle variables that may affect investors' behavior. The traders use general, adaptive strategies (portfolio rules), distributing their current wealth between assets at every period, depending on the observed history of the game and the exogenous random factors. The main goal of the study is to identify investment strategies that guarantee the "long-run survival" of any investor using them, in the sense of keeping a strictly positive, bounded away from zero, share of market wealth over the infinite time horizon, irrespective of the investment strategies employed by the other agents in the market. The main result establishes that Kelly's (1956) famous portfolio rule of "betting your beliefs" possesses this property of unconditional survival. Moreover, we show that the strategy possessing this property is essentially unique: any other strategy of this kind (belonging to a certain class) is asymptotically similar to the Kelly rule. The result on asymptotic uniqueness we obtain may be regarded as an analogue of turnpike theorems[1], stating that all optimal or quasi-optimal paths of economic dynamics converge to each other in the long run.

This work constitutes an attempt to bring together the recent line of studies on evolutionary finance (see, e.g., a review by Blume and Easley 2008) with the older literature on stochastic dynamic games going back to Shapley (1953). The dynamic framework at hand shares some conceptual features with two existing specific classes of dynamic games. One is formed by the classical *games of survival* pioneered by Milnor and Shapley (1957).[2] In a game of survival, two players start with wealth levels (w_1, w_2) such that $w_1 + w_2 = C$ (a fixed constant). At each stage, they play a zero-sum matrix game $B = [b_{ij}]$ wherein the choice of actions i and j would lead to player 2 paying player 1 the amount b_{ij}, causing the state to transit to $(w_1 + b_{ij}, w_2 - b_{ij})$ while the actual stage reward is 0 for both players. The

[1] See, e.g., Nikaido (1968) and McKenzie (1986).

[2] For textbook treatments of this class of games, see Luce and Raiffa (1989, Section A8.4) and Maitra and Sudderth (1996, Section 7.16). For more recent research on similar classes of games see Secchi and Sudderth (2001) and references therein.

game repeats from the new wealth levels ad infinitum, or until $w_1 \leq 0$ (player 1 is bankrupt) or $w_1 \geq C$ (player 2 is bankrupt), with the corresponding payoffs being $(0, 1)$ and $(1, 0)$. In case the game goes on indefinitely, the payoff is defined as $(q, 1 - q)$, with $0 < q < 1$. A game of survival is thus a constant-sum stochastic game that may be viewed as a natural game-theoretic analog of the well-known gambler's ruin decision problem (Dubins and Savage, 1965).

In a similar vein, Shubik and Whitt (1973) consider a dynamic market game with one unit of a durable good per period, N players and a fixed total wealth distributed across the players in exogenous fixed shares. Each player can bid part or all of his current wealth on the durable good, of which he obtains an amount in proportion to his bid. The total bid is then redistributed to the players according to their fixed shares, and play proceeds to the next period. Each player's objective is to maximize the discounted sum of utilities of consumption, using Markov bidding strategies.

While these two classes of dynamic investment games are related to the present model in their general focus, there are important differences. The main difference is that the game solution concept we use here is based on the notion of a survival strategy outlined above, rather than on a Nash equilibrium of any kind. The notion we deal with is defined in terms of a property holding almost surely, rather than in terms of expectations. No utilities, discounted or undiscounted, are involved in the model, which makes the modeling approach closer to applications, where typically quantitative information about investor's preferences is lacking. No conclusions from the rich literature on dynamic games can be directly invoked in the present analysis.

This paper is organized as follows. Section 2 lays out the model description. Section 3 contains the statements of the main results and their discussion. Section 4 provides their proofs.

2 The Model

The model we deal with is a game-theoretic version of evolutionary models of financial markets with one-period assets (Blume and Easley 1992, Evstigneev et al. 2002, Amir et al. 2005, and others). There are $N \geq 2$ *investors* (*traders*) acting in a market where $K \geq 2$ risky *assets* (*securities*) are traded. A *portfolio* of investor i at date $t = 0, 1, ...$ is characterized by a vector $x_t^i = (x_{t,1}^i, ..., x_{t,K}^i) \in \mathbb{R}_+^K$ where $x_{t,k}^i$ indicates the amount ("physical units") of asset k in the portfolio x_t^i. The coordinates of x_t^i are non-negative: short sales are ruled out. We denote by $p_t \in \mathbb{R}_+^K$ the vector of market prices of the securities. For each $k = 1, ..., K$, the coordinate $p_{t,k}$ of $p_t = (p_{t,1}, ..., p_{t,K})$ stands for the price of one unit of asset k at date t. The scalar product $\langle p_t, x_t^i \rangle = \sum_{k=1}^K p_{t,k} x_{t,k}^i$ expresses the market value of investor i's portfolio at date t.

It is supposed that the total amount of each security k available in the market at date 0 is $V_{0,k}$ and in each subsequent time period $t = 1, 2, ...$ it is $V_{t,k}(s^t) > 0$, where s_t is the state of the world at date t and $s^t := (s_1, ..., s_t)$ is the history of the process (s_t) up to time t. The sequence of states of the world $s_1, s_2, ...$ is an exogenous stochastic process with values in a measurable space S. Assets live for one period: they are traded at the beginning of the period and yield payoffs at the end of it; then the cycle repeats. The payoff $A_{t,k}(s^t) \geq 0$ of asset $k = 1, 2, ..., K$ at date $t = 1, 2, ...$ depends, generally, on t and s^t. The functions $A_{t,k}(s^t)$ are measurable and satisfy

$$\sum_{k=1}^{K} A_{t,k}(s^t) > 0 \text{ for all } t, s^t. \tag{1}$$

The last inequality means that in each random situation at least one asset gives a strictly positive payoff.

At date $t = 0$ investors have initial endowments—amounts of money $w_0^i > 0$ $(i = 1, 2, ..., N)$. These initial endowments form the traders' budgets at date 0. Trader i's budget at date $t \geq 1$ is $\langle A_t(s^t), x_{t-1}^i \rangle$, where $A_t(s^t) := (A_{t,1}(s^t), ..., A_{t,K}(s^t))$. It is formed by the payoffs of the assets contained in yesterday's portfolio x_{t-1}^i of investor i. This budget is re-invested in the assets available at date t, which will yield payoffs $A_{t+1,k}(s^{t+1})$, $k = 1, ..., K$, at date $t+1$. Dynamics of this kind reflects some features of markets for real assets related, e.g., to energy, natural resources, etc.

For each $t \geq 0$, each trader $i = 1, 2, ..., N$ selects a vector of *investment proportions* $\lambda_t^i = (\lambda_{t,1}^i, ..., \lambda_{t,K}^i)$ according to which he/she plans to distribute the available budget between assets. Vectors λ_t^i belong to the unit simplex

$$\Delta^K := \{(a_1, ..., a_K) \in \mathbb{R}_+^K : a_1 + ... + a_K = 1\}.$$

The investment proportions at each date $t \geq 0$ are selected by the N traders simultaneously and independently (so that we deal here with a simultaneous-move N-person dynamic game). For $t \geq 1$, they might depend, generally, on the history $s^t := (s_1, ..., s_t)$ of the process of the states of the world and the *market history* $(p^{t-1}, x^{t-1}, \lambda^{t-1})$, where $p^{t-1} = (p_0, ..., p_{t-1})$ is the sequence of asset prices up to time $t-1$,

$$x^{t-1} := (x_l^i),\ \lambda^{t-1} := (\lambda_l^i),\ i = 1, ..., N,\ l = 0, ..., t-1,$$

are the sets of vectors describing the portfolios and the investment proportions of all the traders at all the dates up to $t-1$. A vector $\Lambda_0^i \in \Delta^K$ and a sequence of measurable functions $\Lambda_t^i(s^t, p^{t-1}, x^{t-1}, \lambda^{t-1}),\ t = 1, 2, ...,$ with values in Δ^K form an *investment (trading) strategy* Λ^i of trader i, specifying a *portfolio rule* according to which trader i selects investment proportions at each date $t \geq 0$. This is a general game-theoretic definition of a pure strategy, assuming full information about the market history, including the players' previous actions and the knowledge of all the past and present states of the world. In the class of such general portfolio rules, we will distinguish those for which Λ_t^i depends only on s^t, and not on the market history $(p^{t-1}, x^{t-1}, \lambda^{t-1})$. Such portfolio rules will be called *basic*. They play an important role in this work: the survival strategy we construct is of this kind.

Suppose each investor i at date 0 has selected investment proportions $\lambda_0^i = (\lambda_{0,1}^i, ..., \lambda_{0,K}^i) \in \Delta^K$. Then the amount invested in asset k by trader i will be $\lambda_{0,k}^i w_0^i$, and the total amount invested in asset k will be equal to $\sum_{i=1}^N \lambda_{0,k}^i w_0^i$. The equilibrium price $p_{0,k}$ of each asset k can be determined from the equations

$$p_{0,k} V_{0,k} = \sum_{i=1}^{N} \lambda_{0,k}^i w_0^i,\ k = 1, 2, ..., K. \tag{2}$$

On the left-hand side of (2), we have the total value, expressed in terms of the prevailing price $p_{0,k}$, of the assets of the kth type purchased by the market participants at date 0 (recall that the amount of each asset k at date 0 is $V_{0,k}$). On the right-hand side, we have the total sum of money invested in asset k by all the investors.

The investors' portfolios $x_0^i = (x_{0,1}^i, ..., x_{0,K}^i),\ i = 1, 2, ..., N$, at date 0 can be determined from the equations

$$x_{0,k}^i = \frac{\lambda_{0,k}^i w_0^i}{p_{0,k}},\ k = 1, 2, ..., K,\ i = 1, ..., N, \tag{3}$$

meaning that the current market value $p_{0,k} x_{t,k}^i$ of the kth position of the portfolio x_t^i is equal to the fraction $\lambda_{0,k}^i$ of the trader i's investment budget w_0^i.

The last equation makes sense only if $p_{0,k} > 0$. This condition is guaranteed by the assumption that one of the investors, say the first one, has strictly positive investment proportions $\lambda^1_{t,1}, ..., \lambda^1_{t,K}$ at each moment of time $t \geq 0$. *The strict positivity of the vector of investment proportions of the first investor at all moments of time will be assumed throughout the paper.* Clearly, this assumption implies $x^1_{0,k} > 0$ for all k.

Suppose now that all the investors have chosen their investment proportion vectors $\lambda^i_t = (\lambda^i_{t,1}, ..., \lambda^i_{t,K})$ at date $t \geq 1$. Then the balance between aggregate asset supply and demand implies the formula determining the equilibrium prices

$$p_{t,k} V_{t,k} = \sum_{i=1}^{N} \lambda^i_{t,k} \langle A_t, x^i_{t-1} \rangle, \ k = 1, ..., K, \tag{4}$$

which, in turn, yields the expression for the investors' portfolios $x^i_t = (x^i_{t,1}, ..., x^i_{t,K})$:

$$x^i_{t,k} = \frac{\lambda^i_{t,k} \langle A_t, x^i_{t-1} \rangle}{p_{t,k}}, \ k = 1, ..., K, \ i = 1, ..., N. \tag{5}$$

Here, in contrast with the case $t = 0$, the traders' budgets at date $t \geq 1$ are not given exogenously as initial endowments, rather they are formed by the payoffs of the previous date's portfolios x^i_{t-1}.

When writing equations (5), we have to care about the strict positivity of $p_{t,k}$. To guarantee this, we use our standing hypothesis on the strict positivity of the investment proportions $\lambda^1_{t,k}$ of the first trader. Arguing by induction, we can assume that $x^1_{t-1} > 0$, which implies $\langle A_t, x^1_{t-1} \rangle > 0$ by virtue of (1), which in turn yields $p_{t,k} > 0$ and $x^1_t > 0$ (see (4) and (5)). By summing up equations (5) over $i = 1, ..., N$, we obtain that

$$\sum_{i=1}^{N} x^i_{t,k} = \sum_{i=1}^{N} \frac{\lambda^i_{t,k} \langle A_t, x^i_{t-1} \rangle V_{t,k}}{\sum_{j=1}^{N} \lambda^j_{t,k} \langle A_t, x^j_{t-1} \rangle} = V_{t,k}$$

for every asset k (the market clears).

Given a strategy profile $\Lambda = (\Lambda^1, ..., \Lambda^N)$ of the investors, we can generate a path of the market game by setting $\lambda^i_0 = \Lambda^i_0, \ i = 1, ..., N,$

$$\lambda^i_t = \Lambda^i_t(s^t, p^{t-1}, x^{t-1}, \ \lambda^{t-1}), \ t = 1, 2, ..., \ i = 1, ..., N, \tag{6}$$

and by defining p_t and x_t^i recursively according to equations (2)–(5). The random dynamical system described determines the state of the market at each moment of time $t \geq 1$ as a measurable vector function of s^t:

$$(p_t(s^t); x_t^1(s^t), ..., x_t^N(s^t); \lambda_t^1(s^t), ..., \lambda_t^N(s^t)), \tag{7}$$

where $p_t(s^t), x_t^i(s^t)$ and $\lambda_t^i(s^t)$ are the vectors of equilibrium prices, investors' portfolios and their investment proportions, respectively. For $t = 0$, these vectors are constant. The standing hypothesis $\Lambda_t^1 > 0$ together with the assumption that $w_0^i > 0$ implies by induction that $p_t(s^t) > 0$ and $x_t^1(s^t) > 0$, and so the random dynamical system under consideration is well-defined.

3 The Main Results

Consider a strategy profile $\Lambda = (\Lambda^1, ..., \Lambda^N)$ of the investors and the path (7) of the random dynamical system generated by this strategy profile. Let

$$w_t^i = w_t^i(s^t) := \langle A_t(s^t), x_{t-1}^i(s^{t-1})\rangle \tag{8}$$

denote investor i's wealth at date $t \geq 1$. Note that at every date t investor 1's wealth is strictly positive because $x_{t-1}^1 > 0$. The *total market wealth* is equal to $W_t = \sum_{i=1}^N w_t^i$ (> 0). We are primarily interested in the long-run behavior of the *relative wealth,* or the *market shares,* $r_t^i := w_t^i / W_t$ of the traders, i.e. in the asymptotic properties of the sequence of vectors $r_t = (r_t^1, ..., r_t^N)$ as $t \to \infty$.

Given a strategy profile $(\Lambda^1, ..., \Lambda^N)$, we say that the strategy Λ^i (or investor i using it) *survives* with probability one if $\inf_{t \geq 0} r_t^i > 0$ almost surely (a.s.). This means that for almost all realizations of the process of states of the world (s_t), the market share of investor i is bounded away from zero by a strictly positive random constant. A portfolio rule Λ is called a *survival strategy* if investor i using it survives with probability one regardless of what portfolio rules $\Lambda^j, j \neq i$, are used by the other investors.

To formulate the main result on survival strategies, define the *relative payoffs* by

$$R_{t,k}(s^t) := \frac{A_{t,k}(s^t) V_{t-1,k}(s^{t-1})}{\sum_{m=1}^K A_{t,m}(s^t) V_{t-1,m}(s^{t-1})}. \tag{9}$$

and put $R_t(s^t) = (R_{t,1}(s^t), ..., R_{t,K}(s^t))$. Consider the investment strategy $\Lambda^* = (\lambda_t^*)$ for which

$$\lambda_t^*(s^t) := E_t R_{t+1}(s^{t+1}), \tag{10}$$

where $E_t(\cdot) = E(\cdot|s^t)$ is the conditional expectation given s^t (if $t = 0$, then $E_0(\cdot) = E(\cdot)$). This strategy, depending only on the history s^t of the process (s_t), prescribes to distribute wealth according to the proportions of the conditional expectations of the relative asset payoffs. The portfolio rule (10) is a generalization of the Kelly portfolio rule of "betting your beliefs" well-known in capital growth theory—see Kelly (1956), Breiman (1961), Algoet and Cover (1988), and Hakansson and Ziemba (1995).

Assume that for each $k = 1, 2, ..., K$,

$$E \ln E_t R_{t+1,k}(s^{t+1}) > -\infty. \tag{11}$$

This assumption implies that the conditional expectation $E_t R_{t+1,k} = E(R_{t+1,k}|s^t)$ is strictly positive a.s., and so we can select a version of this conditional expectation that is strictly positive for all s^t. This version, $\lambda_t^*(s^t)$, will be used in the definition of the portfolio rule (10).

A central result is as follows.

Theorem 1. *The portfolio rule Λ^* is a survival strategy.*

In this study we use the notion of a survival strategy as a solution concept for the game under consideration. This notion does not involve explicitly agents' utility functions and Nash equilibrium conditions, as would be standard in game theory. This notion can therefore be applied in those cases when no quantitative information about investors' preferences is available, a realistic feature of financial markets.

Note that the portfolio rule Λ^* belongs to the class of basic portfolio rules: the investment proportions $\lambda_t^*(s^t)$ depend only on the history s^t of the process of states of the world, and do not depend on the market history. The following theorem shows that in this class the survival strategy $\Lambda^* = (\lambda_t^*)$ is essentially unique: any other basic survival strategy is asymptotically similar to Λ^*.

Theorem 2. *If $\Lambda = (\lambda_t)$ is a basic survival strategy, then*

$$\sum_{t=0}^{\infty} ||\lambda_t^* - \lambda_t||^2 < \infty \text{ (a.s.)}. \tag{12}$$

Here, we denote by $||\cdot||$ the Euclidean norm in a finite-dimensional space. Theorem 2 is akin to various *turnpike* results in the theory of economic dynamics, expressing the idea that all optimal or asymptotically optimal paths of an economic system follow in the long run essentially the same route: the turnpike (Nikaido 1968, McKenzie 1986). Survival strategies Λ can be characterized by the property that the wealth w_t^j of any investor j cannot grow infinitely faster (with strictly positive probability) than the wealth of investor i using Λ. The class of such investment strategies is similar to the class of "good" paths of economic dynamics, as introduced by Gale (1967) — paths that cannot be "infinitely worse" than the turnpike. Theorem 2 is a direct analogue of Gale's turnpike theorem for good paths (Gale, 1967, Theorem 8); for a stochastic version of this result see Arkin and Evstigneev (1987, Chapter 4, Theorem 6).

Note that the class of basic strategies is *sufficient* in the following sense. Any sequence of vectors $r_t = (r_t^1, ..., r_t^N)$ $(r_t = r_t(s^t))$ of market shares generated by some strategy profile $(\Lambda^1, ..., \Lambda^N)$ can be generated by a strategy profile $(\lambda_t^1(s^t), ..., \lambda_t^N(s^t))$ consisting of basic portfolio rules. The corresponding vector functions $\lambda_t^i(s^t)$ can be defined recursively by (6). Thus it is sufficient to prove Theorem 1 only for basic portfolio rules; this will imply that the portfolio rule (10) survives in competition with any, not necessarily basic, strategies. Such considerations cannot be automatically applied to the problem of asymptotic characterization of general survival strategies. This problem remains open; it indicates an interesting direction for further research.

4 Proofs

1st step. We begin with the derivation of a system of equations describing the dynamics of the market shares r_t^i. From (2)–(5) and (8), we get

$$p_{t,k} V_{t,k} = \langle \lambda_{t,k}, w_t \rangle, \ x_{t,k}^i = \frac{\lambda_{t,k}^i w_t^i V_{t,k}}{\langle \lambda_{t,k}, w_t \rangle}, \tag{13}$$

where $\lambda_{t,k} := (\lambda_{t,k}^1, ..., \lambda_{t,k}^N)$ and $w_t := (w_t^1, ..., w_t^N)$. Consequently,

$$w_{t+1}^i = \sum_{k=1}^K A_{t+1,k} x_{t,k}^i = \sum_{k=1}^K A_{t+1,k} V_{t,k} \frac{\lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle} \tag{14}$$

By summing up these equations over $i = 1, ..., N$, we obtain

$$W_{t+1} = \sum_{k=1}^{K} A_{t+1,k} V_{t,k} \frac{\sum_{i=1}^{N} \lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle} = \sum_{k=1}^{K} A_{t+1,k} V_{t,k}. \tag{15}$$

Dividing the left-hand side of (14) by W_{t+1}, the right-hand side of (14) by $\sum_{m=1}^{K} A_{t+1,m} V_{t,k}$, and using (15) and (9), we arrive at the system of equations

$$r_{t+1}^i = \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}^i r_t^i}{\langle \lambda_{t,k}, r_t \rangle},\ i = 1, ..., N. \tag{16}$$

2nd step. Observe that it is sufficient to prove Theorem 1 in the case of $N = 2$ investors. Consider the random dynamical system (16) and define

$$\tilde{\lambda}_{t,k}^2(s^t) = \begin{cases} (\lambda_{t,k}^2 r_t^2 + ... + \lambda_{t,k}^N r_t^N)/(1 - r_t^1) & \text{if } r_t^1 < 1, \\ 1/K & \text{if } r_t^1 = 1. \end{cases} \tag{17}$$

Then we have

$$\lambda_{t,k}^2 r_t^2 + ... + \lambda_{t,k}^N r_t^N = (1 - r_t^1) \tilde{\lambda}_{t,k}^2,$$
$$\langle \lambda_{t,k}, r_t \rangle = r_t^1 \lambda_{t,k}^1 + (1 - r_t^1) \tilde{\lambda}_{t,k}^2,$$

and so

$$r_{t+1}^1 = \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}^1 r_t^1}{r_t^1 \lambda_{t,k}^1 + (1 - r_t^1) \tilde{\lambda}_{t,k}^2}. \tag{18}$$

By summing up equations (16) over $i = 2, ..., N$, we obtain

$$1 - r_{t+1}^1 = \sum_{k=1}^{K} R_{t+1,k} \frac{\tilde{\lambda}_{t,k}^2 (1 - r_t^1)}{r_t^1 \lambda_{t,k}^1 + (1 - r_t^1) \tilde{\lambda}_{t,k}^2}. \tag{19}$$

Thus the sequence $(r_t^1(s^t))$ generated by the original N-dimensional system (16) is the same as the analogous sequence generated by the two-dimensional system (18)–(19) corresponding to the game with two investors $i = 1, 2$ whose investment proportions are $\lambda_{t,k}^1(s^t)$ and $\tilde{\lambda}_{t,k}^2(s^t)$, respectively.

3rd step. Assume that $N = 2$ and $\lambda_{t,k}^1 = \lambda_{t,k}^*$. Since $\lambda_{t,k}^* > 0$, our standing hypothesis on the strict positivity of the investment proportions of the first investor is valid. Putting $\kappa_t = \kappa_t(s^t) := r_t^1(s^t)$, we obtain from (16) with $N = 2$:

$$\kappa_{t+1} = \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}^1 \kappa_t}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1 - \kappa_t)}.$$

Observe that the process $\ln \kappa_t$ is a submartingale. Indeed, we have

$$E_t \ln \kappa_{t+1} - \ln \kappa_t = E_t \ln \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1-\kappa_t)} \geq$$

$$E_t \sum_{k=1}^{K} R_{t+1,k} \ln \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1-\kappa_t)} =$$

$$\sum_{k=1}^{K} \lambda_{t,k}^1 \ln \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1-\kappa_t)} =$$

$$\sum_{k=1}^{K} \lambda_{t,k}^1 \ln \lambda_{t,k}^1 - \sum_{k=1}^{K} \lambda_{t,k}^1 \ln[\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1-\kappa_t)] \geq 0 \text{ (a.s.)}.$$

We used here Jensen's inequality for the concave function $\ln x$ and the elementary inequality

$$\sum_{k=1}^{K} a_k \ln a_k \geq \sum_{k=1}^{K} a_k \ln b_k \quad [\ln 0 := -\infty]$$

holding for any vectors $(a_1, ..., a_K) > 0$ and $(b_1, ..., b_K) \geq 0$ with $\sum a_k = \sum b_k = 1$ (see Lemma 2 below).

Further,

$$\kappa_{t+1} = \kappa_t \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1-\kappa_t)} \geq$$

$$\kappa_t \sum_{k=1}^{K} R_{t+1,k} (\min_m \lambda_{t,m}^1) = \kappa_t (\min_m \lambda_{t,m}^1).$$

Since $E \min_m \ln \lambda_{t,m}^1 > -\infty$ by virtue of assumption (11) and κ_0 is a strictly positive non-random number, each of the random variables $0 < \kappa_t \leq 1$ satisfies $E|\ln \kappa_t| < \infty$.

The non-positive submartingale $\ln \kappa_t$ has a finite limit a.s., and so $\kappa_t \to \kappa_\infty$ (a.s.), where κ_∞ is a strictly positive random variable. Consequently, the sequence $\kappa_t > 0$ is bounded away from zero with probability one, which means that investor 1 survives almost surely. □

The proof of Theorem 2 is based on two lemmas.

Lemma 1. Let ξ_t *be a submartingale such that* $\sup_t E\xi_t < \infty$. *Then the series of non-negative random variables* $\sum_{t=0}^{\infty}(E_t\xi_{t+1} - \xi_t)$ *converges a.s.*

Proof. We have $\zeta_t := E_t\xi_{t+1} - \xi_t \geq 0$ by the definition of a submartingale. Further, we have

$$\sum_{t=0}^{T-1} E\zeta_t = \sum_{t=0}^{T-1}(E\xi_{t+1} - E\xi_t) = E\xi_T - E\xi_0,$$

and so the sequence $\sum_{t=0}^{T-1} E\zeta_t$ is bounded because $\sup_T E\xi_T < \infty$. Therefore the series of the expectations $\sum_{t=0}^{\infty} E\zeta_t$ of the non-negative random variables ζ_t converges, which implies $\sum_{t=0}^{\infty} \zeta_t < \infty$ a.s. because $E\sum_{t=0}^{\infty} \zeta_t = \sum_{t=0}^{\infty} E\zeta_t$ (see, e.g., Theorem I.12.3 in Saks 1964). □

Lemma 2. *For any vectors* $(a_1, ..., a_K) > 0$ *and* $(b_1, ..., b_K) \geq 0$ *satisfying* $\sum a_k = \sum b_k = 1$, *the following inequality holds*

$$\sum_{k=1}^{K} a_k \ln a_k - \sum_{k=1}^{K} a_k \ln b_k \geq \frac{1}{4}\sum_{k=1}^{K}(a_k - b_k)^2. \qquad (20)$$

Proof. We have $\ln x \leq x - 1$, which implies $(\ln x)/2 = \ln\sqrt{x} \leq \sqrt{x} - 1$, and so $-\ln x \geq 2 - 2\sqrt{x}$. By using this inequality, we get

$$\sum_{k=1}^{K} a_k(\ln a_k - \ln b_k) = -\sum_{k=1}^{K} a_k \ln\frac{b_k}{a_k} \geq \sum_{k=1}^{K} a_k(2 - 2\frac{\sqrt{b_k}}{\sqrt{a_k}}) =$$

$$2 - 2\sum_{k=1}^{K}\sqrt{a_k b_k} = \sum_{k=1}^{K}(a_k - 2\sqrt{a_k b_k} + b_k) = \sum_{k=1}^{K}(\sqrt{a_k} - \sqrt{b_k})^2.$$

This yields (20) because $(\sqrt{a} - \sqrt{b})^2 \geq (a - b)^2/4$ for $0 \leq a, b \leq 1$. □

Remark. Lemma 2 can be deduced from an inequality between the *Kullback-Leibler divergence* (generalizing the expression on the left-hand side of (20)) and the *Hellinger distance* (which reduces in our context to $[\sum(\sqrt{a_k} - \sqrt{b_k})^2]^{1/2}$) — see, e.g., Borovkov (1998, Section II.31). For the reader's convenience we give a direct and elementary proof of Lemma 2, rather than referring to these general facts.

Proof of Theorem 2. Let $\Lambda = (\lambda_t)$ be a basic survival strategy. Suppose that investors $i = 1, 2, ..., N-1$ use the strategy $\Lambda^* = (\lambda_t^*)$ and investor N uses Λ. By summing up equations (16) with $\lambda_t^i = \lambda_t^*$ over $i = 1, ..., N-1$, we obtain

$$\hat{r}_{t+1}^1 = \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}^* \hat{r}_t^1}{\lambda_{t,k}^* \hat{r}_t^1 + \lambda_{t,k}(1-\hat{r}_t^1)},$$

where $\hat{r}_t^1 := r_t^1 + ... + r_t^{N-1}$ is the market share of the group of investors $i = 1, 2, ..., N-1$ and $1 - \hat{r}_t = r_t^N$ is the market share of investor N. Further, we have

$$1 - \hat{r}_{t+1}^1 = \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}(1-\hat{r}_t^1)}{\lambda_{t,k}^* \hat{r}_t^1 + \lambda_{t,k}(1-\hat{r}_t^1)}.$$

Thus the dynamics of the market shares $\hat{r}_t^1 = r_t^1 + ... + r_t^{N-1}, 1 - \hat{r}_t^1 = r_t^N$ is exactly the same as the dynamics of the market shares $\hat{r}_t^1, \hat{r}_t^2 = 1 - \hat{r}_t^1$ of two investors $i = 1, 2$ ($N = 2$) using the strategies $(\lambda_t^1) = (\lambda_t^*)$ and $(\lambda_t^2) = (\lambda_t)$, respectively. Since (λ_t) is a survival strategy, the random sequence $r_t^N = 1 - \hat{r}_t^1 = \hat{r}_t^2$ is bounded away from zero almost surely.

In the course of the proof of Theorem 1 (step 3), we have shown that the sequence $\ln \kappa_t := \ln \hat{r}_{t+1}^1$ is a non-positive submartingale satisfying

$$E_t \ln \kappa_{t+1} - \ln \kappa_t \geq$$

$$\sum_{k=1}^{K} \lambda_{t,k}^* \ln \lambda_{t,k}^* - \sum_{k=1}^{K} \lambda_{t,k}^* \ln[\lambda_{t,k}^* \kappa_t + \lambda_{t,k}(1-\kappa_t)] \text{ (a.s.).} \tag{21}$$

By virtue of Lemma 1, the series $\sum(E_t \ln \kappa_{t+1} - \ln \kappa_t)$ of non-negative random variables converges a.s., which implies, in view of inequalities (20) and (21), that the sum

$$\sum_{t=0}^{\infty} \sum_{k=1}^{K} [\lambda_{t,k}^* - \lambda_{t,k}^* \kappa_t - \lambda_{t,k}(1-\kappa_t)]^2 = \sum_{t=0}^{\infty} (1-\kappa_t)^2 ||\lambda_t^* - \lambda_t||^2 \tag{22}$$

is finite with probability one. Since $\inf(1-\kappa_t) = \inf \hat{r}_t^2 > 0$ a.s., the fact that the series in (22) converges a.s. yields (12). □

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