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# **Vanishing Liquidity, Market Runs, and the Welfare Impact of TARP\***

by

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## **Vanishing Liquidity, Market Runs, and the Welfare Impact of TARP**

### **Abstract**

I model a financial market that dries out in the wake of premature liquidations. Two main results are obtained. First, liquidity may vanish even if small, risk-neutral buyers could easily compensate the ongoing selling. Thus, more markets are vulnerable to “runs” than suggested by previous work. Second, the scale of premature liquidations is not informative about welfare losses. In fact, market runs may be nearly constrained efficient. The latter finding might suggest an explanation for the recent policy turn of the U.S. Treasury concerning purchases of troubled assets under the Emergency Economic Stabilization Act of 2008 (EESA).

**Keywords:** Liquidity, inventory, market run, welfare, TARP

**JEL-classification:** G1, G2, G21, E44, N2

## 1. Introduction

In a fundamental contribution to the theory of market microstructure, Grossman and Miller (1988) have provided an interpretation of liquidity as the price for immediate execution. In this interpretation, a difference between fundamental value and market valuation of a risky asset comes about because risk-averse market participants are compensated for taking positions that they otherwise would not accept. Building upon this intuition, Bernardo and Welch (2004) have recently proposed a model that, while simple, captures essential elements of a “run” on a financial market.

In short, their framework can be described as follows. There is a population of investors, each of whom owns a single unit of the financial asset. If an investor holds the asset until maturity, it renders a positive expected return. However, with a positive probability, the asset must be liquidated at an interim stage. To avoid the risk of the forced liquidation, some or all investors will sell the asset at an early stage, contributing to the decline in market prices. Thus, there is a “run” on the financial market. An inefficiency occurs here because the allocation of risks in the economy is less efficient than it could be if investors were to wait for better information.

A striking point about this model is its assumption that, apart from the specialists, there are no buyers in the economy. The original contribution mentions this point, but does not conduct a formal analysis.<sup>1</sup> But how restrictive is this “sellers only” perspective? I.e., is a financial market run a true possibility even if there are standby buyers? And how does purchasing power affect welfare? These questions might also be important from a policy perspective. For instance, following the escalation of the subprime credit crunch in fall 2008, U.S. Congress passed the enormous Troubled Asset Relief Program (TARP) that, using taxpayer’s money, would intend to help battered markets by buying and selling illiquid assets. Yet why did the market apparently not work on its own? And of what nature is the efficiency gain that a social planner could hope for by orchestrating trade in the market?

To examine these questions, I complement the basic model of investor fear by

adding a population of risk-neutral investors endowed with cash rather than the risky asset. Investment by these standby buyers may be early or late, and conditional on the state of nature. An equilibrium notion is specified that combines sequential rationality of sellers and buyers with competitive pricing in the specialist sector. The analysis of investors' incentives leads to a unique equilibrium prediction in the most interesting parameter domains. The resulting allocation is then compared, in terms of welfare, to the second-best outcome with forced liquidations.

It turns out that the qualitative properties of the market equilibrium as well as those concerning efficiency depend on the relative size of the buyer population. With moderate purchasing power, the market equilibrium resembles the “run” prediction found in the model without buyers. However, there is one essential distinction, which is that the size of the run, measured in terms of unmatched premature selling orders, is not directly informative about its social loss. In fact, for almost any amount of unmatched premature orders to sell, it is feasible to construct a robust run scenario of this size so that the welfare loss compared to the second-best is arbitrarily small. Thus, in this case, a run may be nearly *constrained* efficient. In contrast, in the case of strong purchasing power, there is never a run to sell. Instead, buyers hurry to invest until prices reflect fundamentals. Moreover, the market equilibrium is always constrained efficient in that case.

The possibility of widespread superfluous liquidations with little social harm is puzzling.<sup>2</sup> To elucidate why a robust run scenario can come arbitrarily close to constrained efficiency under moderate purchasing power, I will elaborate in some detail on the incentives for investors in the dynamic market equilibrium. It turns out that, in anticipation of forced liquidations, the market may “seize up” because buyers submit no orders.<sup>3</sup> Buyers wait in the crash because in contrast to potentially distressed sellers, they would forgive the option value of buying in the bad state. This option value drives a wedge between valuations of buyers and sellers in a financial market run, so that the de facto price process is a martingale for the seller, but

not for the buyer. Sellers, in turn, rationally expect buy orders to arrive too late to make up for the downward price movement. As a consequence, the run on the financial market, i.e., the premature liquidation of the risky asset into the books of risk-averse specialists, is not averted by the presence of risk-neutral liquidity. On the other hand, the run will be socially harmful only to the extent that premature liquidations lead to a long-term misallocation of risks in the economy. This point will be further explored in the main part of the paper.<sup>4</sup>

In direct comparison to Bernardo and Welch (2004), the model is extended in two dimensions. First, as already explained, the present framework will allow for a population of buyers. Second, the specialist sector may, but need not, hold a positive endowment in the risky asset. The consideration of inventory positions allows a better understanding of what happens when buyers rush to invest in the risky asset at depressed price levels. Moreover, with inventory, it is possible to have a run even though risk-neutral liquidity in the market exceeds the total volume of premature liquidations. Otherwise, there are no changes in the underlying assumptions.

There are further links to the existing literature. Liquidation in illiquid financial markets has recently become a topic of broader interest. Morris and Shin (2004) expand the use of global game techniques to runs on illiquid markets. Exogenous loss limits induce proprietary traders to liquidate, yet differences in private information lead investors into preemptive, inefficient liquidations. In contrast to the present paper, however, there are no risk-neutral buyers. Schnabel and Shin (2004) assume illiquid asset markets to model the contagious mechanics of the financial crisis of 1763, yet again without risk-neutral purchasing power. Brunnermeier and Pedersen (2005) identify an intriguing mechanism that can be exploited by strong market players. Specifically, by selling the illiquid asset, “predators” may actively force a market participant into liquidation, and subsequently benefit from depressed asset prices. Such strategic price manipulation is always undesirable from a regulatory perspective. In contrast to this paper, however, investors in my model are

individually too small to affect market prices. Finally, empirical work by Campbell et al. (1993) and by Pastor and Stambough (2003) supports the view taken both by the above studies and by the present paper that market liquidity impacts positively on volume-related return reversals.<sup>5</sup>

An interesting and longstanding debate concerns the question whether rational speculation leads market prices to reflect fundamentals more accurately (cf., e.g., Friedman, 1953). This debate is related to the present analysis since market prices near fundamental values often serve as a proxy for allocational efficiency. Two contributions from this literature are closely related. First, by not stepping in even though prices are below fundamentals, buyers in market runs behave similar to rational speculators in Hart and Kreps (1986). However, in that model, speculators live for only two periods, which is incompatible with the option value that drives investor behavior in the present analysis. Second, in a market run, rational standby investors do not exaggerate price movements, as in De Long et al. (1990), yet they also do nothing to prevent market prices from falling below fundamental values. Generally, the literature represented by these contributions shares with the present analysis the focus on intertemporal incentives for rational investors. However, the nature of forced liquidations clearly adds a new twist to the enquiry.

Finally, the point that financial market runs need not always cause significant social damage has, of course, an analogy in the literature on bank runs. Contributions stressing the possibility of bank runs being efficient are Wallace (1990), Alonso (1996), and Allen and Gale (1998, 2004). However, all these papers focus on institutional design. In contrast, the present analysis is primarily dealing with the efficiency properties of a given financial market.<sup>6</sup>

The rest of the paper is organized as follows. Section 2 outlines the model and discusses its main assumptions. Section 3 describes the equilibrium concept and analyzes the financial market equilibrium in the most interesting case when purchasing power is moderate. Constrained efficient trading is characterized in



Section 4. Section 5 discusses the possibility of nearly welfare-maximizing market runs. The complementary case of strong purchasing power is covered in Section 6. In Section 7, I relate the main results of the present study to the U.S. Treasury's decision, effective from mid November 2008, to put on ice plans of direct purchases of mortgage-related assets. Section 8 concludes by briefly summarizing the paper. The Appendix contains the explicit characterization of the market equilibrium.

## 2. The model

Considered will be a financial market over three dates, date 0, date 1, and the terminal date 2. There are two assets, cash and a risky asset. Cash is risk-free and has a return normalized to zero. The risky asset can be exchanged against cash at the current market price on dates 0 and 1. The true value of the asset  $\tilde{V}$  is revealed and paid out in cash to the holder of the risky asset at date 2. Before date 2, the value of the asset is uncertain, and known to be distributed normally with mean  $V$  and variance  $\Omega > 0$ . Three types of traders are in the market. First, there is a continuum of risk-neutral investors, the *sellers*, that hold the asset but no cash. The size of the population of sellers will be normalized to one. Second, there is a continuum of risk-neutral investors, the *buyers*, who do not hold the asset, but a cash endowment  $e > 0$ . Let  $\beta \geq 0$  denote the size of the population of buyers. Finally, there is a perfectly competitive risk-averse *specialist sector* that clears the market at dates 0 and 1. The specialist sector has a constant coefficient of absolute risk aversion  $\gamma > 0$  and is equipped with an initial endowment, composed of  $X_0 \geq 0$  units of cash and  $Z_0 \geq 0$  units of the risky asset. Immediately before date 1, the state of the world  $\omega \in \{b, g\}$  realizes, and becomes public information. The probability of state  $b$  will be denoted by  $s$ . To avoid uninteresting case distinctions, it will be assumed throughout that  $s < 1$  and  $e, X_0 \geq V$ . This will ensure, in particular, that specialists remain liquid in cash over the entire trading horizon.<sup>7</sup>

Four assumptions are imposed. The first assumption is of a technical nature. Essentially, it reduces the dimension of investors' strategy spaces to a manageable

level.

**Assumption 1.** *At any point in time, an individual investor (i.e., seller or buyer) may hold either one unit or no unit of the risky asset. Moreover, each investor may trade at most once during the trading horizon.*

Under Assumption 1, the set of trading strategies for sellers and buyers can be described as follows. A seller may either sell at date 0, or sell unconditionally at date 1, or sell only at date-state 1b. Thus, there are three trading strategies for the seller. A buyer, on the other hand, chooses from a total of five strategies. She may either (i) invest in the risky asset at date 0, or (ii) invest unconditionally at date 1, or (iii) purchase the risky asset only in date-state 1b, or (iv) buy only in date-state 1g, or (v) hold cash. In fact, it will become clear soon that the interaction between sellers and buyers, given the pricing function used by the specialists, can be considered from a game-theoretic perspective, and that the market equilibrium to be defined is essentially a subgame-perfect equilibrium (cf. Selten, 1965). However, since identities of the agents are not important for the subsequent analysis, it is more convenient to take an aggregate perspective on trading decisions.

So let  $\sigma_0$  and  $\beta_0$ , respectively, denote the size of the subpopulation of sellers and buyers that decide to trade early, i.e., at date 0. Analogously, let  $\sigma_1^\omega$  and  $\beta_1^\omega$ , respectively, denote the size of the subpopulation of sellers and buyers that decide to sell and buy the risky asset at date-state  $1\omega$ , where  $\omega = b, g$ . Then, based on Assumption 1, aggregate investment decisions can be summarized in a *trading statistics*

$$\xi = (\sigma_0, \sigma_1^b, \sigma_1^g, \beta_0, \beta_1^b, \beta_1^g), \quad (1)$$

where it is tacitly understood that  $\xi$  satisfies *non-negativity* constraints

$$\sigma_0 \geq 0, \beta_0 \geq 0, \text{ and } \sigma_1^\omega \geq 0, \beta_1^\omega \geq 0 \text{ for } \omega = b, g, \quad (2)$$

as well as *population accounting* constraints

$$\sigma_0 + \sigma_1^g \leq 1, \text{ and } \beta_0 + \beta_1^\omega \leq \beta \text{ for } \omega = b, g. \quad (3)$$

The second assumption sets the stage for the analysis. At date-state 1b, all sellers that have not yet sold the risky asset are then *forced to liquidate* their positions.

**Assumption 2.**  $\sigma_1^b = 1 - \sigma_0$ .

The next assumption concerns the price formation in the financial market. It is assumed that the expected rent for the specialists is driven to zero by short-term competition both at date 0 and date 1.

**Assumption 3.** *Only orders without limit are accepted by the specialists. Moreover, at date 0 and date 1, respectively, quotes are such that the expected utility for the specialist sector does not change by fulfilling the orders.*

This assumption has two extremely useful implications. The first is the possibility of a price trend. A momentum may obtain in the present set-up simply because specialists already holding risky positions find it unattractive to increase these positions further. Assumption 3 may therefore be considered as a pragmatic way to capture the momentum effect.<sup>8</sup> The other useful implication of Assumption 1 is that specialists make a zero contribution to changes in welfare, which will simplify the efficiency analysis.

The first implication concerning the momentum term is reflected in the price path

$$P(\xi) = (P_0(\xi), P_1^b(\xi), P_1^g(\xi)) \quad (4)$$

of the risky asset as a function of the trading statistics  $\xi$ . Denote by

$$Y_0 = \sigma_0 - \beta_0 \quad (5)$$

the net volume of orders at date 0, and by

$$Y_1^\omega = \sigma_1^\omega - \beta_1^\omega \quad (6)$$

the net volume of orders at date-state  $1\omega$ , where  $\omega = b, g$ . Then the price vector resulting from a trading statistics can be explicitly computed as follows.

**Lemma 1.** *Under Assumption 3, prices  $P(\xi)$  induced by a trading statistics  $\xi$  are given by*

$$P_0(\xi) = V - \lambda(2Z_0 + Y_0), \text{ and} \quad (7)$$

$$P_1^\omega(\xi) = V - \lambda(2Z_0 + 2Y_0 + Y_1^\omega) \quad (\omega = b, g), \quad (8)$$

where  $\lambda = \gamma\Omega/2 > 0$ .

**Proof.** Assume that a net order flow of  $Y_0$  arrives at the specialist sector at date 0. From Assumption 3, the market price  $P_0$  leaves expected utility unchanged, i.e.,

$$E[-\exp\{-\gamma(X_0 + Z_0\tilde{V})\}] = E[-\exp\{-\gamma(X_0 + Z_0\tilde{V} + Y_0(\tilde{V} - P_0))\}]. \quad (9)$$

In the cara-normal framework, equation (9) is equivalent to

$$X_0 - \frac{\gamma\Omega}{2}(Z_0)^2 = X_0 + Y_0(V - P_0) - \frac{\gamma\Omega}{2}(Z_0 + Y_0)^2. \quad (10)$$

Solving for  $P_0$  yields (7). To prove (8), one substitutes parameters  $\{X_0, Z_0, Y_0, P_0\}$  by  $\{X_0 - Y_0P_0, Z_0 + Y_0, Y_1^\omega, P_1^\omega\}$  for  $\omega = b, g$ .  $\square$

Thus, the market price reflects the limited risk-taking capacity of the specialists, which implies the liquidity premium. For instance, when  $Y_0 > 0$ , then there are more sellers than buyers in the short term, which depresses the market price of the risky asset relative to its fundamental value. The momentum effect is reflected in equation (8). Specifically, if  $Y_0 > 0$ , then even if  $Y_1^\omega = 0$ , the market price declines further, i.e.,  $P_1^\omega < P_0$ . Assumptions 1 and 3 will be imposed throughout the paper without explicit mentioning, so that in particular Lemma 1 holds.

A final assumption is made to avoid that the inventory position of the specialist sector is exhausted during the trading period. In the basic model, this assumption is obsolete for the simple reason that the specialist sector could not face a net demand for the risky asset. With buyers in the market, however, this might indeed happen. In fact, at depressed prices, it is natural to expect that buyers will be more inclined to trade than sellers. The following assumption is therefore needed to exclude an *inventory run-out* of the risky asset at price levels quoted by the specialists.

**Assumption 4.**  $Z_0 + Y_0 \geq 0$  and  $Z_0 + Y_0 + Y_1^\omega \geq 0$  for  $\omega \in b, g$ .

A trading statistics  $\xi$  will be called *feasible* provided it reflects forced liquidations and no inventory run-out, i.e., provided Assumptions 2 and 4 hold.

### 3. Runs with standby investors

This section is concerned with the possibility of financial market runs in the presence of risk-neutral liquidity. After defining the equilibrium notion, the main features of the equilibrium will be discussed, with a focus on the case of moderate purchasing power. In particular, I will state conditions necessary and sufficient for a market run with standby investors.

A market equilibrium will be defined as a combination of a trading statistics and a price vector, satisfying two properties. Firstly, the trading statistics will be required to reflect, for the given vector of prices, sequentially rational trading decisions for both sellers and buyers. Specific conditions, referred to as incentive compatibility in the sequel, will be spelt out for optimization at date 0, at date-state 1b, and at date-state 1g. Secondly, the definition of market equilibrium will entail that competitive prices are set by the specialists in response to the dynamics of aggregate order flows.

Formally, consider a feasible trading statistics  $\xi$  and prices  $P = (P_0, P_1^b, P_1^g)$ . *Incentive compatibility for sellers at date-state 1g* is given by conditions

$$P_1^g < V \Rightarrow \sigma_1^g = 0, \text{ and } P_1^g > V \Rightarrow \sigma_1^g = 1 - \sigma_0. \quad (11)$$

where, as a matter of notation, each “ $\Rightarrow$ ” is followed by a necessary condition. For a given state of the world  $\omega \in \{b, g\}$ , *incentive compatibility for buyers at date-state 1 $\omega$*  amounts to conditions

$$P_1^\omega < V \Rightarrow \beta_1^\omega = \beta - \beta_0, \text{ and } P_1^\omega > V \Rightarrow \beta_1^\omega = 0. \quad (12)$$

In line with the logic of dynamic programming, *incentive compatibility for sellers at date 0* requests

$$P_0 < sP_1^b + (1 - s) \max\{P_1^g, V\} \Rightarrow \sigma_0 = 0, \quad (13)$$

and

$$P_0 > sP_1^b + (1 - s) \max\{P_1^g, V\} \Rightarrow \sigma_0 = 1. \quad (14)$$

These conditions already incorporate the limited flexibility of sellers at date-state

1b. *Incentive compatibility for buyers at date 0* says

$$P_0 > s \min\{P_1^b, V\} + (1 - s) \min\{P_1^g, V\} \Rightarrow \beta_0 = 0, \quad (15)$$

and

$$P_0 < s \min\{P_1^b, V\} + (1 - s) \min\{P_1^g, V\} \Rightarrow \beta_0 = \beta. \quad (16)$$

The trading statistics  $\xi$  will be called *incentive compatible at prices  $P$*  if all these conditions are satisfied, i.e., if incentive compatibility holds for sellers at date 0 and date-state 1g, and if incentive compatibility holds for buyers at date 0, at date-state 1b, and at date-state 1g. Finally, a *market equilibrium* is a pair  $(\xi, P)$  consisting of a feasible trading statistics  $\xi$  and a vector of market prices  $P$  such that (i)  $\xi$  is incentive compatible at prices  $P$ , and (ii)  $P$  is set competitively by the specialists given  $\xi$ , i.e., according to price formulas stated in Lemma 1.

The market equilibrium will now be studied. The qualitative nature of the equilibrium depends, of course, on the strength of the purchasing power in the market. When purchasing power is strong, there is little surprising going on. Financial market runs simply cannot occur. In fact, the market is then too liquid to allow any inefficiency relative to the benchmark. This will be explained more carefully in Section 6. For the main part of the paper, however, the focus will be on the scenario with moderate purchasing power. In this case, it turns out that risk-neutral liquidity impacts on the market equilibrium only in a rather indirect way. This is so because, in anticipation of forced liquidation, standby investors will *wait*. The following example illustrates this point.

**Example 1.** Consider a set-up with exogenous parameters

$$s = \frac{1}{3}, \beta = \frac{1}{4}, \lambda = 8, V = 10, \text{ and } Z_0 = 0. \quad (17)$$

I claim that the trading statistics

$$\xi = (\sigma_0, \sigma_1^b, \sigma_1^g, \beta_0, \beta_1^b, \beta_1^g) = \left(\frac{3}{8}, \frac{5}{8}, 0, 0, \frac{1}{4}, \frac{1}{4}\right) \quad (18)$$

describes the unique market equilibrium. This can be seen as follows. To verify that  $\xi$  corresponds to an equilibrium, one checks forced liquidation, inventory conditions, and incentive compatibility with respect to the induced price vector  $P(\xi)$ . Forced liquidation is clearly satisfied since  $\sigma_1^b = 1 - \sigma_0$ . To check the inventory conditions, note that net order flows over the trading horizon are given by

$$(Y_0, Y_1^b, Y_1^g) = \left(\frac{3}{8}, \frac{3}{8}, -\frac{1}{4}\right). \quad (19)$$

Clearly, therefore,  $Z_0 + Y_0 \geq 0$  and  $Z_0 + Y_0 + Y_1^\omega \geq 0$  for  $\omega = b, g$ . Thus, even though there is a net demand from the risk-neutral investors at date-state 1g, there is no inventory run-out under the trading statistics  $\xi$ . Finally, one checks incentive compatibility. It is immediate from Lemma 1 and (19) that the price vector induced by  $\xi$  is given by  $P(\xi) = (7, 1, 6)$ . Now, at date 0, sellers' expectation of realized value

$$V^S \equiv sP_1^b + (1-s)V = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 10 = 7 \quad (20)$$

is equal to the current market price  $P_0$ . So incentive compatibility for sellers at date 0 clearly holds. Similarly, buyers' expectation of the price at date 1 amounts to

$$V^B \equiv sP_1^b + (1-s)P_1^g = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 6 \approx 4.33, \quad (21)$$

which is strictly lower than the current market price  $P_0$ . So incentive compatibility for buyers at date 0 is satisfied because of  $\beta_0 = 0$ . Finally, incentive compatibility at date 1 is immediate. Indeed, given that prices are below fundamentals in both states of the world, the requirements are simply  $\sigma_1^g = 0$  and  $\beta_1^\omega = \beta - \beta_0$  for  $\omega = b, g$ , which can be readily verified. Thus, as claimed,  $\xi$  corresponds to a market equilibrium. The uniqueness of the equilibrium is less straightforward to check directly but follows, given  $\beta < s$ , from the characterization of the market equilibrium given in the Appendix, i.e., from Proposition B.1.

In the market equilibrium discussed in Example 1, some of the sellers choose to sell early, but even though risk-averse specialists increase inventory positions, there are no early purchases by standby investors. To understand why risk-neutral buyers do not step in, note that at date 0, the buyers' expectation of the price at date 1 is  $V^B < P_0$ . In contrast, the sellers' expectation of realized value is  $V^S = P_0$ . Thus, before uncertainty is resolved, the potentially distressed sellers have a higher valuation for the risky asset than standby buyers. The difference in valuations corresponds to the value of the buyers' option to purchase and hold the risky asset in the bad state. This option is unavailable to the sellers, who are forced to sell. Since intertemporal arbitrage is executed through sellers, the buyers find it in their best interest to delay their investment.

The first main result, stated below, confirms the conjecture that the differential valuation of buyers and potentially distressed sellers, as illustrated by Example 1, holds more generally. Whenever purchasing power is moderate, any market equilibrium will have the property that risk-neutral buyers find an early investment not attractive. In fact, a run may occur notwithstanding purchasing power sufficient in the market to match all premature liquidations.

**Proposition 1.** *Assume  $\beta < s$ . Then, provided that a market equilibrium exists, it is unique and satisfies  $\beta_0 = 0$  as well as  $V^B < V^S$ . Moreover, there are threshold values  $0 < \underline{Z} < \widehat{Z} \leq Z^*$  such that*

(i) *the equilibrium exists if and only if either  $Z_0 \leq \widehat{Z}$ , or both  $Z_0 > Z^*$  and  $Z_0 \geq \beta$ ,*

(ii)  *$\sigma_0 > 0$  if and only if  $Z_0 < Z^*$ , and*

(iii)  *$\sigma_0 < \beta$  if and only if  $Z_0 > \underline{Z}$ .*

**Proof.** To apply Proposition B.1 in the Appendix, let

$$Z^* \equiv \frac{s(1-\beta)}{2(1-s)}, \text{ and } \widehat{Z} \equiv \min\{Z^*, 2Z^* - \beta\}. \quad (22)$$

For  $Z_0 < Z_* \equiv Z^* - \frac{1}{2}$ , the unique market equilibrium is given by Lemma B.1. In



this case,  $\beta_0 = 0$  and

$$P_1^g = V - \lambda(2Z_0 + 2 - \beta). \quad (23)$$

Since  $\beta < s < 1$  and  $Z_0 \geq 0$ , one finds  $P_1^g < V$  and therefore  $V^B < V^S$ . For  $Z_* \leq Z_0 \leq \widehat{Z}$ , the unique equilibrium is given by Lemma B.2. In this case,  $\beta_0 = 0$  and, as a short calculation shows,

$$P_1^g = V - \lambda\left\{2(Z^* - Z_0) + \frac{s - \beta}{1 - s}\right\}. \quad (24)$$

Thus, again,  $P_1^g < V$ , and hence,  $V^B < V^S$ . For  $Z_0 > \widehat{Z}$  such that  $Z_0 \leq Z^*$  or  $Z_0 < \beta$ , there is no market equilibrium. Finally, for  $Z_0$  such that both  $Z_0 > Z^*$  and  $Z_0 \geq \beta$  hold, the unique equilibrium is given by Lemma B.4, so that  $\beta_0 = 0$ , and

$$P_1^g = V - \lambda(2Z_0 - \beta). \quad (25)$$

But because  $\beta < s < 1$ , one finds  $2Z_0 - \beta > 2Z^* - \beta > 0$ . Hence, another time  $P_1^g < V$ , and consequently  $V^B < V^S$ . This proves the first assertion of the proposition as well as (i). To prove (ii), note that the equilibrium characterization delivers also

$$\sigma_0 = \begin{cases} 1 & \text{if } Z_0 < Z_* \\ 2(Z^* - Z_0) & \text{if } Z_* \leq Z_0 \leq \widehat{Z} \\ 0 & \text{if } Z_0 > Z^* \text{ and } Z_0 \geq \beta. \end{cases} \quad (26)$$

Hence,  $\sigma_0 > 0$  if and only if  $Z_0 < Z^*$ . To prove (iii), let  $\underline{Z} \equiv Z^* - \beta/2$ . It is immediate that

$$Z^* \geq \widehat{Z} > \underline{Z} > 0, \quad (27)$$

and that  $\underline{Z} > Z_*$ . Hence,  $\sigma_0 < \beta$  if and only if  $Z_0 > \underline{Z}$ .  $\square$

Thus, even with buyers in the market more than capable of preventing the panic, a financial market run may result as an incentive-compatible outcome of the dynamic trading game. Thus, even more markets are susceptible to runs than suggested by Bernardo and Welch's (2004) assertion concerning standby investors, as quoted in Footnote 1 of the present paper. More generally, Proposition 1 shows that the main prediction of the model of investor fear does not depend on the "sellers only"

perspective that has been taken in the literature. The emergence of runs is, however, dependent on the assumption of a zero inventory. Indeed, when inventory is positive, market prices are depressed at date 0 even if there are no premature liquidations. As a consequence, sellers may be better off waiting provided the probability of forced liquidations is not too high.

#### 4. Constrained efficient trading

Market runs are rarely considered desirable. In the model, a run will exacerbate the situation whenever the resulting long-term allocation of risk in the economy is socially inferior to the benchmark outcome without premature liquidations. However, it will be shown in Section 5 that the welfare loss caused by a run need not be significantly larger than that caused by forced liquidations alone, provided that the illiquidity of the market is of an endogenous nature. The present section prepares this result by determining the constrained efficient allocation in the trading game.

The welfare function employed in the subsequent analysis will be specified as the sum of expected aggregate payoffs for the respective populations of buyers and sellers. This is indeed a consistent approach because, as mentioned in Section 2, the specialist sector's contribution to welfare would be constant across trading dates. Consequently, for a pair  $(\xi, P)$ , define *welfare*  $W = W(\xi, P)$  by  $W = W_S + W_B$ , where

$$W_S = \sigma_0(P_0 - V) + s\sigma_1^b(P_1^b - V) + (1 - s)\sigma_1^g(P_1^g - V) \quad (28)$$

is the sellers' aggregate surplus, which is typically negative, and

$$W_B = \beta_0(V - P_0) + s\beta_1^b(V - P_1^b) + (1 - s)\beta_1^g(V - P_1^g) \quad (29)$$

is the buyers' aggregate surplus, which is typically positive. Thus, each investor in the market receives the same weight in the welfare function.

If a hypothetical social planner were able to orchestrate trading behavior in a virtually unconstrained way, yet still subject to objective constraints, then there would be no room for an increase of the aggregate welfare measure. Therefore, the

constrained efficient outcome results if the social planner maximizes welfare subject to forced liquidations, inventory constraints, and the price formulas stated in Lemma 1.<sup>9</sup> Formally, a feasible trading statistics  $\xi$  will be said to be *constrained efficient* if, for the price vector  $P = P(\xi)$  induced by  $\xi$ , the pair  $(\xi, P)$  maximizes welfare subject to feasibility. Solving the problem of the social planner leads to the following explicit description of the constrained efficient trading statistics. For ease of exposition, it will be assumed that the purchasing power in the market is not too high.<sup>10</sup>

**Lemma 2.** *Assume  $s > 0$  and  $\beta \leq Z_0 + 1$ . Then a feasible trading statistics  $\xi$  is constrained efficient if and only if*

$$Y_0 + Y_1^b = 1 - \beta, \text{ and} \quad (30)$$

$$Y_0 + Y_1^g = -\min\{Z_0; \beta\}. \quad (31)$$

Moreover, any feasible trading statistics  $\xi$  such that  $\sigma_0 = \beta_0 = 0$ , and such that incentive compatibility conditions at date 1 are satisfied at prices  $P(\xi)$ , is constrained efficient.

**Proof.** Using Lemma 1, a feasible trading statistics

$$\xi = (\sigma_0, \sigma_1^b, \sigma_1^g, \beta_0, \beta_1^b, \beta_1^g) \quad (32)$$

generating a vector of net demands

$$(Y_0, Y_1^b, Y_1^g) = (\sigma_0 - \beta_0, \sigma_1^b - \beta_1^b, \sigma_1^g - \beta_1^g) \quad (33)$$

is constrained efficient if and only if it maximizes

$$W(\xi, P(\xi)) = Y_0(P_0(\xi) - V) + sY_1^b(P_1^b(\xi) - V) + (1 - s)Y_1^g(P_1^g(\xi) - V) \quad (34)$$

$$= -\lambda\{s(Z_0 + Y_0 + Y_1^b)^2 + (1 - s)(Z_0 + Y_0 + Y_1^g)^2 - (Z_0)^2\} \quad (35)$$

subject to the constraints that define feasible trading statistics, i.e.,

$$\sigma_0 \geq 0 \quad (36)$$

$$\beta_0 \geq 0 \quad (37)$$

$$\sigma_1^\omega \geq 0 \quad \omega = b, g \quad (38)$$

$$\beta_1^\omega \geq 0 \quad \omega = b, g \quad (39)$$

$$\sigma_0 + \sigma_1^b = 1 \quad (40)$$

$$\sigma_0 + \sigma_1^g \leq 1 \quad (41)$$

$$\beta_0 + \beta_1^\omega \leq \beta \quad \omega = b, g \quad (42)$$

$$Z_0 + Y_0 \geq 0 \quad (43)$$

$$Z_0 + Y_0 + Y_1^\omega \geq 0 \quad \omega = b, g. \quad (44)$$

There are two cases. Assume first that  $Z_0 < \beta$ . Take any feasible trading statistics  $\xi$  such that

$$Y_0 + Y_1^b = 1 - \beta \text{ and} \quad (45)$$

$$Y_0 + Y_1^g = -\min\{Z_0; \beta\} = -Z_0. \quad (46)$$

Then, clearly,  $Z_0 + Y_0 + Y_1^b \geq 0$  and  $Z_0 + Y_0 + Y_1^g = 0$ , where the inequality follows from (44). Moreover, for any alternative feasible trading statistics

$$\widehat{\xi} = (\widehat{\sigma}_0, \widehat{\sigma}_1^b, \widehat{\sigma}_1^g, \widehat{\beta}_0, \widehat{\beta}_1^b, \widehat{\beta}_1^g), \quad (47)$$

necessarily from (40) and (42),

$$Z_0 + \widehat{Y}_0 + \widehat{Y}_1^b = Z_0 + 1 - \widehat{\beta}_0 - \widehat{\beta}_1^b \geq Z_0 + 1 - \beta. \quad (48)$$

Thus, in view of (35), the trading statistics  $\xi$  maximizes  $W(\xi, P(\xi))$  under the feasibility constraint. It is claimed now that, conversely, any feasible trading statistics  $\xi$  that solves the social planner's problem in the case  $Z_0 < \beta$  must satisfy  $Y_0 + Y_1^b = 1 - \beta$  and  $Y_0 + Y_1^g = -Z_0$ . Since  $s \in (0; 1)$  by assumption, it suffices to

find a feasible trading statistics such that  $Z_0 + Y_0 + Y_1^g = 0$ , and such that (48) is an equality. Using the assumption  $Z_0 + 1 - \beta \geq 0$ , it is readily verified that

$$\xi^{\text{SB}} = (0, 1, 0, 0, \beta, Z_0), \quad (49)$$

which induces the vector of net demands

$$(Y_0, Y_1^b, Y_1^g) = (0, 1 - \beta, -Z_0), \quad (50)$$

has these properties. This completes the analysis of the first case. Assume now that  $Z_0 \geq \beta$ . Take any feasible trading statistics  $\xi$  such that

$$Y_0 + Y_1^b = 1 - \beta, \text{ and} \quad (51)$$

$$Y_0 + Y_1^g = -\min\{Z_0; \beta\} = -\beta. \quad (52)$$

Then  $Z_0 + Y_0 + Y_1^g \geq 0$ , and for any alternative feasible trading statistics  $\widehat{\xi}$ , from (36), (38), and (42),

$$Z_0 + \widehat{Y}_0 + \widehat{Y}_1^g = Z_0 + \widehat{\sigma}_0 + \widehat{\sigma}_1^g - \widehat{\beta}_0 - \widehat{\beta}_1^g \geq Z_0 - \beta. \quad (53)$$

Using (48), one concludes that  $\xi$  maximizes  $W(\xi, P(\xi))$  under the feasibility constraint. Conversely, if  $\xi$  solves the social planner's problem for  $Z_0 \geq \beta$ , then  $Y_0 + Y_1^b = 1 - \beta$  and  $Y_0 + Y_1^g = -\beta$ . Indeed, in this case,

$$\xi^{\text{SB}} = (0, 1, 0, 0, \beta, \beta) \quad (54)$$

induces the vector of net demands  $Y = (0, 1 - \beta, -\beta)$ . Moreover,  $\xi^{\text{SB}}$  is a feasible trading statistics satisfying  $Y_0 + Y_1^b = 1 - \beta$  and  $Y_0 + Y_1^g = -\beta$ . In view of (48) and (53), and because  $s \in (0, 1)$ , the solution  $\xi$  must indeed satisfy  $Y_0 + Y_1^b = 1 - \beta$  and  $Y_0 + Y_1^g = -\beta$ . This completes the analysis of the second case, and proves the first part of the lemma.

For the second part, consider a feasible trading statistics  $\xi$  such that  $\sigma_0 = \beta_0 = 0$ , and such that incentive compatibility conditions at date 1 hold at prices  $p(\xi)$ . By the first part, noting that  $Y_0 = 0$ , it suffices to show that  $Y_1^b = 1 - \beta$  and  $Y_1^g =$

$-\min\{Z_0; \beta\}$ . Consider first date-state 1b. From (42),  $\beta_1^b \leq \beta$ . Since  $\sigma_1^b = 1$  by (40), one arrives at  $Y_1^b = 1 - \beta_1^b \geq 1 - \beta$ . Hence,

$$2Z_0 + 2Y_0 + Y_1^b = Z_0 + (Z_0 + 1 - \beta_1^b) \geq 0. \quad (55)$$

If the inequality in (55) is strict, then by Lemma 1,  $P_1^b < V$ . Thus, by incentive compatibility for buyers at date-state 1b,  $\beta_1^b = \beta$  and indeed  $Y_1^b = 1 - \beta$ . If, however, (55) holds with equality, then necessarily  $Z_0 = 0$  and  $\beta_1^b = 1$ . But then, by assumption,  $\beta \leq 1$ , therefore  $\beta = \beta_1^b$  and hence,  $Y_1^b = 1 - \beta$ . Consider now date-state 1g. Summing up endowment constraints (43) and (44) for  $\omega = g$ , one obtains  $2Z_0 + 2Y_0 + Y_1^g \geq 0$ . Hence, by Lemma 1,  $P_1^g(\xi) \leq V$ . There are two cases. Assume first  $P_1^g(\xi) < V$ . Then, by incentive compatibility of buyers and sellers at date-state 1g,  $\sigma_1^g = 0$  and  $\beta_1^g = \beta$ . Hence,  $Y_1^g = -\beta$ , and the endowment constraint (44) implies  $Z_0 \geq \beta$ . Thus,  $Y_1^g = -\min\{Z_0, \beta\}$ . Assume now  $P_1^g(\xi) = V$ . Then,  $2Z_0 + 2Y_0 + Y_1^g = 0$  by Lemma 1. Hence,  $Y_1^g = -2Z_0$ . Via (44),  $Z_0 = 0$ , so that, trivially,  $Y_1^g = -\min\{Z_0, \beta\}$ . This proves also the second part of the lemma.  $\square$

Lemma 2 makes two assertions. The first says that a trading statistics is constrained efficient if and only if specialists accept forced liquidations net of the market's purchasing power at date-state 1b, and dissolve any risky position up to the market's purchasing power at date-state 1g. This first assertion will be exploited below to identify the set of parameter constellations that lead to a constrained efficient outcome. The second part of the result clarifies the relationship between constrained efficiency, as used here, and the benchmark outcome used in earlier work. It is found that any equilibrium in a model with trading prohibited at date 0 will result in the benchmark outcome. This makes the findings of the subsequent welfare analysis directly comparable to those of Bernardo and Welch (2004).

## 5. Nearly constrained efficient market runs

Given the characterization of constrained efficient trading, one can now derive necessary and sufficient conditions for the market equilibrium to be constrained efficient.

As mentioned before, it turns out that not all runs are socially undesirable. However, it is only runs that could possibly be socially harmful.

More specifically, the next result says that the market equilibrium is constrained welfare-maximizing provided the initial endowment held by the specialist sector exceeds a certain threshold. In particular, it shows that any non-run equilibrium for moderate purchasing power is constrained efficient.

**Lemma 3.** *Assume  $\beta < s$ . Then any market equilibrium is constrained efficient if and only if  $Z_0 \geq \widehat{Z}$ .*

**Proof.** Let  $\widehat{Z} = \min\{2Z^* - \beta, Z^*\}$  and  $Z_* = Z^* - \frac{1}{2}$ , as in the proof of Proposition 1. Recall that  $Z_* < \widehat{Z} \leq Z^*$ . Assume first  $Z_0 < Z_*$ . By Proposition B.1 and Lemma B.1 in the Appendix, the market equilibrium is characterized by the trading statistics  $\xi = (1, 0, 0, 0, \beta, \beta)$ . Hence,

$$Y_0 + Y_1^g = 1 - \beta > 0 \geq -\min\{Z_0; \beta\}. \quad (56)$$

Thus, using Lemma 2, the market equilibrium is not constrained efficient. Assume now  $Z_* \leq Z_0 \leq \widehat{Z}$ . Then Proposition B.1 in combination with Lemma B.2 implies that the market equilibrium is given by the trading statistics

$$\xi = (\sigma_0, 1 - \sigma_0, 0, 0, \beta, \beta), \quad (57)$$

where  $\sigma_0 = 2(Z^* - Z_0)$ . Therefore,  $Y_0 + Y_1^b = 1 - \beta$  and  $Y_0 + Y_1^g = \sigma_0 - \beta$ . By Lemma 2, constrained efficiency is tantamount to  $\sigma_0 - \beta = -\min\{Z_0; \beta\}$ . Using  $\sigma_0 = 2(Z^* - Z_0)$ , this implies that for  $\beta < Z_0$ , the equilibrium is constrained efficient if and only if  $Z_0 = Z^*$ . Moreover, in this case,

$$Z^* \geq \widehat{Z} \geq Z_0 > \beta, \quad (58)$$

so that  $2Z^* - \beta > Z^*$ , and consequently  $Z^* = \widehat{Z}$ . Thus, the equilibrium is constrained efficient in this case if  $Z_0 = \widehat{Z}$ , and not constrained efficient if  $Z_0 < \widehat{Z}$ . Similarly, for  $\beta \geq Z_0$ , equation  $\sigma_0 = 2(Z^* - Z_0)$  can be used to show that the equilibrium is

constrained efficient if and only if  $Z_0 = 2Z^* - \beta$ . The assertion follows therefore in the case  $\widehat{Z} = 2Z^* - \beta$ . On the other hand, for  $\beta \geq Z_0$  and  $\widehat{Z} = Z^* < 2Z^* - \beta$ , one finds that necessarily

$$Z_0 \leq \beta < Z^* = \widehat{Z} < 2Z^* - \beta, \quad (59)$$

and that the equilibrium cannot be constrained efficient, as claimed for such  $Z_0 < \widehat{Z}$ . By Proposition B.1, there is no equilibrium for any  $Z_0 > \widehat{Z}$  such that  $Z_0 \leq Z^*$  or  $Z_0 < \beta$ . Finally, assume that  $Z_0 > Z^*$  and  $Z_0 \geq \beta$ . Then, Proposition B.1 says that Lemma B.4 describes the market equilibrium. Hence,  $\xi = (0, 1, 0, 0, \beta, \beta)$ , and consequently,

$$Y_0 + Y_1^b = 1 - \beta, \text{ and} \quad (60)$$

$$Y_0 + Y_1^g = -\beta = -\min\{Z_0; \beta\}. \quad (61)$$

Thus, the market equilibrium is constrained efficient also in this case.  $\square$

Lemma 3 allows constructing robust scenarios of non-negligible financial market runs that have an arbitrarily small impact on economic welfare. The next result captures this point. The idea of the proof is to select parameter values close to a limit constellation that would be constrained efficient. From Proposition 1 and Lemma 3, it is known that a parameter constellation that allows a constrained welfare-maximizing run must satisfy  $Z_0 = \widehat{Z} < Z^*$ . One may then use sufficient conditions for equilibrium existence as well as continuity properties of the market equilibrium to construct robust examples of a nearly constrained efficient run. In fact, this construction is possible for runs of any size, provided that at least *some* sellers do not join the herd.

**Proposition 2.** *Let  $\overline{Y}_0 \in (0; 1)$  be any positive amount of unmatched premature liquidations. Then for any  $\overline{\Delta} > 0$ , there is an open set  $U \subset \mathbb{R}_+ \times (0; 1) \times \mathbb{R}_+$  of parameter constellations  $(\beta, s, Z_0)$  such that for any  $(\beta, s, Z_0) \in U$ ,*

*(i) there is a unique market equilibrium  $(\xi, P)$  for  $(\beta, s, Z_0)$ ,*



(ii) in this equilibrium,  $Y_0 > \bar{Y}_0$ , and

(iii) the welfare loss is smaller than  $\bar{\Delta}$ , i.e.,  $W^{\text{SB}} - W(\xi, P) < \bar{\Delta}$ .

**Proof.** Take  $\bar{Y}_0 \in (0; 1)$ . From Proposition B.1 in combination with Lemma B.2 it follows that for parameters  $(\beta, s, Z_0)$  satisfying  $\beta < s$  and  $Z_0 \in [Z_*; \hat{Z}]$ , there is a unique equilibrium trading statistics  $\xi = \xi(\beta, s, Z_0)$  for these parameters. Moreover, the interval  $[Z_*; \hat{Z}]$  has a non-empty interior for all  $\beta, s$  such that  $\beta < s$ . To ensure that  $Y_0 > \bar{Y}_0$  and  $W^{\text{SB}} - W(\xi, P) < \bar{\Delta}$  for equilibria characterized by  $\xi(\beta, s, Z_0)$ , one imposes additional restrictions on the set of parameter constellations, taking account of existence.

First, to guarantee  $Y_0 > \bar{Y}_0$  in  $\xi(\beta, s, \hat{Z})$ , one infers from the equilibrium characterization that the amount of unmatched premature liquidations in  $\xi(\beta, s, \hat{Z})$  is given by  $Y_0 = 2(Z^* - \hat{Z})$ . Since  $Z^* > \hat{Z}$  is possible only for  $Z^* < \beta$ , I focus on the non-empty set of parameters  $\beta, s$  such that  $Z^* < \beta < s$ . But then the definition of  $\hat{Z}$  implies  $\hat{Z} = 2Z^* - \beta$ , and consequently  $Y_0 = 2(\beta - Z^*)$ . Hence, under the current parameter restrictions,  $Y_0 > \bar{Y}_0$  is equivalent to

$$2\beta - \frac{s}{1-s}(1-\beta) > \bar{Y}_0. \quad (62)$$

Ignoring restrictions on parameters for the moment, (62) is clearly satisfied for parameters  $\beta, s$  if  $\bar{Y}_0 < \beta = s < 1$ . By continuity, there are constants  $\underline{\beta}, \bar{\beta}, \underline{s}, \bar{s}$  satisfying

$$\bar{Y}_0 < \underline{\beta} < \bar{\beta} < \underline{s} < \bar{s} < 1, \quad (63)$$

such that for all  $\beta \in [\underline{\beta}; \bar{\beta}]$  and for all  $s \in [\underline{s}; \bar{s}]$ , inequality (62) holds. Since (62) implies  $Z^* < \beta$ , one finds that, without qualification,  $Y_0 > \bar{Y}_0$  in equilibrium for all  $(\beta, s, \hat{Z})$  such that  $\beta \in [\underline{\beta}; \bar{\beta}]$  and  $s \in [\underline{s}; \bar{s}]$ . Moreover, for parameters  $(\beta, s, Z_0)$  such that  $Z_0 \in [Z_*; \hat{Z}]$ , this implies

$$Y_0 = 2(Z^* - Z_0) \geq 2(Z^* - \hat{Z}) > \bar{Y}_0, \quad (64)$$

as desired.

Second, one turns to the welfare dimension. It follows from Proposition 1 and Lemma 3 that for  $\widehat{Z} < Z^*$ , the trading statistics  $\xi(\beta, s, \widehat{Z})$  is a constrained efficient run. Let

$$\bar{\zeta} = \min_{\beta \in [\underline{\beta}; \bar{\beta}], s \in [\underline{s}; \bar{s}]} \{\widehat{Z} - Z_*\}. \quad (65)$$

Using

$$\widehat{Z} - Z_* = \frac{1}{2} \left( 1 - \beta + \frac{s - \beta}{1 - s} \right), \quad (66)$$

it is straightforward to check that  $\bar{\zeta} > 0$ . For

$$\zeta \equiv \widehat{Z} - Z_0 \in [0; \bar{\zeta}], \quad (67)$$

define the welfare loss

$$\Delta(\beta, s, \zeta) = W^{\text{SB}} - W(\xi, P). \quad (68)$$

The function  $\Delta(\cdot)$  is continuous on

$$K \equiv [\underline{\beta}; \bar{\beta}] \times [\underline{s}; \bar{s}] \times [0; \bar{\zeta}]. \quad (69)$$

Indeed, since both  $\beta < s$  and  $Z_* \leq Z_0 \leq \widehat{Z}$  hold within  $K$ , the equilibrium trading statistics

$$\xi = (2(Z^* - Z_0), 1 - 2(Z^* - Z_0), 0, 0, \beta, \beta) \quad (70)$$

is continuous in  $(\beta, s, Z_0)$ , and so is  $\Delta(\cdot)$ . As  $K$  is compact, the function  $\Delta(\cdot)$  is even uniformly continuous on  $K$  by the Heine-Cantor theorem. Hence, there is a  $\delta > 0$  such that for any  $(\beta, s, \zeta), (\beta', s', \zeta') \in K$  satisfying

$$d((\beta, s, \zeta), (\beta', s', \zeta')) < \delta, \quad (71)$$

necessarily

$$|\Delta(\beta, s, \zeta) - \Delta(\beta', s', \zeta')| < \bar{\Delta}, \quad (72)$$

where  $d(\cdot)$  denotes the euclidean distance. Choose now some  $\beta^* \in (\underline{\beta}; \bar{\beta})$ ,  $s^* \in (\underline{s}; \bar{s})$ , and define

$$\begin{aligned} U = \{ & (\beta, s, Z_0) \mid (\beta, s, \widehat{Z} - Z_0) \in \text{int}(K) \\ & \text{s.t. } d((\beta, s, \widehat{Z} - Z_0), (\beta^*, s^*, 0)) < \delta \}, \end{aligned} \quad (73)$$

where  $\text{int}(K)$  denotes the interior of the set  $K$ . Clearly  $U \neq \emptyset$ , and there is a unique equilibrium for any parameter constellation  $(\beta, s, Z_0) \in U$ . Moreover,  $Y_0 > \bar{Y}_0$  and  $W^{\text{SB}} - W(\xi, P) < \bar{\Delta}$  for any  $(\beta, s, Z_0) \in U$ .  $\square$

Proposition 2 shows that with risk-neutral purchasing power in the market, the allocational inefficiency resulting from a financial market run is not directly linked to the size of that run. This result stands in contrast with the intuitive perception of market runs as a clear signal for an inefficiency. As a theoretical limit case, one may even construct a market run that is constrained welfare-maximizing. In fact, the open sets constructed in the proof of Proposition 2 are all close to the parameter constellation that supports this limit as a market equilibrium. The next example illustrates the limit case of a constrained efficient run, and concludes the main part of the theoretical analysis.

**Example 2.** Consider a financial market with parameters

$$s = \frac{1}{3}, \beta = \frac{1}{4}, \lambda = 9, V = 10, \text{ and } Z_0 = \frac{1}{8}. \quad (74)$$

The assumptions of Lemma 2 hold clearly. Therefore, noting  $Z_0 < \beta$ , equation (35) implies that the benchmark welfare amounts to

$$W^{\text{SB}} = -\lambda\{s(Z_0 + 1 - \beta)^2 - (Z_0)^2\} = -\frac{69}{32}. \quad (75)$$

To determine the market equilibrium, note that

$$Z_* = -\frac{5}{16} < \frac{1}{8} = Z_0 = \hat{Z} < \frac{3}{16} = Z^*. \quad (76)$$

By Proposition B.1 in combination with Lemma B.2, the unique market equilibrium is given by the trading statistics

$$\xi = \left(\frac{1}{8}, \frac{7}{8}, 0, 0, \frac{1}{4}, \frac{1}{4}\right). \quad (77)$$

In particular,  $Y_0 > 0$ , i.e., there is a run. Constrained efficiency follows via Lemma

2 from

$$Y_0 + Y_1^b = \frac{3}{4} = 1 - \beta, \text{ and} \quad (78)$$

$$Y_0 + Y_1^g = -\frac{1}{8} = -Z_0. \quad (79)$$

Alternatively, equilibrium welfare can be directly calculated using equation (35) as  $W^* = -\frac{69}{32}$ . Thus, as predicted by Lemma 3, the run is constrained welfare-maximizing.

## 6. Strong purchasing power

This complementary section looks at the case of strong purchasing power, i.e.,  $\beta > s$ .<sup>11</sup> It is natural to expect that especially in this situation, the specialist sector may run out of inventory. The next example illustrates the problem.

**Example 3.** Let  $\varepsilon$  be small in absolute terms throughout this example, and consider exogenous parameters  $s = \frac{1}{4}$ ,  $\beta = \frac{1}{4} + \varepsilon$ , and  $\lambda = 9$ . It is assumed that  $Z_0 = 0$ , i.e., the specialist sector holds a zero inventory in the risky asset. Clearly,  $Z^* = \frac{1}{8} - \frac{\varepsilon}{6}$ , and  $Z_* < 0$ . For  $\varepsilon < 0$ , one notes that  $\hat{Z} = -\frac{4}{3}\varepsilon > 0$ . Hence, for  $\varepsilon < 0$ , Proposition B.1 in the Appendix combined with Lemma B.2 shows that there is a unique equilibrium characterized by the trading statistics

$$\xi(\varepsilon) = \left(\frac{1}{4} - \frac{\varepsilon}{3}, \frac{3}{4} + \frac{\varepsilon}{3}, 0, 0, \frac{1}{4} + \varepsilon, \frac{1}{4} + \varepsilon\right). \quad (80)$$

But for  $\varepsilon > 0$ , the trading statistics  $\xi(\varepsilon)$  does not correspond to a market equilibrium because at date-state 1g, a total of  $\beta_1^g = \frac{1}{4} + \varepsilon$  standby investors would submit buy orders, yet there would be no sell orders, i.e.,  $\sigma_1^g = 0$ . As a consequence, net order flow at date-state 1g would amount to  $Y_1^g = -\frac{1}{4} - \varepsilon$ . However, the market maker's inventory immediately prior to date 1 is only  $Y_0 = \sigma_0 - \beta_0 = \frac{1}{4} - \frac{\varepsilon}{3}$ . Thus, the net inventory position at date-state 1g would be  $Y_0 + Y_1^g = -\frac{4}{3}\varepsilon < 0$ , which is in conflict with Assumption 4. Thus,  $\xi(\varepsilon)$  does not describe a market equilibrium for  $\varepsilon > 0$ . In fact, as Proposition B.1 in the Appendix shows, there does not exist any market equilibrium for  $\varepsilon > 0$ .

The example above captures the intuitive conflict between the necessary inventory constraint and the momentum effect, as caused by competitive pricing in the specialist sector. The point to note is that if market prices exhibit a downwards trend, then good news at date-state 1g does not immediately push prices back to fundamental levels. Demand at the competitive price level may then exceed the inventory position of the specialist sector, so that orders submitted by standby investors could not be fulfilled.

However, as the next result shows, an equilibrium always exists provided that the specialist sector holds a sufficiently large initial endowment in the risky asset. More importantly, in any such equilibrium with strong purchasing power, sellers have no incentive to run the market.

**Proposition 3.** *Assume  $\beta > s$ . Then, for  $Z_0$  sufficiently large, there exists a unique market equilibrium. In any equilibrium,  $Y_0 \leq 0$  and the outcome is constrained efficient.*

**Proof.** Let  $Z_0 \geq \beta$ . Then by Proposition B.1 in the Appendix, there is a unique market equilibrium for  $\beta > s$ . Moreover, this equilibrium is given by Lemma B.6 and corresponds to  $\xi = (0, 1, 0, \beta, 0, 0)$ . Thus,  $Y_0 = -\beta \leq 0$ . For the equilibrium at hand,

$$Z_0 + Y_0 + Y_1^b = Z_0 + 1 - \beta, \text{ and} \quad (81)$$

$$Z_0 + Y_0 + Y_1^g = Z_0 - \beta. \quad (82)$$

Thus, the equilibrium is constrained efficient by Lemma 2. For  $Z_0 < \beta$ , one can see from Proposition B.1 in combination with Lemma B.3 that  $Z_0 \in \{0, 2Z^*\}$  and that  $Y_0 = -Z_0$ . Thus,  $Y_0 \leq 0$  also in these equilibria. Moreover,

$$Z_0 + Y_0 + Y_1^b = Z_0 + 1 - \beta, \text{ and} \quad (83)$$

$$Z_0 + Y_0 + Y_1^g = Z_0, \quad (84)$$

so that again by Lemma 2, constrained efficiency holds.  $\square$

Besides existence, Proposition 3 provides a necessary condition for financial market runs. Specifically, for a run to occur, it is not only necessary that purchasing power is too weak to compensate forced liquidations conditional on their occurrence. It is also necessary that purchasing power is too weak to compensate forced liquidations in expectation. This finding should be intuitive. Indeed, even if all buyers were to invest late, the resulting positive impact on the equilibrium price at date 0 would still more than compensate the impact of forced liquidations, so there is no incentive for sellers to run. Instead, there is an incentive for standby buyers to hurry to invest their cash endowment. The following example illustrates the equilibrium with strong purchasing power.

**Example 4.** Consider a market environment characterized by parameters  $s = \frac{1}{4}$ ,  $\beta = \frac{1}{3}$ ,  $\lambda = 9$ ,  $V = 18$ , and  $Z_0 = \frac{1}{2}$ . Then, Proposition B.1 and Lemma B.6 imply that  $\xi = (0, 1, 0, \frac{1}{3}, 0, 0)$  in combination with the price vector  $P = (12, 6, 15)$  constitutes the unique market equilibrium.

Thus, in contrast to the case with moderate purchasing power, buyers decide to enter the market early, and there are no premature liquidations.

## **Section 7. The Troubled Assets Relief Program**

This section reviews the initial plans of the U.S. Treasury to rescue battered financial markets via direct purchases of distressed assets towards the end of the year 2008. After a description of the subsequently released decision to renounce these plans, the events will be related to the findings from the theoretical analysis.

### *A. The initial plan*

In early October 2008, amidst serious concerns over the impact of the spreading credit crisis on global economic development, U.S. congress enacted a massive bailout plan for the financial sector, the so-called Emergency Economic Stabilization Act of 2008 (EESA). Established by the legislation was the so-called Troubled Assets Relief Program (TARP), an initiative launched by Henry M. Paulson, Jr., Secretary

of the Treasury, and a former Chairman and CEO of Goldman Sachs. The intention behind the plan has been to rescue nothing less than the U.S. financial market by permitting policy makers to purchase up to \$700 billion worth of distressed assets, primarily mortgage-backed securities, from financial institutions, and to eventually resell those assets to the private sector. Alongside with purchases, an insurance program would have to be established to guarantee troubled assets. In fact, at that time, the Treasury still considered purchases the most effective means of revitalizing credit markets.<sup>12</sup>

The terms of the plan have been left extremely unspecified and flexible. For instance, according to the fundamentals of the plan, troubled assets are defined not only as residential or commercial mortgages and any securities, obligations, or other instruments that were originated or issued no later than mid March 2008, but also as any other financial instrument for which the Treasury would decide that the purchase is needed to promote financial market stability. Similarly, eligible sellers are defined as U.S. financial institutions including but not limited to banks, savings associations, credit unions, broker-dealers, or insurance companies. In fact, the eligibility criterion covers also branches and agencies of foreign institutions, and rules out only central banks or institutions owned by foreign governments. Finally, concerning resale of troubled assets, the Treasury may exercise any rights received in connection with purchases, may manage troubled assets, and may sell or enter into any financial transaction in regard to such assets. In sum, provided that the general objective of financial market stability is promoted, there are almost no restrictions on investment under the plan.

Not only the extent of the Secretary's discretion, but also the budget available under the emergency plan is without example. The total of \$700 billion has been split into three installments. \$250bn were available for immediate deployment. Upon a presidential certification of need, the Treasury has received access to an additional \$100 billion. The remainder, i.e., another \$350 billion, has been made available

with the President's written report to Congress detailing the Secretary's further plans. To make the budget viable, an increase of government debt was considered appropriate. In fact, funding for the program is provided exclusively by domestic taxpayer dollars. Accordingly, the federal debt limit was increased, as part of the act, from \$10 trillion to \$11.3 trillion.

*B. The policy turn*

Already in mid October 2008, the Treasury announced the so-called Capital Purchase Program (CaPP), under which the Treasury would purchase up to \$250 billion of preferred securities from U.S. financial institutions on a voluntary basis. Soon afterwards, by October 26, 2008, an amount of \$115 billion had been invested into preferred securities of eight major financial institutions. Secretary Paulson indicated that a significant number of additional applications from banking institutions seeking investments under the CaPP had been received and were essentially approved. It will be noted by the reader that, through its focus on senior preferred shares, i.e., on the capital basis of financial institutions, the CaPP already implied a significant departure from the initial plan to support illiquid markets by direct purchases of troubled assets.

Mid November, Mr. Paulson openly declared a strategy change in the implementation of a rescue package. Specifically, the Treasury had decided that it considered direct purchases of illiquid assets at that time not as the most effective use of the remaining funds. Instead, the bulk of the left-over funds under the TARP were to be used to equip the financial system further with adequate capital.

It is remarkable that the Treasury changed its original plan in such a short period of time. Of course, it must be taken into account that the time frame has been extremely tight throughout the subprime crisis, and especially since September 2008. Secretary Paulson explained the change in the Treasury's views with reference to a considerable worsening of market conditions during the fortnight that was needed to push the legislation through political instances. On November 12, he said "It



was clear to me by the time the bill was signed on October 3rd that we needed to act quickly and forcefully, and that purchasing troubled assets – our initial focus – would take time to implement and would not be sufficient given the severity of the problem.”<sup>13</sup> As a matter of fact, setting up auctions and reverse auctions that would protect taxpayers’ interests as requested in the policy document turned out to be a non-trivial exercise. Still, the reference to timing issues alone may not explain in a fully satisfactory way why the entire idea of purchasing troubled assets was essentially abandoned overnight. After all, in September 2008, i.e., when the first version of the bill was sent to congress, the liquidity crisis had been lingering already for more than a year, so that the proposal of direct purchases has probably not been a blind one. Moreover, numerous difficult decisions were made by U.S. authorities in record time during that period.<sup>14</sup>

### *C. Relating the events to the theoretical analysis*

The theoretical model might shed some light on the Treasury’s policy turn, as discussed above. Specifically, there are two structural similarities that relate the developments in the actual crisis to the key features of the market equilibrium, as studied in Sections 2 through 5.

First, since the beginning of the subprime crisis in August 2007, there has apparently not been a general scarcity of cash-rich investors. Instead, anecdotal evidence suggests that in a very heterogeneous picture, certain long-term investors tended to have excess holdings of liquidity,<sup>15</sup> while another crowd of investors including investment houses, commercial banks, and hedge funds, was almost constantly in search of additional liquidity. In comparison, in the market equilibrium considered in the theoretical analysis, both buyers and sellers tend to hold too much liquidity at date 0, while the specialist sector holds too little cash. In addition, as has been shown in the analysis, an evaporation of liquidity may be unavoidable even if there is non-collusive purchasing power in the market capable of providing that liquidity. Thus, the evidence is broadly in line with the predictions of the model.

Second, the loss in market efficiency that motivated the Treasury's program is of a similar nature as the one featuring in the theoretical welfare analysis. In reality, credit markets clogged by asset illiquidity have led to an insufficient mobility of money and capital, especially to and from households and businesses. But also in a market equilibrium with anticipated forced liquidations, it is an allocational inefficiency that is calling for economic concern. Indeed, as in the real financial system under distress, the price for market risk increases and so investors have to incur a significant discount to get rid of unwanted securities before maturity. This, in turn, leads to a misallocation of risks in the model economy, in line with the evidence.

These similarities between actual developments and the predictions of the equilibrium analysis might rationalize the Treasury's reconsideration. In effect, the TARP may be considered as a regulatory request on tax payers to invest money in illiquid assets. It has been shown in the theoretical analysis that there are robust scenarios of market runs with standby liquidity in which the social planner may be unable to improve upon the unique market equilibrium. Provided that the Treasury perceived the potential efficiency gains from direct purchases as small, this might explain why there was a change in the strategy. Thus, the Treasury's decision to not acquire troubled assets through direct purchases might, at least in parts, be related to concerns that the efficiency gains obtainable through an orchestration of trading in illiquid market might be smaller than hitherto believed.

## **8. Conclusion**

The model of investor fear<sup>16</sup> has been extended by introducing a population of cash-rich investors, and by allowing the specialist sector to hold an ex-ante inventory in the risky asset. Like the sellers, members of the buyer population face uncertainty, and may either invest early or wait for the state of the world to realize. In the latter case, a buyer's investment may be conditional on the state. For the most interesting parameter domains, this model allows a unique, fully tractable market equilibrium.

Two main results have been obtained.

First, there are robust scenarios in which premature liquidations end up in the hands of risk-averse specialists, even though there is risk-neutral liquidity in the market that could easily compensate those liquidations. The adverse outcome is obtained despite each individual buyer having a negligible impact on the market price, which distinguishes the present analysis from Brunnermeier and Pedersen's (2005) predatory model. The finding implies that a much larger class of financial markets and instruments might be vulnerable to runs than suggested by previous analysis. Last but not least, the result is in line with the practitioner's blame on market liquidity as "disappearing just when investors need it most."<sup>17</sup>

Second, for almost any amount of unmatched premature liquidations, a robust scenario of a run could be found that is arbitrarily close, in terms of aggregate welfare, to the constrained efficient outcome. Thus, with risk-neutral buyers in the market, hasty liquidations are not necessarily an indication for an undesirable exaggeration in aggregate investor behavior. Instead, a run on a market may merely reflect an endogenous illiquidity of the asset in anticipation of forced liquidations. Moreover, in such scenarios, essentially no welfare gain can be expected from having a social planner orchestrate the trading. As discussed in the previous section in the context of the Treasury's Troubled Assets Relief Program (TARP), this finding might therefore shed light on intricate policy questions concerning the potential benefits of direct intervention in illiquid asset markets.

### **Appendix. Characterization of the equilibrium**

This appendix consists of two parts. After stating and proving a number of straightforward necessary conditions for equilibrium, the implications of incentive compatibility will be studied. The purpose of the analysis is to obtain a characterization of the market equilibrium. This characterization, achieved in Proposition B.1, is the basis of the proofs of the main results in the body of the paper, i.e., of Propositions 1 through 3.

## A. Necessary conditions

This part of the appendix lists a number of immediate consequences of the definition of the market equilibrium. All of these enter the subsequent analysis of incentive compatibility.

**Lemma A.1.** *In any market equilibrium,  $P_0 \leq V$  and  $P_1^\omega \leq V$  for  $\omega = b, g$ .*

**Proof.** Combining the endowment constraint  $Z_0 + Y_0 \geq 0$  with  $Z_0 \geq 0$  yields  $2Z_0 + Y_0 \geq 0$ . Hence, by (7),  $P_0 \leq V$ . Similarly, adding up endowment constraints  $Z_0 + Y_0 \geq 0$  and  $Z_0 + Y_0 + Y_1^\omega \geq 0$ , for  $\omega = b, g$ , delivers  $2Z_0 + 2Y_0 + Y_1^\omega \geq 0$ , so by (8), one finds  $P_1^\omega \leq V$  for  $\omega = b, g$ .  $\square$

**Lemma A.2.** *If  $P_1^g = V$ , then  $Y_0 = -Z_0$  and  $Y_1^g = 0$ .*

**Proof.** Combining  $P_1^g = V$  and (8) yields  $2Z_0 + 2Y_0 + Y_1^g = 0$ . The endowment constraint at date-state 1g says  $Z_0 + Y_0 + Y_1^g \geq 0$ . Hence,  $Z_0 + Y_0 \leq 0$ . Using the endowment constraint at date 0, one finds  $Z_0 + Y_0 = 0$ . Thus,  $Y_0 = -Z_0$ . The second assertion follows now from  $2Z_0 + 2Y_0 + Y_1^g = 0$ .  $\square$

**Lemma A.3.**  $P_1^b \leq P_1^g$ .

**Proof.** Assume to the contrary that  $P_1^g < P_1^b$ . Then, by Lemma A.1,  $P_1^g < V$ . Incentive compatibility for sellers and buyers at date-state 1g implies  $\sigma_1^g = 0$  and  $\beta_1^g = \beta - \beta_0$ . But, from non-negativity and population accounting,  $\sigma_1^b \geq 0$  and  $\beta_1^b \leq \beta - \beta_0$ . Hence,  $Y_1^g = \sigma_1^g - \beta_1^g \leq \sigma_1^b - \beta_1^b = Y_1^b$ . Equation (8) yields  $P_1^g \geq P_1^b$ . Contradiction.  $\square$

## B. Incentive compatibility

This part of the appendix analyzes the incentive compatibility conditions for buyers and sellers. As a result of the derivation, one arrives at the characterization of the market equilibrium.

The method applied in solving for the equilibrium is the following. Starting from price comparisons, one derives in a first step explicit restrictions both on the trading

statistics, the price vector, and on exogenous parameters. For instance, if the liquidation value  $P_0$  from early selling strictly exceeds the expected liquidation value  $V^S = sP_1^b + (1-s)V$  for selling late, then sellers adhering to incentive compatibility will choose to liquidate early, while buyers adhering to incentive compatibility will wait. Exploiting then pricing equations stated in Lemma 1, and using further arguments, one indeed arrives at relatively tight restrictions on trading statistics and the price vector. Moreover, combined with equilibrium conditions, restrictions on exogenous parameters can be derived. Luckily, this first step generates something very close to a partition of the parameter space. I.e., for a given constellation of exogenous parameters, there is typically at most one equilibrium candidate consistent with those parameters. It is therefore feasible to employ, in a second step, the principle of exclusion to arrive at a typically unique equilibrium candidate. For this candidate, sufficient conditions can then be checked in a straightforward way.

Beginning with the first step, the various cases will now be considered. While all of the subsequent auxiliary claims are needed to apply the principle of exclusion, the claims intuitively most important are Lemmas B.1, B.2, and B.4. These will correspond to the equilibria with moderate purchasing power. Lemma B.6 will capture the most interesting case with strong purchasing power. The other cases correspond to parameter constellations that are not explicitly discussed in the body of the paper. For the subsequent analysis, recall the notation  $Z^* = \frac{1}{2} \frac{s}{1-s} (1 - \beta)$ ,  $\hat{Z} = \min\{Z^*, 2Z^* - \beta\}$ , and  $Z_* = Z^* - \frac{1}{2}$ .

**Lemma B.1.** *Assume  $P_0 > V^S$ . Then  $\xi = (1, 0, 0, 0, \beta, \beta)$ . Prices are given by*

$$P_0 = V - \lambda(2Z_0 + 1), \text{ and} \tag{85}$$

$$P_1^\omega = V - \lambda(2Z_0 + 2 - \beta) \quad (\omega = b, g). \tag{86}$$

Moreover,  $\beta < s$  and  $Z_0 < Z_*$ .

**Proof.** By incentive compatibility for sellers at date 0,  $\sigma_0 = 1$ . From population accounting and non-negativity,  $\sigma_1^b = \sigma_1^g = 0$ . By Lemma A.1,  $P_1^g \leq V$ , so that

$P_0 > V^B$ . Hence, exploiting incentive compatibility for buyers at date 0,  $\beta_0 = 0$ . Thus,  $Y_0 = \sigma_0 - \beta_0 = 1$ , and from (7),  $P_0 = V - \lambda(2Z_0 + 1)$ . From Lemma A.1, also  $P_0 \leq V$ , so that using  $P_0 > V^S$ , one arrives at  $P_1^b < V$ . Using incentive compatibility for buyers at date-state 1b, this proves  $Y_1^b = -\beta_1^b = -\beta$ , hence by (8),

$$P_1^b = V - \lambda(2Z_0 + 2 - \beta). \quad (87)$$

But  $Y_1^g = -\beta_1^g \geq -\beta$ , so  $P_1^g \leq P_1^b$ . On the other hand,  $P_1^g \geq P_1^b$  from Lemma A.3. Thus,  $P_1^g = P_1^b$ . Plugging explicit expressions for prices into  $P_0 > V^S$  yields  $s(2Z_0 + 2 - \beta) > 1 + 2Z_0$ . Re-arranging, one obtains  $Z_0 < Z_*$ . Using  $Z_0 \geq 0$ , this also proves  $\beta < 1$ , and even  $\beta < s$ .  $\square$

**Lemma B.2.** *Assume  $P_0 = V^S$  and  $P_1^g < V$ . Then  $s > 0$  and*

$$\xi = (\sigma_0, 1 - \sigma_0, 0, 0, \beta, \beta),$$

where  $\sigma_0 = 2(Z^* - Z_0)$ . Prices are given by

$$P_0 = V - 2\lambda Z^*, \quad (88)$$

$$P_1^b = V - 2\lambda Z^*/s, \text{ and} \quad (89)$$

$$P_1^g = V - \lambda(4Z^* - 2Z_0 - \beta). \quad (90)$$

Moreover,  $\beta \leq s$  and  $Z_* \leq Z_0 \leq \widehat{Z}$ .

**Proof.** First, it is shown that  $s > 0$ . To provoke a contradiction, assume  $s = 0$ . Then  $P_0 = V$ , and therefore  $2Z_0 + Y_0 = 0$  by Lemma 1. Thus,  $Y_0 \leq 0$ . Since  $P_1^g < V$ , however,  $2Z_0 + 2Y_0 + Y_1^g > 0$  by another application of Lemma 1. Hence,  $Y_0 + Y_1^g > 0$ , and consequently  $Y_1^g > 0$ . By non-negativity,  $\sigma_1^g > 0$ , in contradiction to incentive compatibility for sellers at date-state 1g. Thus,  $s > 0$ . Next, restrictions will be derived, both on the trading statistics and the price vector. Since  $P_1^g < V$  by assumption,  $P_0 = V^S$  in conjunction with  $s < 1$  implies  $P_0 > V^B$ . By incentive compatibility for buyers at date 0,  $\beta_0 = 0$ . Moreover, from Lemma A.3, also  $P_1^b < V$ . Thus, via incentive compatibility for buyers at date 1,  $\beta_1^b = \beta_1^g = \beta$ . Similarly,

$\sigma_1^b = 1 - \sigma_0$  from forced liquidations, and  $\sigma_1^g = 0$  by incentive compatibility for sellers at date-state 1g. Thus,

$$(Y_0, Y_1^b, Y_1^g) = (\sigma_0, 1 - \sigma_0 - \beta, -\beta). \quad (91)$$

Using (7) and (8),  $P_0 = V^S$  implies

$$2Z_0 + \sigma_0 = s(2Z_0 + 1 + \sigma_0 - \beta). \quad (92)$$

Thus,  $\sigma_0 = 2(Z^* - Z_0)$ . Given that the trading statistics is determined, the price equations follow from (7) and (8). Finally, one derives necessary conditions on the parameters  $\beta$ ,  $s$ , and  $Z_0$ . Recall that  $\widehat{Z} = \min\{Z^*, 2Z^* - \beta\}$ . To prove  $Z_0 \leq 2Z^* - \beta$  and in particular  $s \geq \beta$ , start from the endowment constraint  $Z_0 + Y_0 + Y_1^g \geq 0$  and use the explicit expression for  $\sigma_0$ . The pair of inequalities  $Z_* \leq Z_0 \leq Z^*$  is a re-written form of  $0 \leq \sigma_0 \leq 1$ . This completes the proof of the lemma.  $\square$

**Lemma B.3.** *Assume  $P_0 = V^S$  and  $P_1^g = V$ . Then  $\beta \geq s$ . In fact,  $Z_0 = 0$  or  $Z_0 = 2Z^*$ .*

**Proof.** From  $P_1^g = V$  and Lemma A.2,  $Z_0 + Y_0 = 0$ . Hence, using (7),  $P_0 = V - \lambda Z_0$ . Combining this with  $P_0 = V^S$ , there are two cases. If  $s = 0$ , then  $P_0 = V$  and  $Z_0 = 0$ . Moreover, trivially,  $s \leq \beta$ . If  $s > 0$ , one obtains  $P_1^b = V - \lambda Z_0/s$ . Assume first  $Z_0 > 0$ . Then  $P_1^b < V$ , and therefore  $\beta_1^b = \beta - \beta_0$  by incentive compatibility of buyers at date-state 1b. Hence,  $Y_1^b = 1 - \beta + Z_0$ , and therefore

$$1 - \beta + Z_0 = \frac{Z_0}{s}. \quad (93)$$

Re-arranging gives  $Z_0 = 2Z^*$ . Moreover, by non-negativity and population accounting,

$$Z_0 = -Y_0 = \beta_0 - \sigma_0 \leq \beta_0 \leq \beta. \quad (94)$$

Thus,  $2Z^* \leq \beta$ , and therefore  $s \leq \beta$ . On the other hand, for  $Z_0 = 0$ , one finds  $P_0 = P_1^b = V$ . Hence,  $Y_0 = Y_1^b = 0$  by Lemma 1, so that by forced liquidation and population accounting,

$$0 = Y_0 + Y_1^b = 1 - \beta_0 - \beta_1^b \geq 1 - \beta. \quad (95)$$

Hence,  $\beta \geq s$  also for  $Z_0 = 0$ .  $\square$

**Lemma B.4.** *Assume  $V^B < P_0 < V^S$ . Then  $\xi = (0, 1, 0, 0, \beta, \beta)$ . Prices are given by*

$$P_0 = V - 2\lambda Z_0, \quad (96)$$

$$P_1^b = V - \lambda(2Z_0 + 1 - \beta), \text{ and} \quad (97)$$

$$P_1^g = V - \lambda(2Z_0 - \beta). \quad (98)$$

Moreover,  $\beta < s$ . Finally,  $Z_0 > Z^*$  and  $Z_0 \geq \beta$ .

**Proof.** Incentive compatibility for sellers and buyers at date 0 implies  $\sigma_0 = \beta_0 = 0$ . Immediate from the assumption,  $P_1^g < V$ . By Lemma A.3, also  $P_1^b < V$ . Thus, using forced liquidation and incentive compatibility for buyers and sellers at date 1,  $\sigma_1^g = 0$ ,  $\sigma_1^b = 1$ , and  $\beta_1^b = \beta_1^g = \beta$ . Pricing equations (7) and (8) lead to the explicit expressions for  $P_0$ ,  $P_1^b$ , and  $P_1^g$ . Necessary conditions  $\beta < s$  and  $Z_0 > Z^*$  follow from  $V^B < P_0 < V^S$ , exploiting explicit expressions for the prices. Finally, the endowment constraint at date-state 1g implies  $Z_0 \geq \beta$ .  $\square$

**Lemma B.5.** *Assume  $V^B = P_0 < V^S$ . Then  $s = \beta$ . Moreover,  $Z_0 \geq \beta$  and  $Z_0 > 0$ .*

**Proof.** From incentive compatibility for sellers at date 0,  $\sigma_0 = 0$ . The assumption implies  $P_1^g < V$ , and with Lemma A.3, also  $P_1^b < V$ . Incentive compatibility for buyers and sellers at date 1 yields  $\sigma_1^g = 0$  and  $\beta_1^b = \beta_1^g = \beta - \beta_0$ . Using forced liquidation,  $\sigma_1^b = 1$ . Thus,

$$(Y_0, Y_1^b, Y_1^g) = (-\beta, 1 - \beta + \beta_0, -\beta + \beta_0). \quad (99)$$

From  $V^B = P_0$ , it follows then using (7) and (8) that  $\beta = s$ . The claim  $Z_0 \geq \beta$  follows from the endowment constraint  $Z_0 + Y_0 + Y_1^g \geq 0$ . Finally,  $P_0 < V^S$  implies  $Z_0 > Z^* = \beta/2 \geq 0$ .  $\square$



**Lemma B.6.** *Assume  $P_0 < V^B$ . Then  $\xi = (0, 1, 0, \beta, 0, 0)$ . Prices are given by*

$$P_0 = V - \lambda(2Z_0 - \beta), \quad (100)$$

$$P_1^b = V - \lambda(2Z_0 - 2\beta + 1), \text{ and} \quad (101)$$

$$P_1^g = V - \lambda(2Z_0 - 2\beta). \quad (102)$$

Moreover,  $s < \beta \leq Z_0$ .

**Proof.** From incentive compatibility for buyers and sellers at date 0,  $\sigma_0 = 0$  and  $\beta_0 = \beta$ . Moreover, using population accounting and non-negativity,  $\beta_1^b = \beta_1^g = 0$ . By forced liquidation,  $\sigma_1^b = 1$ . It is claimed that  $\sigma_1^g = 0$ . Indeed, for  $P_1^g < V$ , this follows directly from incentive compatibility for sellers at date-state 1g. For  $P_1^g = V$ , Lemma A.2 implies  $Y_1^g = 0$ , so that  $\sigma_1^g = 0$  also in this case. Finally,  $P_1^g > V$  is not feasible by Lemma A.1. Thus,  $\sigma_1^g = 0$ , and in sum,  $\xi = (0, 1, 0, \beta, 0, 0)$ . The price vector follows now from Lemma 1. Using the price equations in  $P_0 < V^B$  and  $P_1^g \leq V$  delivers  $\beta > s$  and  $\beta \leq Z_0$ , respectively. Hence, the lemma.  $\square$

**Proposition B.1.** *For  $\beta < s$ , the market equilibrium is given by Lemma B.1 for  $Z_0 < Z_*$ , by Lemma B.2 for  $Z_* \leq Z_0 \leq \widehat{Z}$ , and by Lemma B.4 if both  $Z_0 > Z^*$  and  $Z_0 \geq \beta$  hold. For  $s < \beta \leq Z_0$ , the market equilibrium is given by Lemma B.6. In these cases the equilibrium is unique. There is no other market equilibrium for  $\beta < s$ . If  $\beta > s$ , there is also no other market equilibrium except for  $Z_0 \in \{0, 2Z^*\}$ .*

**Proof.** *Uniqueness and non-existence:* Fix  $s < 1$ . Then the conditions imposed on the price vector  $P = (P_0, P_1^b, P_1^g)$  in Lemmas B.1 through B.6, respectively, are pairwise exclusive. This is obvious for the case  $P_0 \geq V^S$ , which is covered by Lemmas B.1 through B.3. If  $P_0 < V^S$ , then either  $P_0 \geq V^B$ , covered by Lemmas B.4 and B.5, or  $P_0 < V^B$ , covered by Lemma B.6. This proves the exclusiveness claim. Next, it is claimed that any  $P \in \mathbb{R}^3$  must satisfy at least one of the given sets of conditions. But this is now immediate from  $P_1^g \leq V$ , i.e., from Lemma A.1. Therefore, applying the principle of exclusion, for a given parameter constellation

$$(\beta, s, Z_0) \in \mathbb{R}_+ \times [0; 1) \times \mathbb{R}_+, \quad (103)$$

necessary conditions on the price vector  $P$  and, in fact, also on the trading statistics  $\xi$  can be found by identifying those of Lemmas B.1 through B.6 that are inconsistent with  $(\beta, s, Z_0)$ . To start with, the respective predictions of Lemmas B.3, B.5, and B.6 are not consistent with  $\beta < s$ . Thus, only trading statistics and prices described by Lemmas B.1, B.2, and B.4 remain as equilibrium candidates in the case of moderate purchasing power. Considering now the respective restrictions on the parameter  $Z_0$ , and noting that  $\widehat{Z} \leq Z^*$  by definition, it follows that the market equilibrium in the case  $\beta < s$  is uniquely determined by Lemma B.1 for  $Z_0 < Z_*$ , by Lemma B.2 for  $Z_* \leq Z_0 \leq \widehat{Z}$ , and provided that  $Z_0 \geq \beta$ , by Lemma B.4 for  $Z_0 > Z^*$ . Moreover, there cannot be any other market equilibrium for  $\beta < s$ . Similarly, the predictions of Lemmas B.1, B.2, B.4, and B.5 are inconsistent with  $s < \beta$ . This leaves trading Lemmas B.3 and B.6 as potential descriptions of the equilibrium in the case of strong purchasing power. From the implications of these lemmas on the parameter  $Z_0$ , the equilibrium is uniquely determined by Lemma B.6 for  $s < \beta \leq Z_0$ . Indeed,  $s < \beta \leq Z_0$  is incompatible with  $Z_0 = 0$  and likewise incompatible with

$$Z_0 = 2Z^* = \frac{s}{1-s}(1-\beta), \quad (104)$$

so that the case captured by Lemma B.3 can be excluded. Thus, only the assumptions of Lemma B.6 are compatible with  $s < \beta \leq Z_0$ . With parameter values  $s < \beta$  and  $\beta > Z_0$ , however, only the assumptions of Lemma B.3 are compatible. But then, no market equilibrium is possible unless  $Z_0 \in \{0, 2Z^*\}$ .

*Existence:* Assume first that  $\beta < s$  and  $Z_0 < Z_*$ . It is claimed that the equilibrium candidate  $\xi = (1, 0, 0, 0, \beta, \beta)$ , as specified in Lemma B.1, characterizes a market equilibrium. Clearly, forced liquidation holds. Moreover,  $Z_0 + Y_0 \geq Y_0 > 0$ , and

$$Z_0 + Y_0 + Y_1^\omega \geq Y_0 + Y_1^\omega = 1 - \beta > 1 - s > 0 \quad (105)$$

for  $\omega = b, g$ . Thus, also the inventory conditions are satisfied. Incentive compatibility conditions for both buyers and sellers at date 0 are tantamount to  $P_0 \geq V^S$ .

Using price formulas stated in Lemma B.1, this inequality transforms into  $Z_0 \leq Z_*$ , which holds by assumption. Incentive compatibility conditions for sellers and buyers at date 1 follow immediately from  $P_1^\omega \leq V$  for  $\omega = b, g$ . Thus,  $\xi$  indeed describes the market equilibrium. Assume now that  $\beta < s$  and  $Z_* \leq Z_0 \leq \widehat{Z}$ . It is claimed that, as specified in Lemma B.2, the trading statistics

$$\xi = (\sigma_0, 1 - \sigma_0, 0, 0, \beta, \beta) \quad (106)$$

with  $\sigma_0 = 2(Z^* - Z_0)$  characterizes a market equilibrium. Forced liquidation is obviously satisfied. The inventory condition at date 0 reads

$$Z_0 + Y_0 = 2Z^* - Z_0 \geq \beta \geq 0. \quad (107)$$

The inventory condition at date-state 1b holds as well because

$$Z_0 + Y_0 + Y_1^b = Z_0 + 1 - \beta \geq 0. \quad (108)$$

The condition for date-state 1g is likewise satisfied via

$$Z_0 + Y_0 + Y_1^g = 2Z^* - Z_0 - \beta \geq 0. \quad (109)$$

Incentive compatibility follows from the price formulas given in Lemma B.2. Specifically, incentive compatibility for sellers at date 0 holds since  $P_0 = V^S$ . To prove incentive compatibility for buyers at date 0 and incentive compatibility for both buyers and sellers at date-state 1g, it suffices to show that  $P_1^g \leq V$  for the price

$$P_1^g = V - \lambda(4Z^* - 2Z_0 - \beta).$$

But this follows from  $Z_0 \leq \widehat{Z}$ . Finally, incentive compatibility for buyers at date-state 1b follows from  $Z^* \geq 0$  and from  $P_1^b = V - 2\lambda Z^*/s$ . Thus, again, the market equilibrium has been identified. Assume next that  $\beta < s$ ,  $Z_0 > Z^*$ , and  $Z_0 \geq \beta$ . Consider the equilibrium candidate  $\xi = (0, 1, 0, 0, \beta, \beta)$ , as specified in Lemma B.4. Forced liquidation as well as the endowment constraint  $Z_0 + Y_0 \geq 0$  are trivially satisfied. The endowment constraint  $Z_0 + Y_0 + Y_1^g \geq 0$  at date-state 1g follows

from  $Z_0 \geq \beta$ . Since  $Y_1^b \geq Y_1^g$ , this also validates the endowment constraint in the bad state. It remains to check incentive compatibility. Using the price formulas given in Lemma B.4, it is straightforward to verify that  $\beta < s$  and  $Z_0 > Z^*$  imply  $V^B < P_0 < V^S$ . This proves incentive compatibility for sellers and buyers at date 0. To verify the remaining conditions for date 1, it suffices to show that  $P_1^b \leq V$  and  $P_1^g < V$ . But indeed, under the current restrictions,

$$2Z_0 - \beta \geq Z_0 > Z^* \geq 0. \quad (110)$$

Hence,  $\xi$  is indeed incentive compatible at prices  $P(\xi)$ . Thus, the market equilibrium has again been determined, concluding the analysis of the case of moderate purchasing power. Assume finally that  $s < \beta \leq Z_0$ . It is claimed that  $\xi = (0, 1, 0, \beta, 0, 0)$ , as specified in Lemma B.6, corresponds to a market equilibrium for these parameter values. But indeed, forced liquidation and inventory conditions can be readily verified. Moreover, via price equations given in Lemma B.6,  $s < \beta$  implies  $P_0 < V^B$ . This proves incentive compatibility for buyers. Similarly,  $\beta \leq Z_0$  implies  $P_1^g \leq V$ , so that incentive compatibility for sellers at date-state 1g holds. Moreover,  $P_0 < V^B$  and  $P_1^g \leq V$  imply  $P_0 < V^S$ , so that incentive compatibility for sellers at date 0 is verified. Thus, the market equilibrium has been determined also in the case of strong purchasing power.  $\square$

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## Footnotes

1) Bernardo and Welch (2004, p. 137) write: “Our assumption precludes the presence of enough standby investors who could eliminate any time lag between the exit of liquidity-shocked investors and the entry of more market-making capacity.”

2) To be clear, I am not claiming that market runs are first-best efficient. Instead, it will be shown that there is no guarantee that a measurable welfare gain is achieved by having a central agency administer investor’s trading decisions *subject to second-best constraints*.

3) This term is borrowed from Borio’s (2004) insightful description of market distress.

4) Anecdotal evidence confirms this description of cash-rich investors during market down-turns. For instance, the Financial Times (“The buyer of last resort,” October 29, 2008) begins an article with the question “Where have all the buyers gone? As stock markets around the world are crashed by the big unwind, few investors seem willing to buy on the mother of all dips...” and concludes “...The cash is there, but it will remain sidelined until the shadow of systemic risk is gone.”

5) Weill (2007) studies optimal inventory policies for market makers *following* a crash. In contrast, the present analysis considers conditions that favor a crash.

6) My analysis draws on a zero-inventory example discussed by Ewerhart and Valla (2007). However, neither the equilibrium characterization nor the general welfare analysis is part of that earlier work.

7) The present set-up reduces to the one studied by Bernardo and Welch (2004) for  $\beta = 0$  and  $z_0 = 0$ .

8) Empirically, the momentum anomaly is a widely documented phenomenon (see, e.g., Jegadeesh and Titman, 1993).

9) I.e., from the conditions for the market equilibrium, only incentive compatibility is dropped.

- 10) If  $\beta > Z_0 + 1$ , then constrained efficient trading just empties the inventory of the specialists also in the bad state. However, this point is not needed in the sequel.
- 11) The analysis of the case  $\beta = s$  does not yield additional insights, and is therefore omitted.
- 12) This view has not been consensual, though. See, e.g., Diamond et al. (2008).
- 13) This quote is taken from “Remarks by Secretary Henry M. Paulson, Jr on the Financial Rescue Package and Economic Update,” Press Release of the U.S. Department of the Treasury, November 12, 2008. All press releases issued by the Treasury can be found on [www.treasury.gov](http://www.treasury.gov).
- 14) Officially, the Treasury has continued to examine the relative merits of purchasing illiquid mortgage-related assets. However, as is evident from the statement made by Paulson on November 12, new priorities are clearly set on (a) capital enhancement for banks and non-banks, (b) securitization and consumer credit, and (c) limiting foreclosures. At the time of writing (January 2009), these priorities were still in place.
- 15) For instance, countries such as India and China have been mentioned in the media as potential investors of last resort.
- 16) Cf. Bernardo and Welch (2004).
- 17) See, e.g., Swenson (2000).



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