

# Are We Extracting the True Risk Neutral Density from Option Prices? A Question with No Easy Answer

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James Huang

Department of Accounting and Finance

Lancaster University Management School

Lancaster, UK. LA1 4YX

Tel: +(44) 1524 593633

Fax: +(44) 1524 847321

Email: [James.huang@lancaster.ac.uk](mailto:James.huang@lancaster.ac.uk)

## ABSTRACT

In this paper we raise a question on the theoretical foundation of option implied risk neutral density. We prove that given any number of options, there exist numerous risk neutral densities which are piecewise constant, have only two values, either a lower bound or an upper bound on the true risk neutral density, and price all these options correctly. Similar results are proved with respect to the true risk neutral density's derivatives. These results show how difficult it is to ensure that the risk neutral density we extract from option prices is the true one and how large estimation errors can be.

Keywords: risk neutral density, option pricing, estimation errors, ill-posedness.

JEL Classification Numbers: C13, C14, G13.

## Introduction

There is an extensive literature on option implied risk neutral densities. Ross (1976) is the first to show that we can recover the risk neutral density (hereafter RND) from a complete set of European option prices. Breeden and Litzenberger (1978) are the first to put this idea into practice and show that the RND is equal to the second derivative of option prices with respect to the strike price. Following their seminal work, there have developed a large collection of studies on this topic.<sup>1</sup>

The estimated RNDs have been used in many ways. For example, they are used to price derivatives with the same time to expiration. They are also used to assess the market beliefs about important political and economic events (see, for example, Campa and Chang (1996, 1998), Brenner et al. (1996), McCauley and Melick (1996), Malz (1997), Melick and Thomas (1997), Söderlind (2000), Jondeau and Rockinger (2000), Coutant et al. (2001)). Some authors use them to test market rationality (see, for example, Bondarenko (1997, 2002)).

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<sup>1</sup>See, for example, Jarrow and Rudd (1982), Shimko (1993), Rubinstein (1994), Longstaff (1995), Corrado and Su (1996), Jackwerth and Rubinstein (1996), Melick and Thomas (1997), Ait-Sahalia and Lo (1998), Campa et al. (1998), Rosenberg (1998), Rubinstein (1998), Jackwerth (1999), Jondeau and Rockinger (2001), Rosenberg and Engle (2002), Panigirtzoglou and Skiadopoulos (2004), Yatchew and Härdle (2006)). Jackwerth (1999), Jondeau and Rockinger (2000), and Bliss and Panigirtzoglou (2002) provide excellent reviews over different methods used to extract RNDs from option prices.

Some others use them to manage risk (see, for example, Aït-Sahalia and Lo (2000) and Berkowitz (2001)). Many other authors use option implied RNDs to estimate investors' risk preferences (see, for example, Aït-Sahalia and Lo (2000), Jackwerth (2000), Rosenberg and Engle (2002), Bliss and Panigirtzoglou (2004), Bakshi et al. (2003), and Liu et al. (2007)).

As option implied RNDs are ever more widely used in finance, there is a danger that enough caution is not taken about the limit on what option prices can tell us about the true RND. Although the true RND can be recovered from a *complete* set of (European) option prices, we can never really have such a *complete* set of option prices. After all, strike prices are set at discrete intervals. For example, strikes for the S&P 500 Index options are usually spaced \$5 apart.<sup>2</sup> Thus the uniqueness of option implied RNDs can never be guaranteed. It is surprising that this flaw in the theoretical foundation of option implied RNDs has been largely ignored in the literature.<sup>3</sup>

In this paper we raise this issue and ask the following question: can we really extract the true RND and its derivatives from option prices? We prove that given any number of options, there exist an uncountable number of different RNDs which are piecewise constant, have only two values, either a lower bound

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<sup>2</sup>In fact, as is pointed out by Bondarenko (2003), “in a typical application, the conditional RND is estimated from a cross-section of 10 to 30 option prices (for a given maturity).”

<sup>3</sup>An exception is Kargin (2004) who points out that the problem of extracting RND from option prices is ill-posed. Bondarenko (2003) briefly mentions the ill-posedness of option implied RND while discussing issues in estimating option implied RND. Jackwerth (1999) also cautions against “reading too much information into the graphs of risk-neutral distributions,” however, his concern is more about simple estimation errors.

or an upper bound on the true RND, and price all these options correctly. This shows that however many options we have in the market, there are always numerous RNDs which are consistent with the prices of all these options and the differences between these RNDs and the true RND can be arbitrarily large.

We also prove that given any number of options, there exist an uncountable number of different RNDs consistent with the prices of all these options whose first derivatives are piecewise constant and have only two values, either a lower bound or an upper bound on the true RND's first derivative. This shows that however many options we have in the market, there are always numerous RNDs which price all these options correctly and the differences between the first derivatives of these RNDs and the true RND can be arbitrarily large.

Moreover, define the 0th order elasticity of an RND as itself and its  $N$ th order elasticity as the elasticity of its  $(N - 1)$ th order elasticity. We prove that given any number of options, there exist an uncountable number of different RNDs consistent with the prices of all these options whose  $N$ th order elasticities are piecewise constant and have only two values, either a lower bound or an upper bound on the true RND's  $N$ th order elasticity.

These results show how difficult it is to ensure that the risk neutral density we extract from option prices is the true one and how large estimation errors can be. The implications of these results for the literature on option implied RNDs can not be ignored. The key point raised by these results is that we must justify why a single RND which is estimated from a set of option prices using a particular method should be the true RND while it is only one of the numerous

RNDs consistent with these option prices.

The structure of the paper is as follows: In Section 1 we introduce the theoretical foundation of option implied RNDs laid by Ross (1976) and Breeden and Litzenberger (1978). In Section 2 we show the existence of RNDs which have extreme forms and price any number of options correctly when the true RND is bounded. In Section 3 we show the existence of RNDs which have extreme forms and price any number of options correctly when the true RND's first derivative is bounded. In Section 4 we show the existence of RNDs which have extreme forms and price any number of options correctly when the true RND's elasticity is bounded. In Section 5 we show the existence of RNDs which have extreme forms and price any number of options correctly when the true RND's  $N$ th order elasticity is bounded. The final section concludes the paper.

## 1 RNDs and Option Prices

Let current time be 0 and a time in the future be  $t$ . Assume the risk free interest rate for this period is  $r_t$ . Then the time zero price of a risk-free unit zero coupon bond (hereafter unit bond) matures at  $t$  is  $B_0 = \frac{1}{1+r}$ . Assume there is a stock in the economy with no dividend.<sup>4</sup> The stock's time 0 and time  $t$  prices are denoted by  $S_0$  and  $S$  respectively. Assume there are a complete set of (European) options written on the stock which are traded in the market. Denote the time 0 price and time  $t$  payoff of the option with strike price  $K$  by  $c_0(K)$  and  $c(S; K)$  respectively.

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<sup>4</sup>The model can be routinely extended to the case with dividends

Assume there are no arbitrage opportunities in the market. Then according to the well-known option pricing theory, there exists a unique risk neutral probability measure (or equivalent Martingale measure)  $Q$  such that  $S_0 = B_0 E^Q(S)$  and

$$c_0(K) = B_0 E^Q(c(S; K)).$$

Assume risk neutral probability measure  $Q$  is represented by a RND  $q(S)$ . We call it the true RND in this paper. Then we have

$$c_0(K) = B_0 \int_{\Omega} q(S) c(S; K) dS, \quad (1)$$

where  $\Omega$  is the support of the underlying stock price distribution.

According to Ross (1976), we can recover the true RND if we have a complete set of option prices. Let  $c_0(K)$  denote the time-zero price of the option with strike price  $K$ . Breeden and Litzenberger (1978) show that

$$q(S) =_{a.s.} \frac{d^2 c_0(K)}{dK^2} \Big|_{K=S}, \quad (2)$$

where  $q(S)$  is unique up to a zero-measure set. In applied mathematics, problem (1) is usually called the original problem while problem (2) is usually called the inverse problem. It is well known that the original problem (1) is well-posed while the inverse problem (2) is ill-posed.<sup>5</sup>

In reality it is impossible to have a complete set of option prices. Strike prices of options are set at discrete intervals. Thus the problem we want to solve in the literature on option implied RNDs is to extract the true RND  $q(S)$  from the prices of a series of options. Now suppose we observe the prices of

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<sup>5</sup>See, for example, Engl et al. (2000).

a series of options with strike prices  $K^i$ ,  $i = 1, 2, \dots$ . These strike price are numbered so that  $K_i$  is increasing in  $i$ . We denote the time 0 price and time  $t$  payoff of the  $i$ th option by  $c_0^i$  and  $c^i(S)$  respectively. The question we ask in this paper is: can we extract the true RND and its derivatives from a series of option prices?

## 2 RND is Bounded

In most models of option implied RNDs, the estimated RNDs usually have an upper bound. Now assume that it is true that as in these models the true RND is bounded above by a constant  $\bar{q} > 0$  and below by 0. Assume the support of the true RND is  $[s_0, s_M] \subset R^+$ . Given the prices of the unit bond, the underlying stock, and  $n$  options,  $B_0, S_0, c_0^1, \dots, c_0^n$  respectively, for any admissible pricing kernel  $\phi(S)$  because it prices the stock and the  $n$  observed options correctly, it must satisfy the following admissibility conditions.

$$E(\phi(S)) = 1,$$

$$B_0 E(S\phi(S)) = S_0,$$

$$B_0 E(c^i(S)\phi(S)) = c_0^i, \quad i = 1, \dots, n.$$

We first introduce three lemmas.

**Lemma 1** *Assume two RNDs intersect once, then the RND with fatter left tail gives lower stock price.*

Proof: Let  $q_1(S)$  and  $q_2(S)$  be the two RNDs which intersect once. Without

loss of generality assume  $q_2(S)$  has fatter left tail. Thus there exists  $s_1$ , which is an interior point of the common support of the two RNDs, such that

$$\begin{aligned} q_1(S) - q_2(S) &\leq 0, & S < s_1 \\ q_1(S) - q_2(S) &\geq 0, & S > s_1 \end{aligned}$$

We have

$$\int_{s_0}^{s_M} [S(q_1(S) - q_2(S))]dS = \int_{s_0}^{s_M} [(S - s_1)(q_1(S) - q_2(S))]dS \geq 0.$$

Hence  $q_2(S)$ , which has fatter left tail than  $q_1(S)$ , gives lower stock price. Q.E.D.

**Lemma 2** *Assume two pricing kernels give the same stock price. If they intersect twice, then the pricing kernel with fatter tails gives higher prices of convex payoff contingent claims written on the stock.*

Proof: See Huang (2004a) or Franke et al. (1999).

**Lemma 3** *Assume two RNDs give the same prices of the underlying stock and options with strike prices  $K_1, K_2, \dots, K_n$ , where  $K_1 < K_2 < \dots < K_n$ . Let  $K_0 = s_0$  and  $K_{n+1} = s_M$ . If the two RNDs intersect  $n + 2$  times then the one with fatter left tail will give higher (lower) prices for all options with strike prices between  $(K_{2i-2}, K_{2i-1})$  ( $(K_{2i-1}, K_{2i})$ ),  $i = 1, 2, \dots$*

Proof: See Huang (2004a).

Let  $q_n^{**}(S)$  be a RND which is  $(n + 3)$ -segment and piecewise constant, has value  $\bar{q}$  at odd segments and minimum value zero at even segments, and prices the stock and  $n$  observed options correctly. Thus we have  $q_n^{**}(S) = \bar{q}$ , for

$S \in (s_{n,j}^{**}, s_{n,j+1}^{**})$  and  $j$  is even, and  $q_n^{**}(S) = 0$ , for  $S \in (s_{n,j}^{**}, s_{n,j+1}^{**})$  and  $j$  is odd, where  $0 \leq j \leq n+3$ ,  $s_{n,0}^{**} = s_0$ ,  $s_{n,n+3}^{**} = s_M$ , and  $s_{n,j}^{**}, j = 1, \dots, n+2$ , are determined by the  $n+2$  equations for admissibility.

Let  $q_n^*(S)$  be a RND which is  $(n+3)$ -segment and piecewise constant, has value zero at odd segments and value  $\bar{q}$  at even segments, and prices the stock and  $n$  observed options correctly. Thus we have  $q_n^*(S) = 0$ , for  $S \in (s_{n,j}^*, s_{n,j+1}^*)$  and  $j$  is even, and  $q_n^*(S) = \bar{q}$ , for  $S \in (s_{n,j}^*, s_{n,j+1}^*)$  and  $j$  is odd, where  $0 \leq j \leq n+3$ ,  $s_{n,0}^* = s_0$ ,  $s_{n,n+3}^* = s_M$ , and  $s_{n,j}^*, j = 1, \dots, n+2$ , are determined by the  $n+2$  equations for admissibility.

Let  $q_n^+(S, a)$  be a RND which has the same form as  $q_{n+1}^{**}(S)$ , satisfies  $s_{n+1,n+3}^{**} = a \in [s_{n,n+2}^*, s_M]$ , and prices the underlying stock and  $n$  options with strikes  $K_1, \dots, K_n$  correctly.

Let  $q_n^-(S, b)$  be a RND which has the same form as  $q_{n+1}^*(S)$ , satisfies  $s_{n+1,1}^* = b \in [s_0, s_{n,1}^*]$ , and prices the underlying stock and  $n$  options with strikes  $K_1, \dots, K_n$  correctly. We present the following result.

**Theorem 1** *Assume the support of the stock price distribution is  $[s_0, s_M]$ . Assume the underlying asset and  $T$  options with strike prices  $K_1 < K_2 < \dots < K_T$ ,  $T \geq 1$ , are priced correctly by a bounded RND. Then for all  $T = 0, 1, \dots$ ,  $q_T^{**}(S)$  and  $q_T^*(S)$  exist. For all  $T = 0, 1, \dots$ , all  $a \in [s_{n,n+2}^*, s_M]$ , all  $b \in [s_0, s_{n,1}^*]$ ,  $q_T^+(S, a)$  and  $q_T^-(S, b)$  exist. Moreover,  $q_T^{**}(S)$  ( $q_T^*(S)$ ) gives the optimal upper (lower) bound on options with strike prices between  $K_{2i}$  and  $K_{2i+1}$ , where  $0 \leq 2i \leq T$ , while  $q_T^{**}(S)$  ( $q_T^*(S)$ ) gives the optimal lower (upper) bound on options with strike prices between  $K_{2i-1}$  and  $K_{2i}$ , where  $0 < 2i - 1 \leq T$ .*

Proof: We prove the theorem by induction. Let the price of the unit bond be  $B_0$ , the underlying stock price be  $S_0$ , and the prices of  $n$  options with strike prices  $K_1, K_2, \dots, K_n$  be  $c_0^1, c_0^2, \dots$ , and  $c_0^n$  respectively. When necessary we write  $q_T^{**}(S)$  and  $q_T^*(S)$  which have forms specified in Theorem 1 and price the underlying stock and the  $T$  options correctly explicitly as

$$q_T^{**}(S; s_0, s_M, K_1, \dots, K_T, S_0, c_0^1, \dots, c_0^T) \text{ and}$$

$$q_T^*(S; s_0, s_M, K_1, \dots, K_T, S_0, c_0^1, \dots, c_0^T).$$

We first prove that when  $T = 0$  the theorem is true. Let  $q_{00}^{**}(S)$  be a RND such that  $q_{00}^{**}(S) = \bar{q}$ , for  $S < s_{0,0,1}^{**}$ , and  $q_{00}^{**}(S) = 0$ , for  $S > s_{0,0,1}^{**}$ . Let  $q_{00}^*(S)$  be a RND such that  $q_{00}^*(S) = 0$ , for  $S < s_{0,0,1}^*$ , and  $q_{00}^*(S) = \bar{q}$ , for  $S > s_{0,0,1}^*$ . The existence of the above two RNDs is obvious. Moreover, as  $q_{00}^{**}(S)$  intersect the true RND once and has fatter left tail, from Lemma 1 we conclude that it under-prices the stock. Using the same argument we conclude that  $q_{00}^*(S)$  over-prices the stock.

We now show the existence of  $q_0^{**}(S)$ . It is straightforward that given any  $s_{0,1} \in [s_0, s_{0,0,1}^{**}]$ , there always exists  $s_{0,2} \geq s_{0,1}$  such that  $\tilde{q}(S; s_{0,1})$  is a RND, where  $\tilde{q}_0(S; s_{0,1}) = \bar{q}$ ,  $S < s_{0,1}$ ;  $\tilde{q}_0(S; s_{0,1}) = 0$ ,  $s_{0,1} < S < s_{0,2}$ ;  $\tilde{q}_0(S; s_{0,1}) = \bar{q}$ ,  $S > s_{0,2}$ .

We have  $\tilde{q}(S; s_0) = q_{00}^{**}(S)$  under-prices the stock while  $\tilde{q}(S; s_{0,0,1}^{**}) = q_{00}^*(S)$  over-prices the stock. Hence we conclude that there exists  $s_{0,1} \in [s_0, s_{0,0,1}^{**}]$  such that  $q_0^{**}(S) = \tilde{q}(S; s_{0,1})$  is a RND which prices the stock correctly.

The existence of  $q_0^*(S)$  can be similarly proved.

It is obvious that the true RND can only intersect  $q_0^{**}(S)$  and  $q_0^*(S)$  at most twice. But since they all price the stock correctly, from Lemma 2,  $q(S)$  must intersect  $q_0^{**}(S)$  and  $q_0^*(S)$  at least twice. Thus  $q(S)$  must intersect  $q_0^{**}(S)$  and  $q_0^*(S)$  exactly twice. Now applying Lemma 2, we conclude that  $q_0^*(S)$  underprices all options while  $q_0^{**}(S)$  over-prices them.

We now prove the existence of  $q_0^+(S, a)$  and  $q_0^-(S, b)$ .

Note  $q_0^+(S, a)$  is a RND which has the same form as  $q_1^{**}(S)$ , satisfies  $s_{1,3}^{**} = a \in [s_{0,2}^*, s_M]$ , and prices the underlying stock correctly. Given any decreasing pricing kernel  $q(S)$  which price the underlying stock correctly, we have

$$\begin{aligned} \int_{s_0}^{s_M} q(S) dS &= 1, \\ B_0 \int_{s_0}^{s_M} Sq(S) dS &= S_0. \end{aligned}$$

Let  $q(S) \equiv q_0^*(S)$ . It follows that

$$\begin{aligned} \int_{s_0}^a q(S) dS &= 1, \\ B_0 \int_{s_0}^a Sq(S) dS &= S_0. \end{aligned}$$

Now consider the case where the support of the stock price distribution is  $[s_0, a]$ .

Let  $\hat{q}(S)$  be the truncated probability density function of  $q(S)$  defined on  $[s_0, a]$ , i.e.,  $\hat{q}(S) \equiv q(S)$ ,  $S \in [s_0, a]$ . As the stock price  $S_0$  is given by a bounded RND  $\hat{q}(S)$ , according to the previously made assumption, we must have a RND which is defined on  $[s_0, a]$ , has the specified form as

$$q_0^{**}(S; s_0, a, S_0),$$

and gives the same price for the underlying stock as  $\hat{q}(S)$ . Now let

$$q_0^+(S, a) = \begin{cases} q_0^{**}(S; s_0, a, S_0), & S \geq a \\ 0, & S < a \end{cases}$$

Straightforward calculations show that this RND has the specified form as  $q_0^+(S, a)$  and price the underlying stock correctly. This proves the existence of  $q_0^+(S, a)$  for all  $a \in [s_{0,2}^*, s_M]$ . The existence of  $q_0^-(S, b)$  for all  $b \in [s_0, s_{0,1}^*]$  can be similarly proved.

Assume that given  $n \geq 1$ , the theorem is true for  $T = n - 1$ . We need only to prove that the theorem is true for  $T = n$ .

First assume  $n$  is odd. First we prove the existence of  $q_n^{**}(S)$  and  $q_n^*(S)$ . Note from the assumption we have for all  $a \in [s_{n-1,n+1}^*, s_M]$ ,  $q_{n-1}^+(S, a)$  exists. As  $q_0^+(S, s_{n-1,n+1}^*) = q_{n-1}^*(S)$ ,  $q_0^+(S, s_M) = q_{n-1}^{**}(S)$ , and they give the optimal lower bound and upper bound on the  $n$ th option's price, we must have a pricing kernel which has the form as  $q_n^{**}(S)$  and price the underlying stock and  $n$  options correctly. This proves the existence of  $q_n^{**}(S)$ . The existence of  $q_n^*(S)$  can be similarly proved.

Obviously  $q_n^{**}(S)$  can intersect the true pricing kernel at most  $n + 2$  times. However from Lemma 3, it must intersect all admissible pricing kernels at least  $n+2$  times; otherwise they cannot price all the  $n$  options correctly. Hence  $q_n^{**}(S)$  intersects all admissible pricing kernel exactly  $n + 2$  times. It is obvious that  $q_n^{**}(S)$  has fatter left tail. Thus applying Lemma 3, we immediately conclude that  $q_n^{**}(S)$  gives the optimal upper bound on options with strike prices between  $K_{2i}$  and  $K_{2i+1}$ , where  $0 \leq 2i \leq n$ , and it gives the optimal lower bound on

options with strike prices between  $K_{2i-1}$  and  $K_{2i}$ , where  $0 < 2i - 1 \leq n$ . Similarly, we can show that  $q_n^*(S)$  gives the optimal lower bound on options with strike prices between  $K_{2i}$  and  $K_{2i+1}$ , where  $0 \leq 2i \leq T$ , and it gives the optimal upper bound on options with strike prices between  $K_{2i-1}$  and  $K_{2i}$ , where  $0 < 2i - 1 \leq n$ .

We now prove the existence of  $q_n^+(S, a)$  and  $q_n^-(S, b)$  for all  $a \in [s_{n,n+2}^*, s_M]$  and  $b \in [s_0, s_{n,1}^*]$ .

It is straightforward that  $K_1 > s_{n,1}^*$ . Moreover, without loss of generality we can assume all these  $n$  options are call options. Furthermore, given any decreasing pricing kernel  $q(S)$  which price the underlying stock and  $n$  options with strikes  $K_1, \dots, K_n$  correctly, we have

$$\begin{aligned} \int_{s_0}^{s_M} q(S) dS &= 1, \\ B_0 \int_{s_0}^{s_M} S q(S) dS &= S_0, \\ B_0 \int_{s_0}^{s_M} c^i(S) q(S) dS &= c_0^i, \quad i = 1, \dots, n. \end{aligned}$$

Let  $q(S) \equiv q_n^*(S)$ . It follows that

$$\begin{aligned} \int_b^{s_M} q(S) dS &= 1, \\ B_0 \int_b^{s_M} S q(S) dS &= S_0, \\ B_0 \int_b^{s_M} c^i(S) q(S) dS &= c_0^i, \quad i = 1, \dots, n. \end{aligned}$$

Now consider the case where the support of the stock price distribution is  $[b, s_M]$ .

Let  $\hat{q}(S)$  be the truncated probability density function of  $q(S)$  defined on  $[b, s_M]$ , i.e.,  $\hat{q}(S) \equiv q(S)$ ,  $S \in [b, s_M]$ . As the prices  $S_0, c_0^i$ ,  $i = 1, \dots, n$ , are given by

a bounded RND  $\hat{q}(S)$ , according to the previously made assumption, we must have a RND which is defined on  $[b, s_M]$ , has the specified form as

$$q_n^{**}(S; b, s_M, K_1, \dots, K_n, S_0, c_0^1, \dots, c_0^n),$$

and gives the same prices for the underlying stock and the  $n$  options as  $\hat{q}(S)$ .

Now let

$$q_n^-(S, b) = \begin{cases} q_{n-1}^{**}(S; b, s_M, K_1, \dots, K_n, S_0, c_0^1, \dots, c_0^n), & S \geq b \\ 0, & S < b \end{cases}$$

Some straightforward calculations show that this RND has the specified form as  $q_{n+1}^*(S)$ , where  $s_{n+1,1}^* = b$ , and price the underlying stock and the  $n$  options correctly. This proves the existence of  $q_n^-(S, b)$  for all  $b \in [s_0, s_{n,1}^*]$ .

When  $n$  is even, the proof is similar. Q.E.D.

From the above theorem if  $\bar{q}$  is a strict upper bound on the true RND, then  $q_n^+(S, a)$  and  $q_n^-(S, b)$  will be different from the true RND anywhere in its support. We may also notice that if  $\bar{q}$  is an upper bound then any value which is larger than  $\bar{q}$  will also be an upper bound. As we can increase  $\bar{q}$  arbitrarily, the pointwise differences between these RNDs with extreme forms and the true RND can be made arbitrarily large.

Moreover, from the theorem we can see that there are always RNDs with extremely fat tails and RNDs with extremely thin tails which both price the any number of options correctly. Thus caution must be taken when we interpret the tails of an estimated RND.

Furthermore, as estimations of option implied risk preferences rely on estimations of option implied RNDs, this result also has important implications for the literature on option implied risk preferences.

### 3 RND's First Derivative is Bounded

Assume the true RND is continuous and its first derivative is bounded above by  $\bar{q} > 0$  and below by  $\underline{q} < 0$ . As in the previous section we assume the support of the true RND is  $[s_0, s_M] \subset R^+$ . Moreover, we assume  $s_M$  is sufficiently large, that is, it is as large as we need it to be though it is bounded.

Let  $J_n = \frac{3n}{2} + 3$  when  $n$  is even or  $\frac{3(n+1)}{2} + 2$  when  $n$  is odd. Let  $q_{1,n}^{**}(S)$  be a continuous RND which is  $J_n$ -segment and piecewise constant, which is zero at  $(2 + 3j)$ th segments,  $j \geq 0$ , whose first derivative is equal to  $\underline{q}$  at  $(1 + 3j)$ th segments,  $j \geq 0$ , and  $\bar{q}$  at  $(3 + 3j)$ th segments,  $j \geq 0$ , and prices the stock and  $n$  observed options correctly. Denote the domain of the  $j$ th ( $j \geq 1$ ) segment by  $(s_{n,j-1}^{**}, s_{n,j}^{**})$ . Thus we have  $s_{n,0}^{**} = s_0$ ,  $s_{n,J_n}^{**} = s_M$ ,

$$s_{n,3j+3}^{**} = \frac{\bar{q}s_{n,3j+2}^{**} + \underline{q}s_{n,3(j+1)+1}^{**}}{\bar{q} + \underline{q}}, \quad j \geq 0,$$

and for  $j \geq 0$ ,  $s_{n,3j+1}^{**}$  and  $s_{n,3j+2}^{**}$  are determined by the  $n + 2$  equations for admissibility.

Let  $L_n = \frac{3n}{2} + 4$  when  $n$  is even or  $\frac{3(n+1)}{2} + 2$  when  $n$  is odd. Let  $q_{1,n}^*(S)$  be a continuous RND which is  $L_n$ -segment and piecewise constant, which is zero at  $(1 + 3j)$ th segments,  $j \geq 0$ , whose first derivative is equal to  $\bar{q}$  at  $(2 + 3j)$ th segments,  $j \geq 0$ , and  $\underline{q}$  at  $(3 + 3j)$ th segments,  $j \geq 0$ , and prices the stock

and  $n$  observed options correctly. Denote the domain of the  $j$ th ( $j \geq 1$ ) segment by  $(s_{n,j-1}^*, s_{n,j}^*)$ . Thus we have  $s_{n,0}^* = s_0$ ,  $s_{n,L_n}^* = s_M$ ,

$$s_{n,3j+2}^* = \frac{\bar{q}s_{n,3j+1}^* + \underline{q}s_{n,3(j+1)}^*}{\bar{q} + \underline{q}}, \quad j \geq 0,$$

and for  $j \geq 0$ ,  $s_{n,3j+1}^*$  and  $s_{n,3(j+1)}^*$  are determined by the  $n + 2$  equations for admissibility.

Let  $q_n^+(S, a)$  be a RND which has the same form as  $q_{n+1}^{**}(S)$ , satisfies  $s_{n+1,1}^{**} = a \in [s_0, s_{n,1}^{**}]$ , and prices the underlying stock and  $n$  options with strikes  $K_1, \dots, K_n$  correctly.

Let  $q_n^-(S, b)$  be a RND which has the same form as  $q_{n+1}^*(S)$ , satisfies  $s_{n+1,2}^* = b \in [s_0, s_{n,2}^*]$ , and prices the underlying stock and  $n$  options with strikes  $K_1, \dots, K_n$  correctly. We now present the following result.

**Theorem 2** *Assume the support of the stock price distribution is  $[s_0, s_M]$ . Assume the underlying asset and  $T$  options with strike prices  $K_1 < K_2 < \dots < K_T$ ,  $T \geq 1$ , are priced correctly by a RND with bounded first derivative. Then for all  $T = 1, \dots$ ,  $q_T^{**}(S)$  and  $q_T^*(S)$  exist. For all  $T = 1, \dots$ , all  $a \in [s_0, s_{n,1}^{**}]$ , all  $b \in [s_0, s_{n,2}^*]$ ,  $q_T^+(S, a)$  and  $q_T^-(S, b)$  exist. Moreover,  $q_T^{**}(S)$  ( $q_T^*(S)$ ) gives the optimal upper (lower) bound on options with strike prices between  $K_{2i}$  and  $K_{2i+1}$ , where  $0 \leq 2i \leq T$ , while  $q_T^{**}(S)$  ( $q_T^*(S)$ ) gives the optimal lower (upper) bound on options with strike prices between  $K_{2i-1}$  and  $K_{2i}$ , where  $0 < 2i-1 \leq T$ .*

Proof: The proof is similar to the proof of Theorem 1; thus it is omitted for brevity.

From the above theorem if  $\bar{q}$  and  $\underline{q}$  are strict upper bounds on the true RND's

first derivative, then the first derivatives of  $q_n^+(S, a)$  and  $q_n^-(S, b)$  will be different from the true RND's first derivative anywhere in its support. Moreover, if  $\bar{q}$  is an upper bound then any value which is larger than  $\bar{q}$  will also be an upper bound and if  $\underline{q}$  is a lower bound then any value which is smaller than  $\underline{q}$  will also be a lower bound. As we can increase  $\bar{q}$  and decrease  $\underline{q}$  arbitrarily, the pointwise differences between the first derivatives of these RNDs's with extreme forms and the true RND can be made arbitrarily large.

Furthermore, as estimations of option implied risk aversion involve estimations of the first derivative of option implied RND, this result also has important implications for the literature on option implied risk aversion.

## 4 RND's Elasticity is Bounded

Assume the true RND is continuous and its elasticity is bounded above by  $\bar{e}_1$  and below by  $\underline{e}_1$ . Although in most cases we have  $\bar{e}_1 > 0$  and  $\underline{e}_1 < 0$ , we do not assume that the lower and upper bounds have opposite signs.

Let  $I_O(j) \equiv \frac{1-(-1)^j}{2}$ . That is,  $I_O(j)$  is an indicator function of integers: if integer  $j$  is odd, then  $I_O(j) = 1$ ; if integer  $j$  is even, then  $I_O(j) = 0$ .

Let  $q_{1,n}^{**}(S)$  be a continuous RND which is  $(n+2)$ -segment, whose elasticity is equal to  $\underline{e}_1$  at odd segments and  $\bar{e}_1$  at even segments, and which prices the stock and  $n$  observed options correctly. Denote the domain of the  $j$ th ( $j \geq 1$ ) segment by  $(s_{n,j-1}^{**}, s_{n,j}^{**})$ , where  $s_{n,0}^{**} = s_0$  and  $s_{n,n+2}^{**} = s_M$ . We have

$$q_{1,n}^{**}(S) = \sum_{j=0}^n a_{1,n}^{**} S^{\underline{e}_1(1-I_O(j))+\bar{e}_1 I_O(j)} \prod_{l=1}^j s_{n,l}^{**(\bar{e}_1-\underline{e}_1)(-1)^l} I_{(s_{n,j}^{**}, s_{n,j+1}^{**})}(S),$$

where  $s_{n,j}^{**}$ ,  $1 \leq j \leq n+1$ , and the normalization factor  $a_{1,n}^{**}$  are determined by the  $n+2$  equations for admissibility.

Let  $q_{1,n}^*(S)$  be a continuous RND which is  $(n+2)$ -segment, whose elasticity is equal to  $\bar{e}_1$  at odd segments and  $\underline{e}_1$  at even segments, and which prices the stock and  $n$  observed options correctly. Denote the domain of the  $j$ th ( $j \geq 1$ ) segment by  $(s_{n,j-1}^*, s_{n,j}^*)$ , where  $s_{n,0}^* = s_0$  and  $s_{n,n+2}^* = s_M$ . We have

$$q_{1,n}^*(S) = \sum_{j=0}^n a_{1,n}^* S^{\bar{e}_1(1-I_O(j)) + \underline{e}_1 I_O(j)} \prod_{l=1}^j s_{n,l}^{**(\underline{e}_1 - \bar{e}_1)(-1)^l} I_{(s_{n,j}^*, s_{n,j+1}^*)}(S),$$

where  $s_{n,j}^*$ ,  $1 \leq j \leq n+1$ , and the normalization factor  $a_{1,n}^*$  are determined by the  $n+2$  equations for admissibility.

Let  $q_{1,n}^+(S, a)$  be a RND which has the same form as  $q_{1,n+1}^{**}(S)$ , satisfies  $s_{n+1,n+2}^{**} = a \in [s_{n,n+1}^*, s_M]$ , and prices the underlying stock and  $n$  options with strikes  $K_1, \dots, K_n$  correctly.

Let  $q_{1,n}^-(S, b)$  be a RND which has the same form as  $q_{1,n+1}^*(S)$ , satisfies  $s_{n+1,1}^* = b \in [s_0, s_{n,1}^*]$ , and prices the underlying stock and  $n$  options with strikes  $K_1, \dots, K_n$  correctly. We have the following result.

**Theorem 3** *Assume the support of the stock price distribution is  $[s_0, s_M]$ . Assume the underlying asset and  $T$  options with strike prices  $K_1 < K_2 < \dots < K_T$ ,  $T \geq 1$ , are priced correctly by a RND with bounded elasticity. For all  $T = 1, \dots$ ,  $q_{1,T}^{**}(S)$  and  $q_{1,T}^*(S)$  exist. For all  $T = 1, \dots$ ,  $a \in [s_{n,n+1}^*, s_M]$ ,  $b \in [s_0, s_{n,1}^*]$ ,  $q_{1,T}^+(S, a)$  and  $q_{1,T}^-(S, b)$  exist. Moreover,  $q_{1,T}^{**}(S)$  ( $q_{1,T}^*(S)$ ) gives the optimal upper (lower) bound on options with strike prices between  $K_{2i}$  and  $K_{2i+1}$ , where  $0 \leq 2i \leq T$ , while  $q_{1,T}^{**}(S)$  ( $q_{1,T}^*(S)$ ) gives the optimal lower (upper) bound on*

options with strike prices between  $K_{2i-1}$  and  $K_{2i}$ , where  $0 < 2i - 1 \leq T$ .

Proof: The proof is similar to the proof of Theorem 1; thus it is omitted for brevity.

From the above theorem if  $\bar{\epsilon}_1$  and  $\underline{\epsilon}_1$  are strict upper bounds on the true RND's elasticity, then the elasticities of  $q_n^+(S, a)$  and  $q_n^-(S, b)$  will be different from the true RND's elasticity anywhere in its support. Moreover, if  $\bar{\epsilon}_1$  is an upper bound then any value which is larger than  $\bar{\epsilon}_1$  will also be an upper bound and if  $\underline{\epsilon}_1$  is a lower bound then any value which is smaller than  $\underline{\epsilon}_1$  will also be a lower bound. As we can increase  $\bar{\epsilon}_1$  and decrease  $\underline{\epsilon}_1$  arbitrarily, the pointwise differences between the elasticities of these RNDs's with extreme forms and the true RND can be made arbitrarily large.

## 5 RND's $N$ th order Elasticity is Bounded

Let a RND's 0th order elasticity be itself. We define a RND's  $N$ th order elasticity as the elasticity of its  $(N - 1)$ th order elasticity. Given a RND  $q(S)$ , its first order elasticity  $\nu_1(S) \equiv \frac{dq(S)}{dS} \frac{S}{q(S)}$ . Its second order elasticity  $\nu_2(S) \equiv \frac{d\nu_1(S)}{dS} \frac{S}{\nu_1(S)}$ . Its  $N$ th order elasticity  $\nu_N(S) \equiv \frac{d\nu_{N-1}(S)}{dS} \frac{S}{\nu_{N-1}(S)}$ .

Assume the true RND's  $(N - 1)$ th order elasticity is continuous and its  $N$ th ( $N \geq 1$ ) order elasticity is bounded above by  $\bar{\epsilon}_N$  and below by  $\underline{\epsilon}_N$ .

Assume  $n \geq N - 2$ . Let  $q_{N,n}^{**}(S)$  be a RND which satisfies the following conditions: it has continuous  $(N - 1)$ th order elasticity, it has  $(n - N + 3)$  segments, its  $N$ th order elasticity is equal to  $\bar{\epsilon}_N(1 - I_O(N)) + \underline{\epsilon}_N I_O(N)$  at odd

segments and  $\underline{e}_N(1 - I_O(N)) + \bar{e}_N I_O(N)$  at even segments, and it prices the stock and  $n$  observed options correctly. Denote the domain of the  $j$ th ( $j \geq 1$ ) segment by  $(s_{N,n,j-1}^{**}, s_{N,n,j}^{**})$ , where  $s_{N,n,0}^{**} = s_0$  and  $s_{N,n,n-N+3}^{**} = s_M$ . We have

$$q_{N,n}^{**}(S) = \sum_{j=0}^n \Pi_{l=0}^j a_l \exp(f_j(S; \epsilon_j, r_{N,n,j,1}^{**}, \dots, r_{N,n,j,N-1}^{**})) I_{(s_{N,n,j}^{**}, s_{N,n,j+1}^{**})}(S),$$

where for  $j = 0, \dots, n - N + 2$ ,

$$\epsilon_j = (\bar{e}_N(1 - I_O(N)) + \underline{e}_N I_O(N))(1 - I_O(j)) + (\underline{e}_N(1 - I_O(N)) + \bar{e}_N I_O(N))I_O(j),$$

$$f_j(S; \epsilon_j, r_{N,n,j,1}^{**}, \dots, r_{N,n,j,N-1}^{**}) = \epsilon_j (\ln S)^N + \sum_{t=1}^{N-1} r_{N,n,j,t}^{**} (\ln S)^t,$$

for  $j = 1, \dots, n - N + 2$ ,

$$a_j = \frac{\exp(f_{j-1}(s_{N,n,j}^{**}; \epsilon_{j-1}, r_{N,n,j-1,1}^{**}, \dots, r_{N,n,j-1,N-1}^{**}))}{\exp(f_j(s_{N,n,j}^{**}; \epsilon_j, r_{N,n,j,1}^{**}, \dots, r_{N,n,j,N-1}^{**}))},$$

for  $j = 1, \dots, n - N + 2$ ,  $r_{N,n,j,1}^{**}, \dots, r_{N,n,j,N-1}^{**}$  are determined by  $r_{N,n,j-1,1}^{**}, \dots, r_{N,n,j-1,N-1}^{**}$

such that for all  $l = 1, \dots, N - 1$ ,

$$\begin{aligned} & \epsilon_j N \dots (N - l + 1) (\ln s_{N,n,j}^{**})^{N-l} + \sum_{t=l}^{N-1} t \dots (t - l + 1) r_{N,n,j,t}^{**} (\ln s_{N,n,j}^{**})^{t-l} \\ &= \epsilon_{j-1} N \dots (N - l + 1) (\ln s_{N,n,j-1}^{**})^{N-l} + \sum_{t=l}^{N-1} t \dots (t - l + 1) r_{N,n,j-1,t}^{**} (\ln s_{N,n,j-1}^{**})^{t-l}, \end{aligned}$$

and  $a_0, r_{N,n,0,l}^{**}, s_{N,n,j}^{**}, l = 1, \dots, N - 1, j = 1, \dots, n - N + 2$ , are determined by

the  $n + 2$  equations for admissibility.

Let  $q_{N,n}^*(S)$  be a continuous  $(n - N + 3)$ -segment RND consistent with the market prices of the stock and  $n$  options whose  $N$ th order elasticity is equal to  $\underline{e}_N(1 - I_O(N)) + \bar{e}_N I_O(N)$  at odd segments and  $\bar{e}_N(1 - I_O(N)) + \underline{e}_N I_O(N)$  at even

segments. Denote the domain of the  $j$ th ( $j \geq 1$ ) segment by  $(s_{N,n,j-1}^*, s_{N,n,j}^*)$ ,

where  $s_{N,n,0}^* = s_0$  and  $s_{N,n,n-N+3}^* = s_M$ . We have

$$q_{N,n}^*(S) = \sum_{j=0}^n \Pi_{l=0}^j b_l f(S; \underline{e}_2(1 - I_O(j)) + \bar{e}_2 I_O(j), r_{N,n,j}^*) I_{(s_{N,n,j}^*, s_{N,n,j+1}^*)}(S),$$

where for  $j = 1, \dots, n$ ,

$$b_j = \frac{f(s_{N,n,j}^*; \underline{e}_2(1 - I_O(j-1)) + \bar{e}_2 I_O(j-1), r_{N,n,j-1}^*)}{f(s_{N,n,j}^*; \underline{e}_2(1 - I_O(j)) + \bar{e}_2 I_O(j), r_{N,n,j}^*)}$$

$$r_{N,n,j}^* = \ln s_{N,n,j}^* - (\ln s_{N,n,j}^* - r_{N,n,j-1}^*) \frac{(\underline{e}_2(1 - I_O(j)) + \bar{e}_2 I_O(j))^2}{(\underline{e}_2(1 - I_O(j-1)) + \bar{e}_2 I_O(j-1))^2},$$

and  $b_0, r_{N,n,0}^*, s_{N,n,j}^*, j = 1, \dots, n - N + 2$ , are determined by the  $n + 2$  equations for admissibility.

Let  $q_{N,n}^+(S, a)$  be a RND which has the same form as  $q_{N,n+1}^{**}(S)$ , satisfies  $s_{N,n+1,n-N+3}^* = a \in [s_{N,n,n-N+2}^*, s_M]$ , and prices the underlying stock and  $n$  options with strikes  $K_1, \dots, K_n$  correctly.

Let  $q_{N,n}^-(S, b)$  be a RND which has the same form as  $q_{N,n+1}^*(S)$ , satisfies  $s_{N,n+1,1}^* = b \in [s_0, s_{N,n,1}^*]$ , and prices the underlying stock and  $n$  options with strikes  $K_1, \dots, K_n$  correctly. We have the following result.

**Theorem 4** *Assume the support of the stock price distribution is  $[s_0, s_M]$ . Assume the underlying asset and  $T$  options with strike prices  $K_1 < K_2 < \dots < K_T$ ,  $T \geq 1$ , are priced correctly by a RND whose elasticity's elasticity is bounded . For all  $T = 1, \dots$ ,  $q_{N,T}^{**}(S)$  and  $q_{N,T}^*(S)$  exist. For all  $T = 1, \dots$ , all  $a \in [s_{N,n,n-N+2}^*, s_M]$ , all  $b \in [s_0, s_{N,n,1}^*]$ ,  $q_{N,T}^+(S, a)$  and  $q_{N,T}^-(S, b)$  exist. Moreover,  $q_{N,T}^{**}(S)$  ( $q_{N,T}^*(S)$ ) gives the optimal upper (lower) bound on options with strike prices between  $K_{2i}$  and  $K_{2i+1}$ , where  $0 \leq 2i \leq T$ , while  $q_{N,T}^{**}(S)$  ( $q_{N,T}^*(S)$ ) gives*

*the optimal lower (upper) bound on options with strike prices between  $K_{2i-1}$  and  $K_{2i}$ , where  $0 < 2i - 1 \leq T$ .*

Proof: The proof is similar to the proof of Theorem 1; thus it is omitted for brevity.

From the above theorem if  $\bar{e}_N$  and  $\underline{e}_N$  are strict upper bounds on the true RND's  $N$ th order elasticity, then the  $N$ th order elasticities of  $q_n^+(S, a)$  and  $q_n^-(S, b)$  will be different from the true RND's  $N$ th order elasticity anywhere in its support. Moreover, if  $\bar{e}_N$  is an upper bound then any value which is larger than  $\bar{e}_N$  will also be an upper bound and if  $\underline{e}_N$  is a lower bound then any value which is smaller than  $\underline{e}_N$  will also be a lower bound. As we can increase  $\bar{e}_N$  and decrease  $\underline{e}_N$  arbitrarily, the pointwise differences between the  $N$ th order elasticities of these RND's with extreme forms and the true RND can be made arbitrarily large.

## 6 Conclusions

In this paper we have raised a serious question on the theoretical foundation of option implied RNDs. We have shown that there always exist numerous RNDs which have extreme forms and price any number of options correctly. The pointwise differences between these RNDs and the true RND can be made arbitrarily large. We have also shown that there always exist numerous continuous RNDs consistent with the prices of any number of options whose first derivatives have extreme forms. The pointwise differences between the first derivatives of these

RNDs and the true RND can be made arbitrarily large. Similar results are obtained with respect to higher order derivatives. These results not only show that the RND implied by a given series of option prices is not unique but also show how different the RNDs consistent with the serie of option prices can be. Thus caution must be taken when we use estimated RNDs for various purposes.

Of course, we may argue that if we assume that RNDs must have continuous  $N$ th derivatives, then the RNDs given in the theorems in this papers with discontinuous  $N$ th derivatives will simply disappear. But then we have to justify why RNDs must have continuous  $N$ th derivatives. Considering that we even allow discrete distributions in option pricing models, we may not find such assumptions easy to justify.

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