

Jumps in Equilibrium Prices and Market Microstructure Noise

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Abstract

Asset prices we observe in the financial markets combine two unobservable components: equilibrium prices and market microstructure noise. In this paper, we study how to tell apart large shifts in underlying equilibrium prices from noise using high frequency data. We propose a new nonparametric test which allows us to asymptotically remove the noise from observable price data and to discover jumps in fundamental asset values. We provide its asymptotic distribution to decide when such jumps occur. In finite samples, our test offers reasonable power for distinguishing between noise and jumps. Empirical evidence found from tick-by-tick stock trades indicates that it is necessary to incorporate the presence of jumps in equilibrium prices.

JEL classification: C12, C14, G14

Key words: jumps, noise, nonparametric tests, high frequency data

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1 Introduction

Asset prices we observe in financial markets are determined by two important, unobservable components. One is equilibrium prices, which reflect demand and supply of assets, and they are also called as efficient prices, incorporating investors' thoughts on market information. The other is market microstructure noise induced by frictions with which actual trades take place. Examples of such frictions are tick size, discrete observation, bid-ask spread, and other trading mechanics.¹ Given that both components are essential ingredients for trading, as also indicated in Black (1986), researchers have sought a better understanding of both and of their interactions. In particular, in recent years, with the availability of databases consisting of observations sampled at ultra-high frequency up to every second, extensive research that takes advantage of such data for better volatility and noise estimation has appeared, and their economic implications have also been investigated in many studies.²

In this paper, we are motivated to question the assumptions imposed by most of the aforementioned studies for log equilibrium prices to follow diffusion processes. Although it is simpler to study this issue under such assumptions, it is widely known in the asset pricing literature that financial markets experience jumps in prices that are too large to be explained by pure diffusion processes, and their presence has been incorporated in numerous theoretical and empirical studies.³ Obviously, one can argue that all the evidence of jumps documented in the previous

¹Other examples include institutional structure, transaction costs, adverse selection due to asymmetric information for different traders, trading size, volume, liquidity, dealer's inventory control, among others. [See O'Hara (1995) and Hasbrouck (2004) and the reference therein.]

²See Andersen, Bollerslev, and Diebold (2003), Bai, Russell, and Tiao (2000), Aït-Sahalia, Mykland, and Zhang (2006), Barndorff-Nielsen and Shephard (2003), Bandi and Russell (2005), Zhang, Mykland, and Aït-Sahalia (2005), and Hansen and Lunde (2006).

³See Bates (1996), Bakshi, Cao, and Chen (1997), Aït-Sahalia (2002), Andersen, Benzoni, and Lund (2002), Pan

asset pricing literature based on discretely sampled data are due to noise and hence, a diffusion assumption for efficient prices would be valid since noise indeed creates discreteness in recorded prices and it is difficult to tell through existing empirical methods if there are fundamental shifts in underlying asset values.⁴ Nonetheless, distinguishing jumps in efficient prices from noise is important, first because, if there were in fact dramatic changes (jumps) in the fundamental values of underlying assets but these were neglected, as noted in various studies, their implications for financial management such as pricing and hedging would be significant. Secondly, we believe that discovering jumps in efficient prices apart from noise and understanding their interactions should give us a better tool for event studies, which we often employ in empirical investigations of market trading behavior.

Specifically, we propose a new empirical test that suggests preprocessing price level data for the purpose of de-noising and makes a distinction between jumps in efficient prices and noise. Assuming that noise has an additive effect on equilibrium prices, we first take local averages of observed prices over an upcoming local window in the preprocessing. This local averaging allows us to asymptotically remove the noise and approximate the true underlying prices. (The device has earlier been studied by Jacod, Li, Mykland, Podolskij, and Vetter (2008) and Podolskij and Vetter (2008) for estimating volatility). Therefore, evidence based on this test becomes about the efficient prices. In order for econometricians to determine the rejection regions for claiming jump arrivals, we offer a limiting distribution of our test statistics. To execute the test, noise variance is needed as an input. We also suggest a noise variance estimator, which is asymptotically immune

(2002), Chernov, Gallant, Ghysels, and Tauchen (2003), Eraker, Johannes, and Polson (2003), Johannes (2004), Barndorff-Nielsen and Shephard (2005a).

⁴Many empirical methods to test jumps in asset prices using high frequency observations do not take into consideration the presence of market microstructure noise. See Barndorff-Nielsen and Shephard (2005a), Aït-Sahalia and Jacod (2006), Mancini (2001), and Lee and Mykland (2006), among others.

to the presence of jumps in efficient prices. Our test is designed to take full advantage of an ultra-high frequency database. Hence, as long as high frequency price data are available for analysis, it can be used to find the behavior of both unobservable price processes and noise processes for any kind of asset price series. In addition, the outcome of our test is robust to model specification, because the suggested procedure is nonparametric.

After presenting asymptotic theories of inference, we discuss finite sample performance using Monte Carlo simulation. We first show that when there are jumps in equilibrium prices, the noise variance can be estimated better by multi-power variations than other existing estimators such as quadratic power variations or bi-power variations studied by Bandi and Russell (2005), Zhang, Mykland, and Aït-Sahalia (2005), and Oomen (2002), among others. Then, we present the size and power properties of our test and show that detectable jumps tend to depend on magnitude of noise variance. When the noise variance level is high, the test tends to detect jumps that are greater in size. For a given jump size, however, we can maximize the power of the test by increasing the frequency of observations.

Finally, we apply our new test of jumps in equilibrium prices and estimation procedure for noise variance to August 2007 IBM stock trade data from the TAQ database. In order for the asymptotic results of theoretical inference to be most effective in data analysis, we use all tick-by-tick data available sampled at the highest frequencies. Noise variance estimates for IBM trades are around 0.01% and found to be greater at opening time (09:30-10:00) and closing time (15:00-16:00) on trading days. Based on our new jump test that takes into account the general form of dependent noise in the market, we strongly reject the null hypothesis of no jump models for equilibrium prices, which suggests evidence in favor of pricing models with jumps.

The rest of the paper is organized as follows. We start in Section 2 by setting up a theoretical

framework for equilibrium prices and specify a model of microstructure noise due to market imperfection. In Section 3, we explain the intuition behind the development of our test and introduce the definition of our test. In Section 4, we discuss the asymptotic behavior of our test and the noise estimator. Section 5 illustrates the finite sample performance of noise estimator and of our test under general assumptions on noise. After our empirical study is discussed in Section 6, we conclude in Section 7. All the proofs are in the Appendix.

2 Theoretical Model

This section sets up a theoretical framework to test the presence of jumps in equilibrium prices, using market price data which include noise from market microstructure. We first fix a complete probability space $(\Omega, \mathcal{F}_t, \mathcal{P})$, where Ω is the set of events in a financial market, $\{\mathcal{F}_t : t \in [0, T]\}$ is right-continuous information filtration for market participants, and \mathcal{P} is a data-generating measure.

We denote as $P(t)$ the unobservable log-equilibrium price at t , in which we test the presence of jumps. Under the null hypothesis, the continuously compounded return $dP(t)$ is represented as

$$dP(t) = \mu(t)dt + \sigma(t)dW(t), \quad (1)$$

where $W(t)$ is an \mathcal{F}_t -adapted standard Brownian Motion, and the drift $\mu(t)$ and diffusion $\sigma(t)$ coefficients are \mathcal{F}_t -adapted random processes, so that the underlying process is an Itô process that has continuous sample paths. Under the alternative hypothesis with the presence of jumps, the return is characterized by a jump diffusion process as

$$dP(t) = \mu(t)dt + \sigma(t)dW(t) + Y(t)dJ(t), \quad (2)$$

where $dJ(t)$ is a jump counting process with a stochastic intensity of $\lambda(t)$ independent of $W(t)$, and

$Y(t)$ is its jump size. The $dJ(t)$ term is an indicator of jump arrival. Jump size $Y(t)$, independent and identically distributed, has its mean $\mu_y(t)$ and standard deviation $\sigma_y(t)$. This $P(t)$ describes the asset price evolution under a perfectly frictionless market, where there is costless trading or an infinitely liquid market.

For simplicity but without loss of generality, we set the drift $\mu(t)$ to 0 and do our econometric analysis, following the Girsanov's Theorem, as in Karatzas and Shreve (1991). It allows us to carry out our econometric analysis under a measure-theoretically equivalent probability space, which shares the same null sets with the original space \mathcal{P} . Analysis with and without the drift gives us asymptotically identical empirical results.⁵ Hence, for now, we consider the null hypothesis as

$$dP(t) = \sigma(t)dW(t), \tag{3}$$

and the alternative hypothesis as

$$dP(t) = \sigma(t)dW(t) + Y(t)dJ(t). \tag{4}$$

Econometricians observe market data for the above process through either quoted or transaction prices under market friction due to physical limits on observing data only at discrete times or to various types of other market noise. The transaction or quote price observed at t_i , denoted as $\tilde{P}(t_i)$ in this paper, is determined by the efficient price $P(t_i)$ as well as market microstructure noise $U(t_i)$. As in most of the empirical and theoretical market microstructure literature including Black (1986) and Stoll (2000), among others, we take a model with additive effect of noise on log equilibrium prices, so that

$$\tilde{P}(t_i) = P(t_i) + U(t_i). \tag{5}$$

Now, we impose the following assumptions on observation times, latent price processes, and noise,

⁵See Mykland and Zhang (2007) for a more detailed justification.

throughout this paper.

Assumption A

A.1: Ultra High Frequency Observation Times

We set the grids \mathcal{G}_n over the fixed time horizon $[0, T]$. Each observation time is set as $t_i = t_{n,i}$ and belongs to $\mathcal{G}_n = \{0 = t_{n,0} < t_{n,1} < \dots < t_{n,n} = T\}$. The distance between two successive observations, $\Delta t_{n,i} = t_{n,i} - t_{n,i-1}$, is not necessarily fixed and can change over time depending on i . We assume

$$\max_{1 \leq i \leq n} |\Delta t_{n,i}| = O_p(n^{-\beta}) \text{ for any } \beta > 0, \quad (6)$$

so that the grid becomes dense in $[0, T]$ as $n \rightarrow \infty$. The subscript n is normally suppressed in our discussion.⁶

A.2: Equilibrium Price Process

The volatility $\sigma(t)$ is *càdlàg* (right continuous with left limit), bounded away from zero and $\max_{t \in [0, T]} \sigma(t) < \infty$. Moreover, for any β and ϵ such that $0 < \epsilon < \beta/2$,

$$\max_{t_i \in \mathcal{G}_n} \left| \int_{t_{i-1}}^{t_i} \sigma(u) dW(u) \right| = O_p(n^{\epsilon - \frac{\beta}{2}}). \quad (7)$$

A.3: Market Microstructure Noise

The noise distribution is given by

$$U(t_i) \sim \mathcal{S}(0, q^2), \quad (8)$$

⁶We use O_p notation throughout this paper to mean that for random vectors $\{X_n\}$ and non-negative random variable $\{d_n\}$, $X_n = O_p(d_n)$ if for each $\epsilon > 0$, there exists a finite constant M_ϵ such that $P(|X_n| > M_\epsilon d_n) < \epsilon$ eventually.

where \mathcal{S} denotes a stationary Gaussian process with its mean 0 and standard deviation q , which is also called as market quality parameter and $U(t_i) = O_p(1)$.⁷

Assumption A.1 implies that the distance between two successive observations can be irregular, which is the usual characteristic of ultra high frequency data, for example, data available in the TAQ database. Assumption A.2 implies that the spot volatility can be stochastic, display jumps, and have high persistence, nonstationarity, and leverage effect. Furthermore, it can depend on the price process $P(t)$. The motivation for imposing Assumption A.2 is to cover most of the continuous-time assets pricing models existing in the literature that incorporate jumps in financial markets. The motivation for imposing Assumption A.3 is to allow a dependent structure for noise by stationarity so that we cover most of the models existing in the market microstructure literature. We call q in Assumption A.3 a market quality parameter to describe how noisy the market is. $q = 0$ is equivalent to a frictionless market where equilibrium prices $P(t)$ can be observed. Thus, q represents the degree of market imperfection or quality of trading exchange. Approximately, if we use a mid point quote as the observed price, we can interpret the magnitude of noise as the difference between the mid-point quote and the corresponding equilibrium price. Its justification can also be found in Hasbrouck (2004) and the reference therein.

In this paper, we perform our analysis with a locally fixed market quality parameter q . We carry out our formal study with this simplified assumption on the noise parameter as a first step to theoretically refine our understanding on the impact of the noise. This assumption can be

⁷Barndorff-Nielsen and Shephard (2005b), on the other hand, assume a component model for noise with $U(t_i) = U_1(t_i) + U_2(t_i)$ where $U_1(t_i) = o_p(\sqrt{\Delta t_i})$ and $U_2(t_i) = O_p(1)$. Since $U_2(t_i)$ will dominate as $\Delta t_i \rightarrow 0$, this assumption is asymptotically equivalent to ours. Gloter and Jacod (2000) allows the variance of noise to decrease with n , hence, $U(t_i) = O_p(\sqrt{\Delta t_i})$.

easily relaxed to accommodate the seasonality or time-variation of the noise process. We further study more general cases using simulation in a later section.

3 Intuition and Definition of Test

This section explains the intuition behind the development of our test and its definition. In order to understand the interaction between jump in equilibrium prices and microstructure noise, we first consider the null hypothesis, where there is no jump in equilibrium price process as in Equation (3) and we observe its data with noise. If econometricians calculate the log returns using recorded prices at high frequency, as the distance between two successive observation time stamps gets smaller so that our observation time becomes closer to continuous time: $\max_{0 \leq i \leq n} |\Delta_{n,i}| \rightarrow 0$, the statistics based on these observed log returns will be about noise, not about the latent price process. This is because noise, for example bid-ask spread, does not disappear in observed prices, even if $\max_{0 \leq i \leq n} |\Delta_{n,i}| \rightarrow 0$, but the effect of the Brownian motion process disappears theoretically. In other words, noise plays a dominant role at such highest frequencies.⁸

Now, how about the alternative hypothesis where there are jumps in equilibrium prices as in Equation (4) and we observe data from it with noise? As before, the effect of the Brownian motion disappears, as $\max_{0 \leq i \leq n} |\Delta t_{n,i}| \rightarrow 0$. But this time, two kinds of discreteness remains in the observed returns. One is noise as explained above, and the other is jump in latent equilibrium prices. Even if $\max_{0 \leq i \leq n} |\Delta_{n,i}| \rightarrow 0$, these two will not disappear both theoretically and empirically. This is where the distinction becomes difficult because when we have very large changes in

⁸This is noted in Zhang, Mykland, and Aït-Sahalia (2005) and Bandi and Russell (2005), suggesting not using most frequently observed returns but using less frequently observed returns in order to make better volatility $\sigma(t)$ estimation. They also offer optimal sampling frequency for sample selection. But these studies assume that there is no jump in equilibrium prices.

observed prices, this could be due to noise or to jumps in efficient prices.

In order to tell apart jumps in equilibrium prices from noise, we suggest preprocessing the raw price level data. Instead of using observed prices directly for return calculation, we first average observed prices over an upcoming block of size M . This technique of averaging observed prices with an appropriate M allows us to asymptotically remove the noise from the price data which are contaminated by the noise while keeping information about underlying prices.⁹ Formally, we write the preprocessing procedure and the test statistic for jumps in equilibrium prices as in Definition 1.

Definition 1. *Let M be the block size such that $M \rightarrow \infty$ as $n \rightarrow \infty$, and $M = O_p(n^\gamma)$ with $0 < 3\gamma < \beta - \delta$ for some positive δ . The preprocessed price for de-noising, $\widehat{P}(t_j)$, is the averaged log price over the block of size M such that $\widehat{P}(t_j) = \frac{1}{M} \sum_{i=j}^{j+M-1} \widetilde{P}(t_i)$, where $\widetilde{P}(t_i)$ is the log price observed at time t_i . The statistic \mathcal{L}_j , to test the presence of jumps in equilibrium price between t_{j+M} to t_j , is defined as*

$$\mathcal{L}_j \equiv \widehat{P}(t_{j+M}) - \widehat{P}(t_j) \tag{9}$$

with the observation time $t_j \in \mathcal{G}_n$ for all j .

4 Theory of Inference for Equilibrium Price with Noise

This section explains how the preprocessing of local averaging prices can asymptotically remove noise. Also, we discuss how to estimate noise variance when there are jumps in equilibrium prices.

⁹This pre-averaging technique has been proposed for volatility estimation for diffusion processes in the presence of noise. [See Jacod, Li, Mykland, Podolskij, and Vetter (2008) and the references therein.]

4.1 Asymptotic Behavior of Test

In this subsection, we discuss the asymptotic behavior of our test statistic and how to set up the rejection region to detect jumps in equilibrium prices. We first study in Lemma 1 the asymptotic behavior of changes in averaged log prices, which converge to zero under the null hypothesis of the no jump model. This specifically states that its limiting distribution is Gaussian, given that the noise process is Gaussian as well.

Lemma 1. *If there is no jump in efficient prices under the null hypothesis as in Equation (3), for a given j and $M \rightarrow \infty$ as $n \rightarrow \infty$ such that $M = O_p(n^\gamma)$ with $0 < 3\gamma < \beta - \delta$ for some positive δ ,*

$$|\widehat{P}(t_{j+M}) - \widehat{P}(t_j)| \xrightarrow{P} 0, \quad (10)$$

as $n \rightarrow \infty$. More precisely, if we set

$$\mathcal{X}(j) = \frac{1}{\sqrt{2q}} \frac{1}{\sqrt{M}} \left(\sum_{i=j+M}^{j+2M-1} U(t_i) - \sum_{i=j}^{j+M-1} U(t_i) \right), \quad (11)$$

then, $\mathcal{X}(j)$ is a stationary Gaussian process with $E\mathcal{X}(j) = 0$ and $E\mathcal{X}^2(j) = 1$ for all j , and for some $\eta > 0$,

$$\sup_j |\sqrt{M} \left(\widehat{P}(t_{j+M}) - \widehat{P}(t_j) \right) - \sqrt{2q}\mathcal{X}(j)| = O_p(n^{-\eta}). \quad (12)$$

In particular,

$$\sqrt{M} \left(\widehat{P}(t_{j+M}) - \widehat{P}(t_j) \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 2q^2), \quad (13)$$

as $n \rightarrow \infty$.

Given the Lemma 1, the following Lemma 2 suggests that in the presence of a stationary

Gaussian noise process, we can use our test to detect jump arrivals in efficient prices by considering the limiting distribution of its extremes.

Lemma 2. *Let $\mathcal{X}(j)$ be a stationary Gaussian process, so that $E\mathcal{X}(j) = 0$ and $E\mathcal{X}^2(j) = 1$ for all $j = 0, 1, \dots, n$. Furthermore, its covariance sequence $\rho_k = E\mathcal{X}(0)\mathcal{X}(k)$ with $\sum_{k=1}^{\infty} \rho_k^2 < \infty$, or $\lim_{k \rightarrow \infty} \rho_k \log k = 0$. Then, as $n \rightarrow \infty$,*

$$\frac{\max_{0 \leq j \leq n} |\mathcal{X}(j)| - A_n}{B_n} \xrightarrow{\mathcal{D}} \xi, \quad (14)$$

where ξ follows a standard Gumbel distribution whose cumulative distribution function $P(\xi \leq x) = \exp(-e^{-x})$ ¹⁰,

$$A_n = (2 \log n)^{1/2} - \frac{\log \pi + \log(\log n)}{2(2 \log n)^{1/2}} \text{ and } B_n = \frac{1}{(2 \log n)^{1/2}}. \quad (15)$$

Specifically, the above Lemma 2 implies that in the presence of dependent noise, one can find maximums for the absolute differences in averages of log prices sampled at the highest frequencies available and use the Gumbel variable for the purpose of testing. A similar lemma without the general stationarity assumption on the noise process was used in Lee and Mykland (2006), which does not take the presence of noise into account for jump detection. We state this more formally in Theorem 1 as follows.

Theorem 1. *Let \mathcal{L}_j be as in Definition 1 and Assumption A is satisfied. Suppose there are no jumps in equilibrium price processes in $[0, T]$ under the null hypothesis as in Equation (3) and*

¹⁰This standard Gumbel distribution has its probability density function $P(\xi = x) = e^{-x} \exp(-e^{-x})$ with the mean Euler-Mascheroni constant approximately 0.577 and standard deviation $\pi/\sqrt{6} \approx 1.2825$. The generalized version of Gumbel distribution is called Fisher-Tippett distribution and also known as log-Weibull distribution.

observed prices are from Equation (5). Then, as $n \rightarrow \infty$,

$$\hat{\xi}_n = B_n^{-1} \left(\frac{\sqrt{M}}{\sqrt{2q}} \max_{t_i \in \mathcal{G}_n} |\mathcal{L}_j| - A_n \right) \xrightarrow{\mathcal{D}} \xi, \quad (16)$$

where ξ follows the standard Gumbel distribution in (14) and A_n and B_n are as in (15). In other words,

$$\max_{t_j \in \mathcal{G}_n} |\mathcal{L}_j| \stackrel{\mathcal{D}}{\approx} \frac{\sqrt{2q}}{\sqrt{M}} (B_n \xi + A_n). \quad (17)$$

This Theorem 1 provides us with the threshold to reject the null hypothesis of no jumps in equilibrium prices. For example, if we choose the significance level at 1%, then the threshold becomes the 99th percentile of the Gumbel distribution after relocating and scaling. Now, we study in the following Theorem 2 how this test would react to jumps in equilibrium prices.

Theorem 2. *Let \mathcal{L}_j be as in Definition 1 and Assumption 1 is satisfied. Also suppose that*

$\frac{A_n}{\sqrt{M}} \rightarrow 0$. If there are jumps at times $\tau_k \in [0, T]$ for a finite k , then,

$$\max_j |\mathcal{L}_j| \approx \max_k |Y(\tau_k)|, \quad (18)$$

where $Y(\tau_k)$ is the jump size at the jump time τ_k .

As stated in Theorem 2, the test statistic would be close to the maximum jump size over the interval within which we would like to test the jumps in equilibrium prices. Therefore, this test will detect the presence of jumps (which can be single or multiple) in the interval under consideration.

4.2 Consistent Estimation of Noise Variance in the Presence of Jumps

One can notice in Theorem 1 that in order to apply our test, we need a consistent estimator for q . The following proposition suggests using multi-power variation to estimate it, regardless of the

presence of jumps in efficient prices.

Proposition 1. *Suppose the noise follows a k -dependent stationary Gaussian process with $0 \leq k < \infty$. Its variance estimator over the interval $[0, T]$, \widehat{Q} , is defined as*

$$\widehat{Q} \equiv \left[\frac{1}{(n - 2g + 2)} \frac{1}{c_r^g} \sum_{j=2g-1}^n \left(\prod_{m=1}^g |\widetilde{P}(t_{j-k(2m-2)}) - \widetilde{P}(t_{j-k(2m-1)})|^r \right) \right]^{1/gr}, \quad (19)$$

for any $r > 0$ and any integer $g \geq 1$, and c_r is defined as in

$$c_r = E|u|^r = \pi^{-1/2} 2^{r/2} \Gamma\left(\frac{r+1}{2}\right), \quad (20)$$

where u is a standard normal variable. Then, regardless of the presence of jump, as Δt goes to 0,

$$\widehat{Q} \xrightarrow{P} \sqrt{2}q. \quad (21)$$

Therefore, q can be estimated by $\hat{q} = \widehat{Q}/\sqrt{2}$.

Under both hypotheses on the presence of jumps, the realized (second or higher lagged, depending on the order of autocorrelation of the noise process) multi-power variation estimator does not converge in probability to the integrated variance itself. Rather, it converges to a quantity that explains variance of noise. In this paper, we assume that q is locally constant and can be estimated by this estimator and plugged into the calculation of $\hat{\xi}_n$.

5 Simulation for Finite Sample Behavior

Our asymptotic arguments require infinite sampling, which is not completely achieved in practice, though we have enough high frequency data available. In this section, we examine by Monte Carlo

simulation the finite sample performance of our test in terms of both size and power of the test. We first show the superior performance of multipower variation as a noise variance estimator.

We consider various levels of market quality parameter q reported in previous empirical studies, and present the impact of magnitude of noise. For all series generation, we used the Euler-Maruyama Stochastic Differential Equation (SDE) discretization scheme in Kloeden and Platen (1992), an explicit order 0.5 strong and order 1.0 weak scheme. We discard the burn-in period – the first part of the whole series – to avoid the starting value effect every time we generate each series. As shown, overall simulation results consistently support our theory.

5.1 Performance of Noise Variance Estimator

The limiting distribution of our test depends on the performance of the noise variance estimator, for which we suggest using multi-power variation. As a nonparametric estimator for noise variance, the quadratic variation has been suggested in Zhang, Mykland, and Aït-Sahalia (2005) and Bandi and Russell (2005), among others, assuming that there are no jumps in efficient price processes. In this subsection, we study by simulation how the quadratic variation (QV) as a noise variance estimator performs in the the presence of jumps in efficient prices. We also compare this to the performance of bi-power variation (BPV) and multi-power variation (MPV).

We simulate 500 series of efficient prices from a jump diffusion process over a day with 5 second frequency for both Figure 1 and Table 1. We set the market quality parameter q at 0.01%.¹¹ The jump intensity is set at 5% and 10% per year, and we consider two jump size standard deviations σ_y at 3 and 5 times σ . $U(t_i)$ is assumed to be normal with its standard deviation q . In Figure 1, we show the noise variance estimates, according to QV, BPV, and three kinds of MPV with

¹¹We also perform the same analysis for q at different levels such as $q = 0.001\%$, 0.01% , and 0.1% and reach similar results.

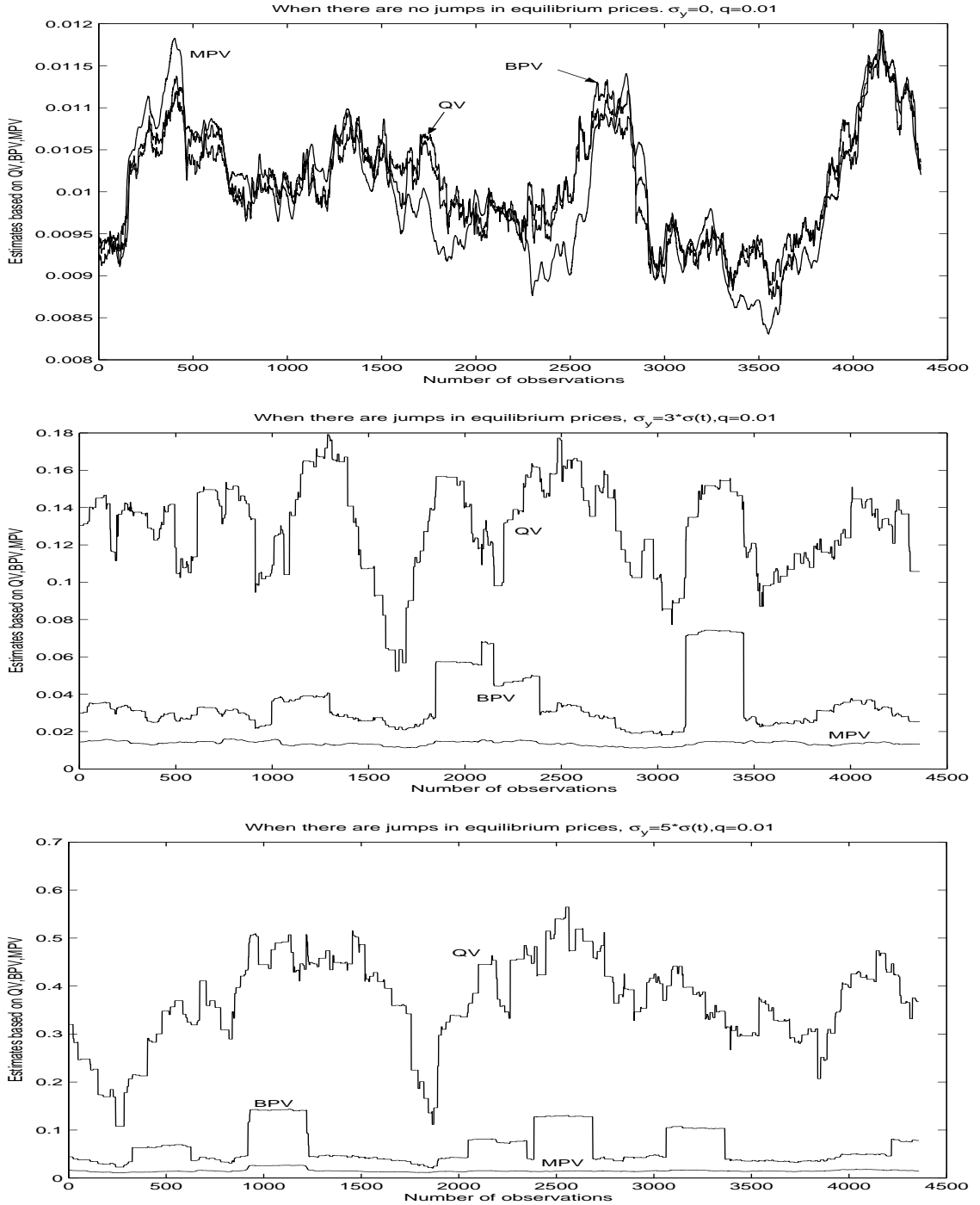
Table 1: **RMSEs of Noise Variance Estimator**[†]

σ_y	$1 \times \sigma$	$2 \times \sigma$	$3 \times \sigma$	$4 \times \sigma$	$5 \times \sigma$
RMSE	$\lambda = 5\%$ and $\sigma = 30\%$				
QV	5.5956e-005	0.0023	0.00141	0.0472	0.1190
BPV	7.8630e-006	1.4791e-004	8.0697e-004	0.0026	0.0064
MPV(6,1/3)	1.9458e-006	1.0359e-005	2.4583e-005	4.4288e-005	6.5475e-005
MPV(8,1/4)	1.6430e-006	7.5720e-006	1.6280e-005	2.6530e-005	3.7450e-005
MPV(10,1/5)	1.4909e-006	6.2853e-006	1.2827e-005	1.9635e-005	2.7141e-005
RMSE	$\lambda = 10\%$ and $\sigma = 30\%$				
QV	1.5567e-004	0.0051	0.0296	0.0975	0.2438
BPV	2.9241e-005	5.9403e-004	0.0031	0.0103	0.0248
MPV(6,1/3)	7.7429e-006	4.9734e-005	1.2660e-004	2.4127e-004	3.8672e-004
MPV(8,1/4)	6.5969e-006	3.6343e-005	8.2744e-005	1.4336e-004	2.1484e-004
MPV(10,1/5)	5.9916e-006	3.0225e-005	6.4407e-005	1.0623e-004	1.5276e-004

[†] This table presents the Root Mean Squared Error of the noise variance estimators. Estimators based on power variation (PV), bipower variation (BPV), and three kinds of multi-power variations (MPV) are considered. g and r denotes the number of products and power of absolute values used in the MPVs. Five different levels of jump sizes relative to volatility level have been considered.

$r = \frac{1}{5}, \frac{1}{4},$ and $\frac{1}{3}$ and $g = 10, 8,$ and $6,$ respectively. The upper, middle, and lower panels of Figure 1 present results for cases with diffusion price processes without jumps and jump diffusion price processes with σ_y at 3 and 5 times σ , respectively. Table 1 explicitly shows numerical values for Root Mean Squared Error of the three estimators for noise variance. Simulation evidence in both Figure 1 and Table 1 shows that if there are jumps, the bias gets increased most in quadratic variation and least in multi-power variation. Hence, we conclude that multi-power variation is most desirable as an estimator for q for our purpose.

Figure 1: Noise Variance Estimates Based on Power Variations



The upper, middle, and lower panels include the time series of the noise variance estimates calculated according to estimators based on quadratic variation (QV), bi-power variation (BPV), and multi-power variation (MPV). The model for the upper panel is $d \log S(t) = \sigma(t) dW(t)$ and the models used for the middle and lower panel are $d \log S(t) = \sigma(t) dW(t) + Y dJ(t)$ where $W(t)$ is a Brownian motion process, $J(t)$ is a Poisson-type counting process with its intensity, and Y is a jump size with its standard deviation σ_y . Constant volatility is set at $\sigma(t)$ at 30%. q is chosen at 0.01%, which is the averaged estimates for hourly q we found for IBM stocks (see Section 6 for more details).

Table 2: **Size Properties of Test**[†]

Nominal size of test = 0.01				
Independent Noise				
Frequency	n	$q = 0.001\%$	$q = 0.01\%$	$q = 0.1\%$
5 second	720	0.0067	0.0167	0.0200
3 second	1200	0.0033	0.0100	0.0100
2 second	1800	0.0033	0.0033	0.0067
1 second	3600	0.0033	0.0033	0.0040
Dependent Noise				
Frequency	n	$q = 0.001\%$	$q = 0.01\%$	$q = 0.1\%$
5 second	720	0.0040	0.0133	0.0130
3 second	1200	0.0033	0.0100	0.0100
2 second	1800	0.0033	0.0033	0.0067
1 second	3600	0.0033	0.0100	0.0033

[†] This table presents the size of our test under both independent and dependent noise. For independent noise, we generate $U(t_i)$ from a normal distribution, $\mathcal{N}(0, q^2)$. For dependent noise, we generate $U(t_i)$ from a model studied by Engle and Sun (2006), who estimate the model using tick-by-tick data on an individual equity, and we use their parameter estimates reported as significant at 5%. Specifically, we simulate noise series from $U(t_i) = \theta_0 \int_{t_{i-1}}^{t_i} \sigma dW(s) + \theta_1 \int_{t_{i-2}}^{t_i} \sigma dW(s) + X(t_i)$, where $X(t_i)$ is a normal variable with mean 0 and standard deviation q and θ_0 and θ_1 are 0.0861 and 0.06, respectively. Instead of using their q estimate, we use the market quality parameter q 's chosen at three different levels for both independent and dependent noise, following Ait-Sahalia, Mykland, and Zhang (2005) and Bandi and Russell (2005). The equilibrium prices are generated from a diffusion process $dP(t) = \sigma(t)dW(t)$ with a fixed $\sigma(t)$ at 20% per a year. The significance level α used is 1%. n is number of observations over one trading hour, in this study chosen at $n = 720, 1200, 1800, 3600$, which are equivalent to sample observations at every 5 second, 3 second, 2 second, and 1 second frequency. For $q = 0.001\%$, $q = 0.01\%$, and $q = 0.1\%$, we choose $M = A_n^2/8, A_n^2/4$, and A_n^2 , respectively.

5.2 Independent and Dependent Noise Specifications

In this subsection, we discuss specifications for both independent and dependent noise. For independent noise, we generate $U(t_i)$ from a normal distribution, $\mathcal{N}(0, q^2)$. However, as discussed in Engle and Sun (2006), a more realistic noise model should incorporate its various characteristic such as stationarity and cross-correlation between noise and equilibrium prices. Because the information flow affect both components of transactions, for example, it is likely that market microstructure noise is correlated with market equilibrium price changes. Price determination by adverse selection under asymmetric information can also create various type of dependence [see O'Hara (1995)].

In order to incorporate such general properties of noise, we use the general noise model employed by Engle and Sun (2006). We use their parameter estimates for an individual U.S. equity reported as significant at 5%. Specifically, the cross-correlated model which we employ for our simulation, relating current and lagged innovation in equilibrium prices to noise, is

$$U(t_i) = \theta_0 \int_{t_{i-1}}^{t_i} \sigma(t) dW(s) + \theta_1 \int_{t_{i-2}}^{t_i} \sigma(t) dW(s) + X(t_i), \quad (22)$$

where $X(t_i)$ is a normal variable with standard deviation q and θ_0 and θ_1 are set at 0.0861 and 0.06, respectively. Though they also have estimates for q , we consider q at three different levels in order to see the impact of noise magnitude on the performance of our test. These q 's are chosen around the estimates reported by Ait-Sahalia, Mykland, and Zhang (2005) and Bandi and Russell (2005). To study size and power properties in the following subsections, we add these two types of noise both under the null and alternative hypotheses for price processes.

5.3 Size of Test

To calculate size, we generate the equilibrium prices from a diffusion process $dP(t) = \sigma(t)dW(t)$ with a fixed $\sigma(t)$ at 20% per a year. The significance level α used is 1%. n is number of observations over one trading hour, in this study chosen at $n = 720, 1200, 1800, 3600$, which are equivalent to sample observations at every 5 second, 3 second, 2 second, and 1 second intervals. The number of simulations is 300. It is important in application of our test to choose proper block size M . This simulation study shows that users can choose block size M using a function $M = f(A_n^2, q)$. For Table 1, $q = 0.001\%$, $q = 0.01\%$, and $q = 0.1\%$, we choose $M = A_n^2/8, A_n^2/4$, and A_n^2 , respectively.

We report in Table 2 the probability of rejecting the null hypothesis of no jump in price processes, when there is actually no jump. In the empirical applications using ultra high frequency data, it is important to check first whether any test detects the presence of jumps spuriously and does not detect microstructure noise as jumps, because as explained in our introduction and in the intuition behind our test, asymptotically, both noise and jumps can be regarded similarly in that both of them are $O_p(1)$. As long as M is chosen properly, our test does not present spurious detection problems.

5.4 Power of Test

In order to examine the power of the test, the equilibrium prices are generated from a jump diffusion process $dP(t) = \sigma(t)dW(t) + Y(t)dJ(t)$ with a fixed volatility $\sigma(t)$ at 20% per year and a standard deviation σ_y of jump size distribution relative to volatility level. The significance level α used for detection is 1%, and we choose the same block size M we choose in Table 2 for size to be close to the significance level under the null hypothesis. Table 3 and 4 includes results

Table 3: **Power of Test under Independent Noise with Finite Variance** $O_p(1)^\dagger$

Market quality parameter ($q = 0.001\%$)				
Jump Size σ_y relative to σ	0.06σ	0.07σ	0.08σ	0.09σ
5 second (720)	0.1533	0.4400	0.8200	0.9767
3 second (1200)	0.5133	0.9033	1.000	1.000
2 second (1800)	0.8768	1.000	1.000	1.000
1 second (3600)	1.000	1.000	1.000	1.000
Market quality parameter ($q = 0.01\%$)				
Jump Size σ_y relative to σ	0.07σ	0.08σ	0.09σ	0.10σ
5 second (720)	0.3100	0.5833	0.8400	0.9600
3 second (1200)	0.3667	0.7067	0.9400	1.0000
2 second (1800)	0.3933	0.8267	0.9867	1.0000
1 second (3600)	0.9633	1.0000	1.0000	1.0000
Market quality parameter ($q = 0.1\%$)				
Jump Size σ_y relative to σ	0.10σ	0.12σ	0.14σ	0.16σ
5 second (720)	0.3460	0.6380	0.9040	0.9740
3 second (1200)	0.4600	0.8440	0.9840	0.9980
2 second (1800)	0.5480	0.9200	1.0000	1.0000
1 second (3600)	0.7840	0.9980	1.0000	1.0000

[†] This table reports the finite sample performance of our test in terms of detecting power for jumps in equilibrium prices in the presence of independent noise $U(t_i)$ with finite variance. Noise are generated from a normal distribution, $\mathcal{N}(0, q^2)$. The market quality parameter q 's are chosen at various levels around values shown in Ait-Sahalia, Mykland, and Zhang (2005) and Bandi and Russell (2005) based on U.S. equity markets. The equilibrium prices are generated from a jump diffusion process $dP(t) = \sigma(t)dW(t) + Y(t)dJ(t)$. The number of simulations was 300. We consider fixed $\sigma(t)$ at 20% per a year. σ_y denotes the standard deviation of jump size distribution, and we choose the levels relative to volatility level σ of the underlying price process. The significance level α used is 1%. We use the same M 's as in Table 2.

Table 4: **Power of Test under Dependent Noise**[†]

Market quality parameter ($q = 0.001\%$)				
Jump Size σ_y relative to σ	0.06σ	0.07σ	0.08σ	0.09σ
5 second (720)	0.1357	0.4100	0.7933	0.9600
3 second (1200)	0.4967	0.9200	0.9667	1.000
2 second (1800)	0.8733	1.000	1.000	1.000
1 second (3600)	1.0000	1.000	1.000	1.000
Market quality parameter ($q = 0.01\%$)				
Jump Size σ_y relative to σ	0.07σ	0.08σ	0.09σ	0.10σ
5 second (720)	0.2967	0.6000	0.8700	0.9700
3 second (1200)	0.3400	0.7833	0.9633	0.9967
2 second (1800)	0.5067	0.9067	1.0000	1.0000
1 second (3600)	0.9767	1.0000	1.0000	1.0000
Market quality parameter ($q = 0.1\%$)				
Jump Size σ_y relative to σ	0.10σ	0.12σ	0.14σ	0.16σ
5 second (720)	0.2900	0.6433	0.8867	0.9700
3 second (1200)	0.4667	0.8367	0.9800	1.0000
2 second (1800)	0.5433	0.9267	1.0000	1.0000
1 second (3600)	0.7933	1.0000	1.0000	1.0000

[†] This table reports performance of our test for jumps in equilibrium prices in the presence of noise $U(t_i)$ generated from the dependent model studied by Engle and Sun (2006). They estimated the model using tick-by-tick data on randomly picked U.S. individual equities, and we use their parameter estimates reported as significant at 5%. In particular, we simulate noise series from $U(t_i) = \theta_0 \int_{t_{i-1}}^{t_i} \sigma dW(s) + \theta_1 \int_{t_{i-2}}^{t_i} \sigma dW(s) + X(t_i)$, where $X(t_i)$ is a normal variable with mean 0 and variance q , and θ_0 and θ_1 are set at their estimates, which are 0.0861 and 0.06, respectively. The equilibrium prices are generated from a jump diffusion process $dP(t) = \sigma(t)dW(t) + Y(t)dJ(t)$. The number of simulations was 300. We both consider fixed $\sigma(t)$ at 20% per a year. σ_y in the table denotes the standard deviation of the jump size distribution, and we choose the levels relative to volatility level σ of the underlying price process. The significance level α used for detection is 1%. We use the same M 's as in Table 2.

for independent noise with finite variation and dependent noise, respectively, as specified in the previous subsection. The overall results regarding the power of the test indicate that detectable jumps in equilibrium prices depends on noise level. If the magnitude of noise is greater, detectable jump sizes in equilibrium prices are greater, and hence, the detecting power for small sized jumps gets decreased. And as in our asymptotic argument in Lemma 2, dependence through stationarity does not appear to lower the power of the test. However, increasing frequency helps to improve it.

6 Empirical Analysis for IBM Stock Trades

We apply our new test of jumps in equilibrium prices and an estimation procedure for noise variance to actual stock trades. In order to make our asymptotic result most effective in our analysis, it is best using tick-by-tick transaction data sampled at the highest frequency.

6.1 Data

Data are collected from the TAQ database, and we only consider transactions on the New York Stock Exchange (NYSE) to be consistent in terms of trading mechanism for all trades under investigation. The sample period is August in 2007. Due to interrupted trading in the NYSE overnight, all trades before 9:30am or after 4:00pm are discarded. We also exclude the first trade after 9:30am for each trading day, which is the usual way of avoiding the overnight effect [see Engle and Sun (2006), for example]. For trades that happen at the same time and hence have multiple prices at one time, we take averaged observed price, which removes all transactions with zero duration. We discard all recording errors such as zero prices (if any). In order to eliminate *bounce-back* type data errors as noted in Aït-Sahalia, Mykland, and Zhang (2006), we remove

obvious outliers and only keep data with log returns within the range of its 7 standard deviation around its mean. Therefore, the total number of tick-by-tick observations used in our analysis is 167,595.

In Table 5, we include summary statistics for the number of trades, durations in seconds, log returns in basis points, and prices in dollars. We have 23 trading days for August 2007 and 6.5 trading hours for each trading day. We take the time horizon for our test T to be an hour after 10am till 4pm and 30 minutes for opening half hours every day. Columns in Table 5, for example 11-12, include information about trades after 11am (inclusive) and before 12pm (exclusive). Though there is seasonality of number of trades, we have enough number of trades within all horizons for our asymptotic results to be effective. Durations between two consecutive trades Δt_i have averages below 5 second which makes our simulation study in the previous section informative. $\Delta \log \tilde{P}(t_i)$ is the first difference of observed log prices sampled at the highest frequencies available.

6.2 Empirical Results

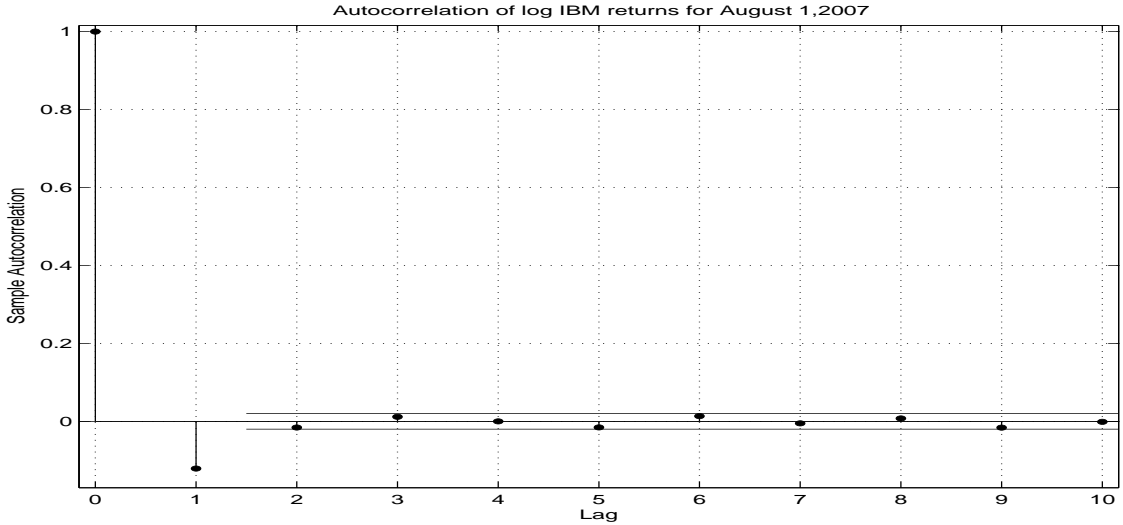
As noted in both Section 2 and 3, our test needs the input of market quality parameter q , which is the standard deviation of the market microstructure noise process in Equation (8). For the application of the noise variance estimator in Equation (19), we need to determine k for the serial dependence of noise. Given that observed log returns sampled at the highest frequency would give us information about noise dependence, we first calculate their serial correlation functions for every horizon and apply the usual significance test at 5%, as in Figure 2, to determine the number of dependent lags. Figure 2 shows one representative sample autocorrelation function of most frequently sampled log returns on August 1, 2007. The two solid horizontal lines in the graph for the lags of 2 and beyond make the 95% confidence band. If the dot is inside the band, it means

that the corresponding lag is insignificant. We obtain similar patterns in the autocorrelations for other time horizons as well.

Using k 's selected according to autocorrelation functions, we estimate the noise variance and report its summary statistics in Table 6. Results indicate that \hat{q} 's are greater in the opening hours such as 9:30-10 and closing hours of 15-16, though the magnitudes are similar in other hours. Based on the estimates of q found in each horizon, we also calculate $\hat{\xi}_n$ in Equation (16). Using the significance level of 1%, we count how many times we reject the null hypothesis of no jumps in equilibrium prices and determine whether there are jumps in each corresponding trading hour of the day, and calculate the annualized λ estimates. With the significance level of 1%, the threshold for ξ is 4.6001. We found that likelihoods of jump arrivals are in the similar magnitude across different trading hours a day, when we take market noise into consideration. As in simulation section, we choose the size of blocks over which we take the averaging of prices. We use the same method we used in the simulation section, in order to ensure that we do not have over or under detection problems.

In Figure 3, we also graph the empirical distribution of IBM trade noise variance estimates \hat{q} 's. For each trading day, we have 7 different time horizons and we calculate the time-varying noise variance by separately estimating the quantities over different time horizons. Different colors for each bin in Figure 3 indicate different trading hours. In particular, dark blue, regular blue, light blue, green, yellow, orange, and red represent trading hours of 09:30-10, 10-11, 11-12, 12-13, 13-14, 14-15, and 15-16, respectively. As also reported in Table 6, estimates of \hat{q} are centered around 0.01%. Figure 3 also graphically shows that we tend to have higher values in the 9:30-10 interval with the dark blue bars and lower values in the 11-12 and 12-13 intervals with light blue and green bars.

Figure 2: **Sample Autocorrelation of IBM Stock Returns during August 2007**



The figure includes a representative sample autocorrelation function of returns from IBM stocks traded on the New York Stock Exchange (NYSE). This graph is for August 1, 2007 and we have qualitatively similar figures for other days and hours during the whole month of August 2007. We calculate this sample autocorrelation of returns sampled at the highest frequency and employ the significant lag number for k in a multi-power variation calculation in Equation (19). The two solid horizontal lines in this graph for the lags of 2 and beyond make the 95% confidence band. If the dot is inside the band, it means that the corresponding lag is insignificant.

Finally, in Figure 4, we compare graphically the asymptotic distribution and empirical distribution of $\hat{\xi}_n$. The asymptotic distribution is graphed with simulated data under the null hypothesis of no jump in equilibrium according to Equation (16) in Theorem 1. The left panel in Figure 4 includes the histogram of simulated ξ , which we would expect to see from data if there is no jumps in equilibrium prices. The number of simulations is 300. The right panel includes the histogram of $\hat{\xi}_n$ using our sample. As can be seen, we have different ranges in the distribution, which indicates strong rejection of models for no jump in equilibrium prices. Therefore, one could conclude from this case study that models with jumps in the underlying prices can capture better intra-day dynamics of asset market behavior.

Table 5: Descriptive Statistics of IBM Stock Trades during August 2007[†]

Trading Hour	9:30-10	10-11	11-12	12-13	13-14	14-15	15-16
Min No. of trades	494	884	622	553	556	626	1052
Max No. of trades	1107	1783	1510	1750	1505	1814	2323
Ave No. of trades	677	1184	1024	875	848	1056	1622
Std No. of trades	147	274	257	268	228	283	352
Min Δt_i (second)	1	1	1	1	1	1	1
Max Δt_i (second)	35	60	59	61	63	69	41
Ave Δt_i (second)	2.701	3.180	3.729	4.397	4.512	3.646	2.324
Std Δt_i (second)	2.865	3.636	4.253	5.236	5.313	4.245	2.433
Min $\Delta \log \tilde{P}(t_i)$ (1.0e-004)	-0.17	-0.15	-0.14	-0.12	-0.15	-0.17	-0.18
Max $\Delta \log \tilde{P}(t_i)$ (1.0e-004)	0.18	0.16	0.14	0.12	0.17	0.13	0.16
Ave $\Delta \log \tilde{P}(t_i)$ (1.0e-004)	0.0105	0.0030	0.0019	-0.0018	-0.0013	-0.0020	-0.0033
Std $\Delta \log \tilde{P}(t_i)$ (1.0e-004)	0.0278	0.0197	0.0168	0.0165	0.0168	0.0172	0.0175
Min $\tilde{P}(t_i)$	108.76	108.08	108.18	107.44	106.94	107.50	106.58
Max $\tilde{P}(t_i)$	116.27	116.47	116.63	116.94	116.76	116.93	117.34
Ave $\tilde{P}(t_i)$	112.11	112.21	112.32	112.45	112.38	112.32	112.30
Std $\tilde{P}(t_i)$	1.002	1.001	1.001	1.001	1.001	1.001	1.002

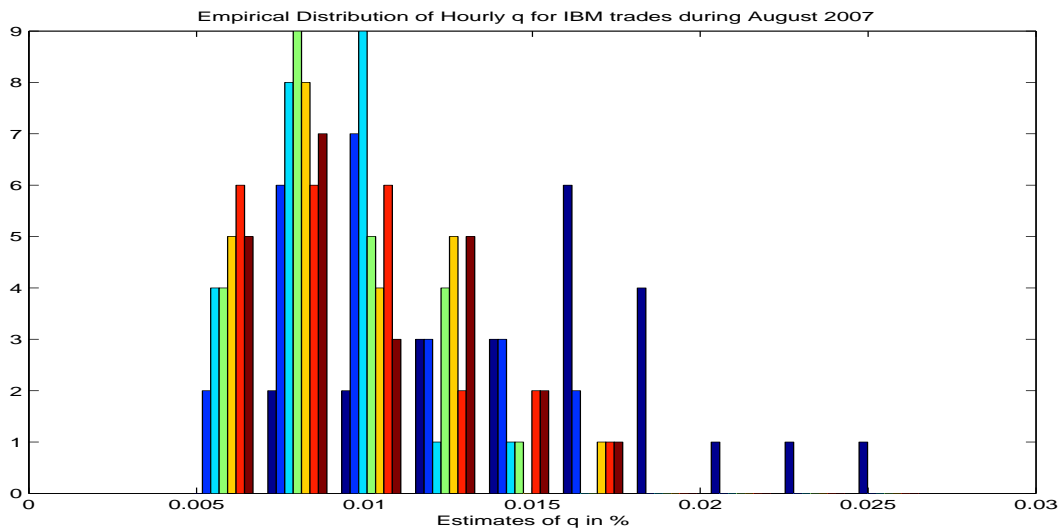
[†] The table contains summary statistics for the number of trades, durations in seconds, log returns in basis points, and prices in dollars for IBM stock during the whole month of August 2007. The total number of tick-by-tick observations used is 167,595. Data are collected from the TAQ database and for transactions on the New York Stock Exchange (NYSE). All trades before 9:30am or after 4pm and the first trade after 9:30am are discarded due to NYSE trading hours and mechanism. Each trading hour column, for example 11-12, includes information about trades after 11am (inclusive) and before 12pm(exclusive). All trades that have multiple prices at the same time are counted once and the averaged price over the multiple trades is used.

Table 6: **Empirical Evidence on IBM Stock Trades during August 2007[†]**

Trading Hour	9:30-10	10-11	11-12	12-13	13-14	14-15	15-16
Min \hat{q} (%)	0.0085	0.0066	0.0050	0.0054	0.0056	0.0051	0.0047
Max \hat{q} (%)	0.0267	0.0173	0.0143	0.0154	0.0166	0.0176	0.0162
Ave \hat{q} (%)	0.0159	0.0108	0.0089	0.0090	0.0093	0.0094	0.0096
Std \hat{q} (%)	0.0047	0.0030	0.0023	0.0025	0.0027	0.0032	0.0033
Min $\hat{\xi}$	1.0000	0.6097	0.5165	0.3403	-0.9288	0.8723	3.7936
Max $\hat{\xi}$	22.5160	12.9025	19.4117	12.6851	20.3557	13.5240	28.6731
Ave $\hat{\xi}$	8.8887	6.5714	7.1562	5.6440	5.6228	6.1350	11.4127
Std $\hat{\xi}$	6.2424	3.4036	4.8278	2.9327	4.6326	3.5974	7.2727
Annualized $\hat{\lambda}(t)$	0.0543	0.0616	0.0543	0.0543	0.0471	0.0543	0.0761

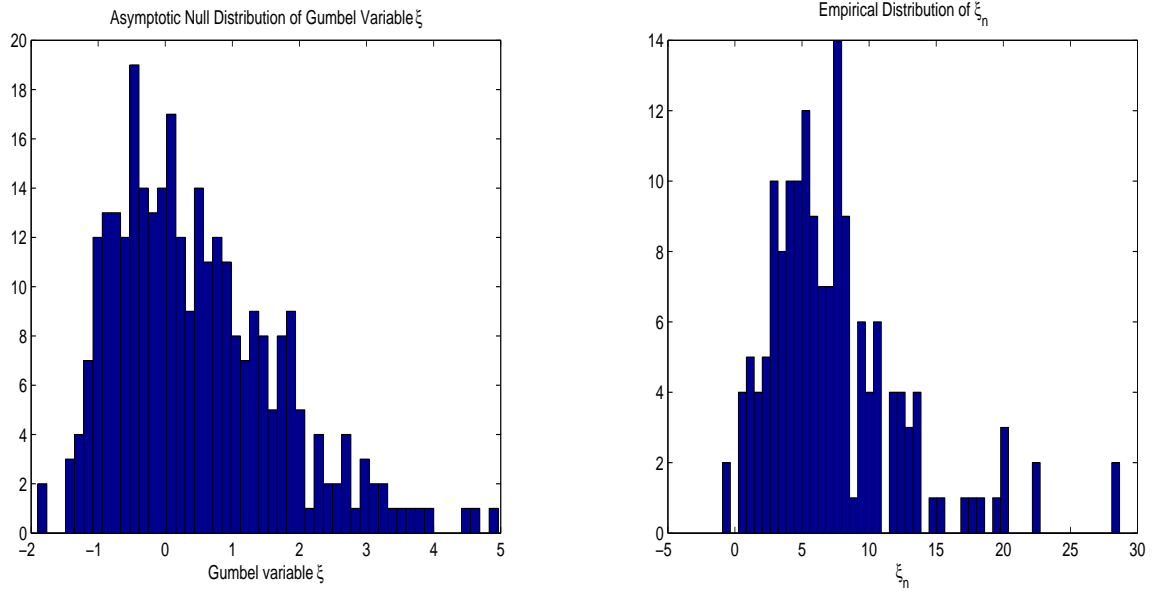
[†] The table contains summary statistics for estimated market quality parameter q , which is the dispersion measure of market microstructure noise as in Equation (8), and estimated Gumbel variables ξ as in Equation (16), and the annualized λ estimates over the time horizon indicated in the top row. For λ calculation, we choose the significance level of 1%, which makes the threshold for jump counting 4.6001. We use IBM stock data during the whole month of August 2007 and the total number of tick-by-tick observations used is 167,595. Data are collected from the TAQ database and for transactions on the New York Stock Exchange (NYSE). All trades before 9:30am or after 4pm and the first trade after 9:30am are discarded due to NYSE trading hours and mechanism. Each trading hour column, for example 11-12, includes information about trades after 11am (inclusive) and before 12pm(exclusive). All trades that have multiple prices at the same time are counted once and the averaged price over the multiple trades is used.

Figure 3: Empirical Distribution of Hourly q for IBM Trades during August 2007



This figure includes histograms of hourly q estimated according to the multipower variation as in Equation (19). For each calculation during every horizon, we calculate the sample autocorrelation to determine the lag k in Equation (19), using the highest available frequency. We use IBM stock data during the whole month of August 2007 and the total number of tick-by-tick observations used is 167,595. Data are collected from the TAQ database and for transactions on the New York Stock Exchange (NYSE). All trades before 9:30am or after 4pm and the first trade after 9:30am are discarded due to NYSE trading hours and mechanism. Different colors for each bin indicate different trading hours. Dark blue, regular blue, light blue, green, yellow, orange, and red represent trading hours of 09:30-10, 10-11, 11-12, 12-13, 13-14, 14-15, and 15-16, respectively. Each trading hour, for example 11-12, includes information about trades after 11am (inclusive) and before 12pm (exclusive). Trades that have multiple prices at the same time are counted once and the averaged price over the multiple trades is used.

Figure 4: Comparison of Limiting and Empirical Distributions



The left panel includes the histogram of simulated Gumbel variables ξ , which is expected under the null hypothesis of no jumps in equilibrium prices. The number of simulations is 300. The right panel includes the histogram of $\hat{\xi}_n$ for IBM stock trades during the whole month of August 2007. The total number of tick-by-tick observations used is 167,595. Descriptions of data such as time horizon T used, number of observations n during $[0, T]$, durations Δt_i over each hour, and price $\tilde{P}(t_i)$ and log return $\Delta \log \tilde{P}(t_i)$ levels are reported in Table 5. Data are collected from the TAQ database and for transactions on the New York Stock Exchange (NYSE). All trades before 9:30am or after 4pm and the first trade after 9:30am are discarded due to NYSE trading hours and mechanism. Trades that have multiple prices at the same time are counted once and the averaged price over the multiple trades is used.

7 Conclusion

Despite the empirical evidence of jumps documented in the asset pricing literature and the popularity of jump diffusion models to accommodate such evidence, the empirical market microstructure literature often ignores their presence in studies using high frequency data. This can be due to the difficulty of distinguishing two unobservable components of observable data: noise and jumps in efficient prices. In this paper, we contribute to the literature by proposing new empirical methods which allow us to find evidence of jumps in underlying efficient price processes which is immune to the presence of noise and to offer new empirical evidence. The evidence produced by our test and estimation for noise variance is expected to be used in various context such as event studies and arbitrage trading strategies as well as portfolio and risk managements, among others.

Since we design this test to take full advantage of ultra high frequency price data, the test can be applied to all sorts of price level data for local averaging as long as high frequency observations are available, so that our asymptotic arguments with a large number of observations in fixed time intervals are valid in the application. We suggest nonparametric methods, which would give evidence robust to model specification. It is important to note that we can investigate the equilibrium price jumps in the presence of general dependent noise processes, which is a crucial feature of noise patterns in financial markets. This general assumption on dependence in noise process distinguishes our test from other existing jump tests.

In finite samples, our suggested noise variance estimator reduces bias due to the presence of jumps more than other existing estimators. Our test has reasonable finite sample properties as long as block size for preprocessing is appropriately chosen. Our empirical study using IBM stock trades on the New York Stock Exchange indicates that there is need to incorporate the presence of jumps in underlying pricing models.

Finally, in this paper, we only study how to identify jumps with finite activity in the presence of noise using high frequency data. It would be interesting to investigate the case of Lévy type jumps with infinite activity to determine whether evidence of Lévy jumps is due to the presence of noise or not.

8 Appendix

8.1 Proof of Lemma 1

Let $\widehat{P}(t_j) = \check{P}(t_j) + \check{U}(t_j)$. Then, with $\check{P}(t_j) = \frac{1}{M} \sum_{i=j}^{j+M-1} P(t_i)$ and $\check{U}(t_j) = \frac{1}{M} \sum_{i=j}^{j+M-1} U(t_i)$,

$$\widehat{P}(t_{j+M}) - \widehat{P}(t_j) = (\check{P}(t_{j+M}) + \check{U}(t_{j+M})) - (\check{P}(t_j) + \check{U}(t_j)) = (\check{P}(t_{j+M}) - \check{P}(t_j)) + (\check{U}(t_{j+M}) - \check{U}(t_j)).$$

With $\mathcal{X}(j)$ defined in (11),

$$\begin{aligned} |\sqrt{M} \left(\widehat{P}(t_{j+M}) - \widehat{P}(t_j) \right) - \sqrt{2q} \mathcal{X}(j)| &= \left| \frac{1}{\sqrt{M}} \sum_{i=j}^{j+M-1} (P(t_{i+M}) - P(t_i)) \right| \\ &= \left| \frac{1}{\sqrt{M}} \sum_{i=j}^{j+M-1} \int_{t_i}^{t_{i+M}} \sigma(u) dW(u) \right| \end{aligned}$$

Since

$$\sum_{i=j}^{j+M-1} \int_{t_i}^{t_{i+M}} \sigma(u) dW(u) = \int_0^t \sigma(u) \sum_{i=j}^{j+M-1} I(t_i < u < t_{i+M}) dW(u) \equiv \mathcal{H}_j$$

and we know that its quadratic variation

$$\langle \mathcal{H}_j, \mathcal{H}_j \rangle_t = \int_0^t \sigma^2(u) \sum_{i=j}^{j+M-1} (I(t_i < u < t_{i+M}))^2 du \leq M^2 \max_{t_i \in \mathcal{G}_n} |\Delta t_i| \max_{0 \leq u \leq t} \sigma^2(u) = M^2 \times O_p(n^{-\beta})$$

And according to Lemma 1 in Mykland and Zhang (2006), we have that for any $0 < \epsilon < \beta/2$,

$$\frac{1}{\sqrt{M}} \sup_j |\mathcal{H}_j| = \frac{1}{\sqrt{M}} M^2 \times O_p(n^{-\beta/2+\delta}) = O_p(n^{\frac{3\gamma-\beta}{2}+\epsilon}) = O_p(n^{-\eta}) \xrightarrow{P} 0,$$

if $\eta = -(\frac{3\gamma-\beta}{2} + \epsilon) > 0$. Therefore, γ needs to satisfy $0 < 3\gamma < \beta - \delta$ for some positive δ . This proves the convergence of probability, which implies the convergence in law as stated.

8.2 Proof of Lemma 2

Proof is in Berman (1964) under the general asymptotic mixing condition as stated. As also mentioned in Ljung (1993).

8.3 Proof of Theorem 1

Under the null hypothesis, as $n \rightarrow \infty$,

$$\begin{aligned} \max \mathcal{L}|_{null} &= \max_{0 \leq j \leq n-2M} \sqrt{M} |\widehat{P}(t_{j+M}) - \widehat{P}(t_j)| \sim \max_{0 \leq j \leq n-2M} \sqrt{M} |U(t_{j+M}) - U(t_j)| \\ &\stackrel{\mathcal{D}}{\sim} \max_{0 \leq j \leq n-2M} |\mathcal{N}(0, 2q^2)| = \sqrt{2q^2} \max_{0 \leq j \leq n-2M} |\mathcal{N}(0, 1)| \\ &\xrightarrow{\mathcal{D}} \sqrt{2q^2} (B_n \xi + A_n) \end{aligned} \quad (23)$$

The first convergence in law follows from Lemma 1 and the second convergence in law follows from Lemma 2.

8.4 Proof of Theorem 2

Under the alternative when the jump time is $\tau \in (t_j, t_{j+M-1})$ for some j , we have

$$\begin{aligned} \max \mathcal{L}|_{alternative} &= \max_{0 \leq j \leq n-2M} |\widehat{P}(t_{j+M}) - \widehat{P}(t_j)| \\ &= \max_{0 \leq j \leq n-2M} |O_p(n^{\epsilon-\beta/2+3\gamma/2}) + O_p(n^{-\gamma/2}) + \frac{1}{M} \sum_{i=j}^{j+M-1} \int_{t_i}^{t_{i+M}} Y(s) dJ(s)| \\ &= |Y(\tau)| \times \frac{\text{Number of times}(t_i \in (t_j, t_{j+M-1}), t_i \leq \tau)}{M}. \end{aligned} \quad (24)$$

8.5 Proof of Proposition 1 under the null

Denote u_i , u , and n_i are all normal variables with a mean of 0 and a variance of 1. Using

$$P(t_{j-k(2m-2)}) - P(t_{j-k(2m-1)}) = \sigma \sqrt{k\Delta t} n_i \text{ and}$$

$$U(t_{j-k(2m-2)}) - U(t_{j-k(2m-1)}) = q (u_{j-k(2m-2)} - u_{j-k(2m-1)}),$$

$$\begin{aligned}
\text{plim}_{\Delta t \rightarrow 0} \widehat{Q}^{gr} &= \text{plim}_{\Delta t \rightarrow 0} \left(\frac{1}{n-2g+2} \frac{1}{c_r^g} \sum_{j=2g-1}^n \prod_{m=1}^g |\widetilde{P}(t_{j-k(2m-2)}) - \widetilde{P}(t_{j-k(2m-1)})|^r \right) \\
&= \text{plim}_{\Delta t \rightarrow 0} \frac{1}{n-2g+2} \frac{1}{c_r^g} \sum_{j=2g-1}^n \prod_{m=1}^g |P(t_{j-k(2m-2)}) - P(t_{j-k(2m-1)}) + U(t_{j-k(2m-2)}) - U(t_{j-k(2m-1)})|^r \\
&= \text{plim}_{\Delta t \rightarrow 0} \frac{1}{n-2g+2} \sum_{j=2g-1}^n \frac{1}{c_r^g} \prod_{m=1}^g |\sigma \sqrt{k \Delta t} n_i + q(u_{j-k(2m-2)} - u_{j-k(2m-1)})|^r \\
&= \frac{q^{gr}}{c_r^g} E \prod_{m=1}^g |(u_{j-k(2m-2)} - u_{j-k(2m-1)})|^r = \frac{q^{gr}}{c_r^g} \prod_{m=1}^g E |\sqrt{2}u|^r = (\sqrt{2}q)^{gr}
\end{aligned}$$

8.6 Proof of Proposition 1 under the alternative

We assume there are N number of rare jumps in the efficient price process with N being finite over any fixed time horizon. The presence of jumps now comes into our efficient prices as $P(t_{j-k(2m-2)}) - P(t_{j-k(2m-1)}) = \sigma \sqrt{k \Delta t} n_i + O_p(1) I_{\tau \in [t_{j-k(2m-2)}, t_{j-k(2m-1)}]}$, where τ is the jump arrival time. Then,

$$\begin{aligned}
\text{plim}_{\Delta t \rightarrow 0} \widehat{Q}^{gr} &= \text{plim}_{\Delta t \rightarrow 0} \left(\frac{1}{n-2g+2} \frac{1}{c_r^g} \sum_{j=2g-1}^n \prod_{m=1}^g |\widetilde{P}(t_{j-k(2m-2)}) - \widetilde{P}(t_{j-k(2m-1)})|^r \right) \\
&= \text{plim}_{\Delta t \rightarrow 0} \frac{1}{n-2g+2} \sum_{j=2g-1}^n \frac{1}{c_r^g} \prod_{m=1}^g |O_p(1) I_{\tau} + q(u_{j-k(2m-2)} - u_{j-k(2m-1)})|^r \\
&= \text{plim}_{\Delta t \rightarrow 0} \frac{1}{n-2g+2} \underbrace{\sum_{\text{with jump}} \frac{1}{c_r^g} \prod_{m=1}^g |O_p(1) I_{\tau} + q(u_{j-k(2m-2)} - u_{j-k(2m-1)})|^r}_{= \frac{N}{n-2g+2} O_p(1) \text{ gets smaller } \rightarrow 0 \text{ as } n \rightarrow \infty \text{ and } N \text{ is fixed.}} \\
&\quad + \text{plim}_{\Delta t \rightarrow 0} \frac{1}{n-2g+2} \sum_{\text{without jump}} \frac{1}{c_r^g} \prod_{m=1}^g q(u_{j-k(2m-2)} - u_{j-k(2m-1)})|^r \\
&\approx \frac{q^{gr}}{c_r^g} E \prod_{m=1}^g |(u_{j-k(2m-2)} - u_{j-k(2m-1)})|^r = \frac{q^{gr}}{c_r^g} \prod_{m=1}^g E |\sqrt{2}u|^r = (\sqrt{2}q)^{gr}
\end{aligned}$$

We assume the jump arrival rate is low enough so that the number of terms that include jumps is much less than n that satisfies $n \rightarrow \infty$ as $\Delta t \rightarrow 0$.

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