

Optimal liquidation in dark pools

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Abstract

We consider a finite time horizon, multi-asset optimal liquidation problem in discrete time for an investor having access to both a traditional trading venue and a dark pool. Our model captures the price impact of trading in transparent traditional venues as well as the execution uncertainty of trading in a dark pool. We prove existence and uniqueness of optimal trading strategies for risk averse mean-variance traders and find that dark pools change optimal trading strategies and can significantly reduce trading costs. Their effect can be reduced by adverse selection and trading restrictions.

1 Introduction

In the last years, equity trading has been transformed by the advent of so called dark pools. These alternative trading venues differ significantly from classical exchanges. Although dark pools vary in a number properties (see, e.g, Butler (2007)), they generally share the following two stylized facts. First, dark pools do not determine prices. Instead, dark pools monitor the prices determined by the primary (classical) exchanges and settle trades in the dark pool only if possible at these prices. Thus trades in the dark pool have no price impact. Second, dark pools are dark. The liquidity available in a dark pool is not quoted, making trade execution in the dark pool uncertain and unpredictable. In this paper, we analyze trade execution when trading is possible both at the classical exchange as well as in the dark pool. We first propose a model that captures the stylized facts above for trading in such a market. Then we determine the optimal trade execution strategy in a discrete time framework.

Our model for trading and price formation at the classical exchange is a general price impact model. Trade execution can be enforced by selling aggressively, which however results in higher execution costs due to a stronger price impact. The dark pool in our model provides a limited and previously unknown amount of liquidity that can be used for trade execution without price impact. Trades in the dark pool are executed until the liquidity is exhausted, and there is no way to achieve trade execution in the dark pool for larger orders. The split of orders between dark pool and exchange is thus driven by the trade-off between execution uncertainty and price impact costs. While there is no feedback from the dark pool to the price determined at the exchange, the two venues can be connected since liquidity in the dark pool and price movements at the primary exchange can be dependent. For example, liquidity on the bid side of the dark pool might be unusually high exactly when prices move up. By allowing for such correlation in our model, we can incorporate and analyze adverse selection in the dark pool. We believe that the market model is an important part of this paper since it is to our knowledge the first mathematical framework for trade execution in dark pools. By design it is a partial equilibrium model where price impact and dark pool liquidity are assumed to be exogenously given.

In our model, we prove the existence and uniqueness of optimal execution strategies that trade simultaneously at the primary exchange and in the dark pool. Subsequently, we consider a specific multi-asset market model with linear price impact. The model can be specified by a small set of parameters which can be directly estimated from trade execution data. In this market model, we obtain a linear solution of the optimal trade execution problem which can be computed recursively. This recursive scheme makes the model tractable for practicable applications and allows us to investigate several examples. When liquidating a position in a single asset, the current asset position is at all times being offered in the dark pool, while it is liquidated in parallel at

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the primary exchange. The opportunity to trade in the dark pool leads to a slower liquidation at the primary exchange compared to a market without a dark pool. This observation has several consequences. First, trade execution algorithms need to be fundamentally adjusted when a dark pool is introduced. It is not sufficient to use an algorithm that was optimal for execution at the exchange and to add a component that also places trades in the dark pool; instead, trading at the exchange needs to be adjusted during the entire trade execution time interval. Second, the delayed trading at the primary exchange could potentially have an adverse effect on the price quality and should thus be of central interest to supervisory agencies. This effect might be mitigated by adverse selection in the dark pool, which leads to reduced orders in the dark pool and can even lead to a complete avoidance of the dark pool.

While in a single-asset setting the entire asset position is placed in the dark pool, this is not true if a multi-asset portfolio is to be liquidated. If the portfolio is balanced and thus only exposed to little market risk, then a complete liquidation of the position in one of the assets is unfavorable and thus only a fraction of the entire portfolio is placed in the dark pool. This highlights again that overly simple adjustments to existing trade execution algorithms are exposed to potential pitfalls. For dark pools, the reluctance of traders to place balanced portfolios in a dark pool is an incentive to offer balanced executions in order to attract more liquidity.

Building on empirical investigations of the market impact of large transactions, a number of theoretical models of illiquid markets have emerged. One part of these models focuses on the underlying mechanisms for illiquidity effects, e.g., Kyle (1985) and Easley and O'Hara (1987). We follow a second line that takes the liquidity effects as given and derives optimal trading strategies within such a stylized model market. Several market models have been proposed for classical exchanges, e.g., Bertsimas and Lo (1998), Almgren and Chriss (2001), Almgren (2003), Obizhaeva and Wang (2006) and Alfonsi, Schied, and Schulz (2007). Since the advantages and disadvantages of these models are still a topic of ongoing research, we propose a discrete-time market model that generalizes all of these models. As a special case, we analyze the linear price impact model by Almgren and Chriss (2001) for two reasons. First, it captures both the permanent and temporary price impacts of large trades, while being sufficiently simple to allow for a mathematical analysis. It has thus become the basis of several theoretical studies, e.g., Rogers and Singh (2007), Almgren and Lorenz (2007), Carlin, Lobo, and Viswanathan (2007) and Schöneborn and Schied (2008). Second, it demonstrated reasonable properties in real world applications and serves as the basis of many optimal execution algorithms run by practitioners (see e.g., Kissell and Glantz (2003), Schack (2004), Abramowitz (2006), Simmonds (2007) and Leinweber (2007)).

Trade execution in dark pools has raised significant interest of practitioners (see, e.g., Conrad, Johnson, and Wahal (2003), Abrokwah and Sofianos (2006), Leinweber (2007), Almgren and Harts (2008)). While the effect of dark pools on price quality has been analyzed by academics (see, e.g., Tse and Hackard (2004) and Hendershott and Jones (2005)), we are not aware of any academic research on optimal trade execution in dark pools.

The remainder of this paper is structured as follows. In Section 2, we introduce the market model. This consists of a model for the primary trading venue (Subsection 2.1) and for the dark pool (Subsection 2.2). Furthermore, we describe the trader's objective function (Subsection 2.3). Existence and uniqueness of optimal trade execution strategies are established in Section 3. In Section 4, we propose a specific tractable market model and analyze its properties. Finally, we discuss adverse selection in Section 5, practical trading restrictions in Section 6 and an alternative to the transaction price of the dark pool in Section 7. All proofs are presented in Appendix A.1.

2 Model description

The market we consider consists of a risk-free asset and n risky assets. For simplicity of exposition, we assume that the risk-free asset does not generate interest. Large transactions are usually executed within a few hours or at most a few days; the effect of discounting is therefore marginal, and we will not consider it in this paper. We analyze a discrete-time model, i.e., we assume that trades can be executed at the (not necessarily equidistant¹) time points t_0, t_1, \dots, t_N . At each of these time points, we assume that the seller as well as a number of noise traders execute orders. We denote the orders of the seller at time t_i at the primary venue by $x_i = (x_i^{(1)}, \dots, x_i^{(n)}) \in \mathbb{R}^n$ and in the dark pool by $y_i = (y_i^{(1)}, \dots, y_i^{(n)}) \in \mathbb{R}^n$, where positive entries denote sell orders and negative entries denote buy orders. In the following Subsections 2.1 and 2.2, we describe the different

¹For example, the distance can be taken in volume time to adjust for the U-shaped intraday pattern of market volatility and liquidity.

effects of the orders x_i and y_i : The execution of the order x_i at the primary venue is *guaranteed*, but has an adverse effect on the market price. The execution of the order y_i in the dark pool is *uncertain*, but has no price impact (irrespective of whether it is executed or not). In Subsection 2.3, we define the trading objective of the investor and specify the set of admissible strategies.

2.1 Transaction price and impact of primary venue orders x_i

We assume that the transaction price $P_i \in \mathbb{R}^n$ at the exchange at time t_i can be decomposed into the price impact of the primary venue trades x_j of the large trader and the “fundamental” asset price $\tilde{P}_i \in \mathbb{R}^n$ that would have occurred in the absence of large trades. We model the fundamental asset price \tilde{P}_i as an arbitrary stochastic process with independent increments $\epsilon_i \in \mathbb{R}^n$:

$$\tilde{P}_{i+1} = \tilde{P}_i + \epsilon_{i+1}. \quad (1)$$

To avoid technical difficulties, we assume that the underlying probability space Ω is finite². Then we only need to require that the ϵ_i are independent. We do not make assumptions on the distribution of the ϵ_i . In particular, they can have different distributions. The random price changes ϵ_i reflect the noise traders’ actions as well as all external events, e.g., news. The assumption of independence of the ϵ_i implies that the random price changes do not exhibit autocorrelation. The results we derive are sensitive to this assumption; autocorrelation will in principle have an effect on the proceeds of any dynamic trading strategy. Including autocorrelation in the market model however shifts the focus from optimal liquidation to optimal investment: even without any initial asset position, the mathematical model will recommend high-frequency trading to exploit the autocorrelation. But this effect is not related to the original question of optimal execution. Furthermore, many investors do not have an explicit view on autocorrelation and thus choose an execution algorithm that is optimal under the assumption of independence of price increments. Finally, for realistic parameters the effect of autocorrelation on the optimal execution strategy and the resulting execution cost is marginal as was demonstrated by Almgren and Chriss (2001). For these reasons, we will not include autocorrelation in our market model.

We allow a general form of the impact of the trades x_0, x_1, \dots, x_i on the transaction price P_i :

$$P_i = \begin{array}{ccc} \tilde{P}_i & - & f_i(x_0, \dots, x_i). \\ \uparrow & & \uparrow \\ \text{“Fundamental”} & & \text{Price impact} \\ \text{asset price} & & \text{of seller} \end{array} \quad (2)$$

By allowing f_i to depend on x_i , we allow the order x_i to influence its own execution price (e.g., in the form of a temporary price impact). We define the “price impact cost of trading” as

$$\sum_i x_i^\top f_i(x_0, \dots, x_i) \quad (3)$$

and assume that it is strictly convex, i.e., that for any two trading trajectories $x \neq x'$ and $0 < t < 1$ the following inequality holds:

$$\begin{aligned} & \sum_i (tx_i + (1-t)x'_i)^\top f_i(tx_0 + (1-t)x'_0, \dots, tx_i + (1-t)x'_i) \\ & < t \left(\sum_i x_i^\top f_i(x_0, \dots, x_i) \right) + (1-t) \left(\sum_i x'_i{}^\top f_i(x'_0, \dots, x'_i) \right). \end{aligned} \quad (4)$$

Furthermore, we require that the price impact cost of trading grows superlinearly, i.e., that

$$\lim_{|(x_0, \dots, x_N)| \rightarrow \infty} \frac{\sum_i x_i^\top f_i(x_0, \dots, x_i)}{|(x_0, \dots, x_N)|} = \infty. \quad (5)$$

This framework generalizes most of the existing market impact models of liquidity. For example, the model suggested by Almgren and Chriss (1999) and Almgren and Chriss (2001) is equivalent to assuming that the ϵ_i

²The results of this paper also hold for infinite Ω if the price increments ϵ_i satisfy suitable conditions and the set of admissible strategies is chosen appropriately.

are identically normally distributed and describing the market impact as

$$f_i(x_0, \dots, x_i) = \left(\sum_{j=0}^{i-1} \text{PermImp}(x_j) \right) + \text{TempImp}(x_i). \quad (6)$$

Here, $\text{PermImp}, \text{TempImp} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are functions describing the permanent and temporary market impact of a trade. If these functions are linear, we have

$$f_i(x_0, \dots, x_i) = \Gamma \left(\sum_{j=0}^{i-1} x_j \right) + \Lambda x_i \quad (7)$$

for constant matrices $\Gamma, \Lambda \in \mathbb{R}^{n \times n}$. Our framework also includes the limit order book model introduced by Obizhaeva and Wang (2006) as a single asset example if we again assume that the ϵ_i are identically normally distributed and that the price impact is given by

$$f_i(x_0, \dots, x_i) = \gamma \left(\sum_{j=0}^{i-1} x_j + x_i/2 \right) + \lambda \left(\sum_{j=0}^{i-1} e^{-\rho(t_i-t_j)} x_j + x_i/2 \right) \quad (8)$$

for constants $\gamma, \lambda, \rho \in \mathbb{R}^+$. Alfonsi, Schied, and Schulz (2007) suggested an extension of this model. The special case of independent increments of the fundamental price process can also be described in our framework with the following price impact function:

$$f_i(x_0, \dots, x_i) = \begin{cases} \left(\int_0^{x_i} g \left(\sum_{j=0}^{i-1} e^{-\rho(t_i-t_j)} x_j + y \right) dy \right) / x_i & \text{if } x_i \neq 0 \\ g \left(\sum_{j=0}^{i-1} e^{-\rho(t_i-t_j)} x_j \right) & \text{if } x_i = 0 \end{cases} \quad (9)$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is a function determined by the shape of the limit order book.

2.2 Trade execution in the dark pool

Contrary to the primary venue, the dark pool does not guarantee trade execution, since it only provides limited liquidity. We introduce the random variables $a_i, b_i \in [0, \infty]^n$ that model the liquidity that can be drawn upon at time t_i for buy orders (a_i , ask side of the dark pool) and for sell orders (b_i , bid side of the dark pool). The amount $z_i = (z_i^{(1)}, \dots, z_i^{(n)}) \in \mathbb{R}^n$ that is executed in the dark pool between time t_i and t_{i+1} is then given by

$$z_i^{(k)} = \begin{cases} \min(y_i^{(k)}, b_i^{(k)}) & \text{if } y_i^{(k)} \geq 0 \\ -\min(-y_i^{(k)}, a_i^{(k)}) & \text{if } y_i^{(k)} < 0. \end{cases} \quad (10)$$

We assume that a_i and b_i are independent of historical liquidity in the dark pool (a_0, \dots, a_{i-1} and b_0, \dots, b_{i-1}) as well as of previous price moves ($\epsilon_1, \dots, \epsilon_i$). However, the liquidity parameters a_i and b_i and the price move ϵ_{i+1} at time t_i can depend on each other, which allows us to incorporate the simultaneous occurrence of price jumps and liquidity in the dark pool which can lead to adverse selection (see Section 5). In order to ensure uniqueness of the optimal strategy, we assume that expected market moves are perfectly rank correlated with dark pool liquidity, i.e., that for all $p > q \geq 0$ we have

$$\mathbb{E}[\epsilon_{i+1}^{(k)} | a_i^{(k)} = p] \leq \mathbb{E}[\epsilon_{i+1}^{(k)} | a_i^{(k)} = q], \quad (11)$$

$$\mathbb{E}[\epsilon_{i+1}^{(k)} | b_i^{(k)} = p] \geq \mathbb{E}[\epsilon_{i+1}^{(k)} | b_i^{(k)} = q]. \quad (12)$$

Economically, this means that the higher the demand in the dark pool, the stronger the price will move upwards, and the stronger the supply in the dark pool, the stronger the price will move downwards. We also assume that execution in the dark pool is not guaranteed, i.e., for all i and k

$$\mathbb{P}[a_i^{(k)} = 0] > 0, \quad \mathbb{P}[b_i^{(k)} = 0] > 0. \quad (13)$$

While the dark pool has no impact on prices at the primary exchange, it is less clear to which extent the price impact f_i of the exchange is reflected in the trade price of the dark pool. If for example the price impact f_i is

realized predominantly in the form of a widening spread, then the impact on dark pools that monitor the mid quote can be much smaller than f_i . In Sections 3 to 6, we will make the simplifying assumption that trades in the dark pool are not influenced by the price impact f_i at all, i.e., that they are executed at the fundamental price \tilde{P}_i . If trading in the dark pool reflects the price impact f_i , then market manipulating strategies become profitable. We investigate this phenomenon in Section 7.

2.3 The liquidation problem

We consider an investor who has executed trades x_0, \dots, x_{i-1} at times t_0, \dots, t_{i-1} and needs to liquidate a portfolio $X_i = (X_i^{(1)}, \dots, X_i^{(n)}) \in \mathbb{R}^n$ of risky assets within a finite time-horizon $[t_i, t_N]$. For $X_i^{(k)} > 0$, this implies liquidating a long position in asset k (selling), whereas $X_i^{(k)} < 0$ implies liquidating a short position in asset k (buying). In both cases, we will speak of the “liquidation” or “sale”. We require that at all times $t_j \geq t_i$ the investor’s orders x_j, y_j are predictable, i.e., that they only depend on past information $(\epsilon_1, \dots, \epsilon_j, a_0, \dots, a_{j-1}$ and $b_0, \dots, b_{j-1})$. This includes deterministic (also called static) strategies, i.e., strategies that do not depend on *any* ϵ_l, a_l or b_l . Since the portfolio has to be liquidated by time t_N , we require that in all scenarios $\omega \in \Omega$ that

$$\sum_{j=i}^N (x_j + z_j) = X_i. \quad (14)$$

We recursively define

$$X_{j+1} := X_j - x_j - z_j. \quad (15)$$

Requirement (14) is then equivalent to $X_{N+1} = 0$. In order to ensure uniqueness of the optimal strategy, we assume that the investor only submits orders in the dark pool that have a positive probability of complete execution³, i.e., that satisfy for all j and k

$$y_j^{(k)} \in \left[-\max_{\omega \in \Omega} a_j^{(k)}(\omega), \max_{\omega \in \Omega} b_j^{(k)}(\omega) \right]. \quad (16)$$

Due to order submission fees, this restriction is natural in practice. We call the set of trading strategies fulfilling the above conditions the *set of admissible strategies* and denote it by $\mathbb{A}_i(X_i)$.

For an admissible strategy $(x, y) = (x_i, \dots, x_N, y_i, \dots, y_N)$ the trader’s cost of execution is

$$\mathcal{R}_i = X_i^\top \tilde{P}_i - \sum_{j=i}^N (x_j^\top P_j + z_j^\top \tilde{P}_j) = \sum_{j=i}^N \left(x_j^\top (\tilde{P}_i - \tilde{P}_j + f_j(x_0, \dots, x_j)) + z_j^\top (\tilde{P}_i - \tilde{P}_j) \right). \quad (17)$$

\mathcal{R}_i is a random variable; depending on market moves ϵ_j and dark pool trading opportunities a_j and b_j , the liquidation proceeds can vary significantly. The trade-off between expected proceeds and risk is an important driver of optimal liquidation and has been the focus of several investigations including Almgren and Chriss (1999), Dutilleul (2002), Almgren and Lorenz (2007), Schied and Schöneborn (2008) and Schied and Schöneborn (2009). In this paper, we assume that the investor wants to minimize the following mean-variance function of execution cost:

$$J_i(x_0, \dots, x_{i-1}, X_i) := \min_{(x, y) \in \mathbb{A}_i(X_i)} \left\{ \mathbb{E}[\mathcal{R}_i] + \alpha \cdot \mathbb{E} \left[\sum_{j=i}^N X_j^\top \Sigma_{j+1} X_j \right] \right\}. \quad (18)$$

Here, $\alpha \in \mathbb{R}_0^+$ is the coefficient of risk aversion and Σ_j is the covariance matrix of the increments $\epsilon_j^{(k)}$. We call a strategy (x, y) *optimal* if it realizes the minimum in Equation (18). Note that our optimization criterion penalizes risk due to market moves ϵ_j , but not the risk due to execution uncertainty in the dark pool. Since the market risk usually outweighs the liquidity risk, disregarding the latter should not lead to significantly different results while at the same time simplifying the analysis considerably. “Selective risk aversion” focusing only on market risk and disregarding liquidity risk has been applied before by Walia (2006) and Rogers and Singh (2007) in the contexts of stochastic liquidity and hedging.

³Due to our assumption of independence of dark pool liquidity a_i, b_i , the range of potential dark pool liquidity from time t_i up to time t_{i+1} is independent of the information available at time t_i .

3 Optimal liquidation

The following theorem establishes the existence and uniqueness of an optimal trading strategy that exploits both the trading opportunities at the primary exchange and the dark pool.

Theorem 3.1. *If the random variables ϵ_i, a_i, b_i fulfill the assumptions in Section 2, then there exists a unique optimal strategy, i.e., there exists a unique $(x, y) \in \mathbb{A}_i(X_i)$ realizing the minimum in Equation (18).*

The theorem is a direct consequence of the following propositions.

Proposition 3.2. *An optimal liquidation strategy exists.*

Proposition 3.3. *The optimal liquidation strategy is unique. More precisely, for each $(x_0, \dots, x_{i-1}, X_i)$ there exists a unique admissible strategy $(x, y) \in \mathbb{A}_i(X_i)$ that realizes the minimum $J_i(x_0, \dots, x_{i-1}, X_i)$ in Equation (18). Furthermore, the expression*

$$\left(\sum_{j=0}^{i-1} x_j^\top f_j(x_0, \dots, x_j) \right) + J_i(x_0, \dots, x_{i-1}, X_i) \quad (19)$$

is strictly convex in $(x_0, \dots, x_{i-1}, X_i) \in \mathbb{R}^{i+1}$, i.e., for any two vectors $(x_0, \dots, x_{i-1}, X_i) \neq (\tilde{x}_0, \dots, \tilde{x}_{i-1}, \tilde{X}_i)$ and $0 < s < 1$ the following inequality holds:

$$\begin{aligned} & \left(\sum_{j=0}^{i-1} (sx_j + (1-s)\tilde{x}_j)^\top f_j(sx_0 + (1-s)\tilde{x}_0, \dots, sx_j + (1-s)\tilde{x}_j) \right) \\ & + J_i(sx_0 + (1-s)\tilde{x}_0, \dots, sx_{i-1} + (1-s)\tilde{x}_{i-1}, sX_i + (1-s)\tilde{X}_i) \\ & < s \left(\left(\sum_{j=0}^{i-1} x_j^\top f_j(x_0, \dots, x_j) \right) + J_i(x_0, \dots, x_{i-1}, X_i) \right) \\ & + (1-s) \left(\left(\sum_{j=0}^{i-1} \tilde{x}_j^\top f_j(\tilde{x}_0, \dots, \tilde{x}_j) \right) + J_i(\tilde{x}_0, \dots, \tilde{x}_{i-1}, \tilde{X}_i) \right). \quad (20) \end{aligned}$$

4 Solution of a tractable model

In the previous section, we have dealt with a general market model. It is hard to derive any additional results in full generality. In this section, we therefore turn to a special tractable case of the general market model. In Subsection 4.1, we specify this model in terms of its price impact functions f_i , the fundamental price process \tilde{P} and the liquidity in the dark pool $a_0, b_0, \dots, a_N, b_N$. In Subsection 4.2 the value function J_i and the optimal orders x_i, y_i at times t_i are proven to be of quadratic respectively linear form and are derived in the form of an explicit backward induction.

In Subsections 4.3 and 4.4, we examine the effects of dark pools for liquidation of a single asset position and a two asset portfolio, respectively.

4.1 Model specification

In this section, we assume that price impact is linear and purely temporary, i.e.,

$$f_i(x_i) = \Lambda x_i \quad (21)$$

for all i , where the *price impact matrix* $\Lambda = (\lambda_{i,j}) \in \mathbb{R}^{n \times n}$ is a positive definite matrix. The price impact f_i then fulfills Conditions (4) and (5). This type of market impact is called *temporary* since the function f_i only depends on the trade x_i at time t_i and not on past trades x_0, \dots, x_{i-1} . Note that Λ can be chosen to be symmetric, since for the purposes of optimal execution only the product $x_i^\top f_i(x_i) = x_i^\top \Lambda x_i$ is relevant. One special case are diagonal matrices with strictly positive entries, which correspond to markets without cross asset price impact.

For the dark pool, we assume that orders are executed fully or not at all, i.e., $a_i^{(k)}, b_i^{(k)} \in \{0, \infty\}$ for all i, k . Furthermore, we assume that the liquidity $((a_i, b_i))_i$ in the dark pool is identically distributed and independent of the price moves ϵ_{i+1} . While the order execution probability in asset k can depend on the liquidity of the other assets, we assume a symmetry between liquidity on the buy and sell side of the dark pool:

$$\begin{aligned} \mathbb{P} \left[a_i^{(k)} = \infty \mid a_i^{(1)}, \dots, a_i^{(k-1)}, a_i^{(k+1)}, \dots, a_i^{(n)}, b_i^{(1)}, \dots, b_i^{(k-1)}, b_i^{(k+1)}, \dots, b_i^{(n)} \right] \\ = \mathbb{P} \left[b_i^{(k)} = \infty \mid a_i^{(1)}, \dots, a_i^{(k-1)}, a_i^{(k+1)}, \dots, a_i^{(n)}, b_i^{(1)}, \dots, b_i^{(k-1)}, b_i^{(k+1)}, \dots, b_i^{(n)} \right]. \end{aligned} \quad (22)$$

The price process \tilde{P} is assumed to be a martingale (i.e., $\mathbb{E}[\epsilon_i] = 0$) with a constant covariance matrix (i.e., $\Sigma_i = \Sigma_j =: \Sigma$).

As a consequence of the above assumptions, we directly obtain that $J_i(x_0, \dots, x_{i-1}, X_i)$ is independent of x_0, \dots, x_{i-1} and that

$$J_i(X_i) := J_i(x_0, \dots, x_{i-1}, X_i) = \min_{(x, y) \in \mathbb{A}_i(X_i)} \left\{ \mathbb{E} \left[\sum_{j=i}^N x_j^\top \Lambda x_j \right] + \alpha \cdot \mathbb{E} \left[\sum_{j=i}^N X_j^\top \Sigma X_j \right] \right\}. \quad (23)$$

4.2 Optimal liquidation

The following theorem establishes that the optimal orders x_i placed at the primary venue and y_i placed in the dark pool are linear functions of the portfolio X_i at any time t_i . Furthermore the value function J_i is quadratic in X_i .

Theorem 4.1. *For $0 \leq i \leq N$, there exist matrices $A_i, B_i \in \mathbb{R}^{n \times n}$, such that the optimal strategy (x, y) is given by $x_i = A_i X_i$, $y_i = B_i X_i$ for portfolio X_i at time t_i . Furthermore, there exists a matrix $C_i \in \mathbb{R}^{n \times n}$ such that*

$$J_i(X_i) = X_i^\top C_i X_i. \quad (24)$$

A_i, B_i and C_i are given recursively by $A_N = I, B_N = 0, C_N = \Lambda + \alpha \cdot \Sigma$ and $A_i = g_A(C_{i+1}), B_i = g_B(C_{i+1})$ and $C_i = g_C(C_{i+1})$ for functions $g_A, g_B, g_C : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ independent of i . Furthermore, C_i is positive definite.

4.3 Liquidating a single-asset position

The easiest case to analyze is the liquidation of a position X_0 in a single asset, i.e., $n = 1$. As some of the most interesting effects of using dark pools can already be observed in this simple case, we will study it in more depth in this subsection.

In order to understand the effects of dark pools on portfolio liquidation, it is desirable to compare the optimal strategies we obtained in the previous subsection with optimal strategies for traders that do not use dark pools. If p denotes the probability that the order in the dark pool is executed in a time-interval $[t_i, t_{i+1})$, this corresponds to the case $p \rightarrow 0$. Solutions to the optimal liquidation problem for this case are known, see, e.g., Almgren and Chriss (1999), Almgren and Chriss (2001).

For the remainder of the subsection we let $X_0 > 0$. The following proposition gives explicit formulae for the optimal strategy and the value function matrix. As a side product of the proof we also obtain recursive formulae.⁴

Proposition 4.2. *Let*

$$\kappa_p := \operatorname{arcosh} \left(\frac{\sqrt{1-p}}{2} \left(\frac{\alpha \Sigma}{\Lambda} + 1 + \frac{1}{1-p} \right) \right). \quad (25)$$

Then the optimal orders at time t_i are given by $x_i = A_i(p)X_i$ and $y_i = B_i(p)X_i$ with

$$A_i(p) = 1 - \frac{\sinh(\kappa_p(N-i))}{\sqrt{1-p} \sinh(\kappa_p(N+1-i))}, \quad (26)$$

⁴Originally we obtained the explicit formulae in a similar way as Almgren and Chriss (2001) obtained those for trading without dark pool, i.e., we computed the optimal asset position in time via a second order linear difference equation.

$$B_i(p) = 1 - A_i(p) = \frac{\sinh(\kappa_p(N-i))}{\sqrt{1-p} \sinh(\kappa_p(N+1-i))} < 1, \quad (27)$$

in particular $0 < x_i, y_i < X_i$ for $i \neq N$. The value function is given by $J_i(X_i) = C_i(p)X_i^2$ with

$$C_i(p) = \frac{\Lambda}{1-p} \cdot \left(\frac{\sqrt{1-p} \sinh(\kappa_p(N+2-i))}{\sinh(\kappa_p(N+1-i))} - 1 \right). \quad (28)$$

Corollary 4.3. *Assume that the optimal liquidation strategy is being pursued. If no dark pool order has been executed until t_i , then the asset position at time t_i is*

$$X_i^{\text{ne}}(p) := X_i = \frac{1}{\sqrt{1-p}^i} \frac{\sinh(\kappa_p(N+1-i))}{\sinh(\kappa_p(N+1))} X_0 \quad (29)$$

and the optimal orders at time t_i are given by

$$x_i^{\text{ne}}(p) := x_i = A_i(p)X_i = \frac{1}{\sqrt{1-p}^{i+1}} \frac{\sqrt{1-p} \sinh(\kappa_p(N+1-i)) - \sinh(\kappa_p(N-i))}{\sinh(\kappa_p(N+1))} X_0, \quad (30)$$

$$y_i^{\text{ne}}(p) := y_i = B_i(p)X_i = \frac{1}{\sqrt{1-p}^{i+1}} \frac{\sinh(\kappa_p(N-i))}{\sinh(\kappa_p(N+1))} X_0 = X_i^{\text{ne}}(p) - x_i^{\text{ne}}(p). \quad (31)$$

If a dark pool order has been executed before t_i , then $X_i = x_i = y_i = 0$.

As a consequence of Proposition 4.2 and Corollary 4.3, we can compare the optimal strategies with and without dark pools.

Corollary 4.4 (Properties of the optimal strategy). *The optimal strategy stated in Proposition 4.2 has the following properties:*

1. *Trading costs are small when the dark pool is liquid, i.e., for $i \in \{1, \dots, N-1\}$, $C_i(p)$ is strictly decreasing in p .*
2. *The higher the probability of dark pool execution, the slower we will initially trade in the primary venue. More precisely, for $i \in \{1, \dots, N-1\}$, $X_i^{\text{ne}}(p)$ is strictly increasing in p . In particular*

$$X_i^{\text{ne}}(p) > X_i^{\text{ne}}(0) \quad \forall p > 0. \quad (32)$$

3. *Let $\mathbb{E}_p[X_i]$ be the expected asset position at time t_i if the probability of execution in the dark pool is p and the optimal strategy is applied. Then:*

$$\mathbb{E}_p[X_i] = (1-p)^i X_i^{\text{ne}}(p) = \frac{\sqrt{1-p}^i \sinh(\kappa_p(N+1-i))}{\sinh(\kappa_p(N+1))} X_0 \quad (33)$$

and $\mathbb{E}_p[X_i]$ is strictly decreasing in p for $i \in \{1, \dots, N-1\}$, in particular

$$\mathbb{E}_p[X_i] < \mathbb{E}_0[X_i] \quad \forall p > 0. \quad (34)$$

Proposition 4.2 and Corollary 4.4 answer the questions of how to use a dark pool optimally and how to adjust the trading strategy in the primary venue for liquidating a single asset position. Intuitively, one would answer these questions as follows:

- As we do not pay price impact in the dark pool, it should always be optimal to post the largest possible order in the dark pool.
- As we want as much as possible to be executed in the dark pool, we should slow down the trading speed in the primary venue. If we are not executed towards the end, we have to speed up in order to finish the liquidation until time T .

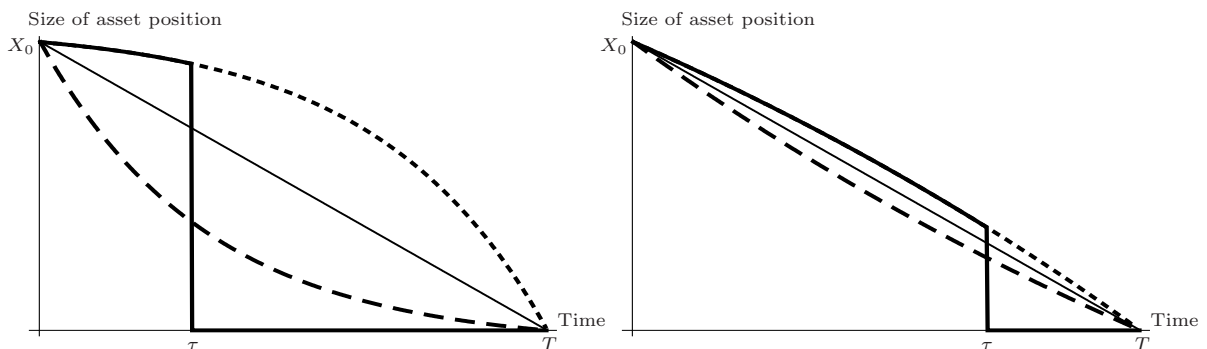


Figure 1: Comparison of the portfolio evolution for the optimal strategies of a risk-neutral trader ($\alpha = 0$) of a stock with large (left) and small (right) probability of execution p in the dark pool. The task of the trader is to liquidate a position $X_0 = 1$ in $N + 1 = 501$ trading times. Furthermore: $\Lambda = 500$, $\Sigma = \frac{1}{500}$ and $p = \frac{3}{500}$ (left graph), $p = \frac{1}{1000}$ (right graph). The solid lines show the scenario, where a trade in the dark pool is executed in the τ^{th} trading period (left $\tau = 150$, right $\tau = 350$).

By (27) and (32) both intuitions are confirmed by our model. Furthermore, the higher the probability of execution in the dark pool, the slower we trade in the primary venue. Conversely, the higher the probability of execution, the smaller the expected asset position at each point in time. Figures 1 and 2 illustrate how the dark pool changes the optimal strategy in the primary venue. In all pictures, the optimal strategy without using the dark pool is denoted by the thin line. When the dark pool is used, then the portfolio evolution is stochastic and depends on the liquidity found in the dark pool. We illustrate the stochastic portfolio evolution with three lines. The solid line shows portfolio evolution in the scenario where liquidity is found in the dark pool at time τ . If there is no liquidity found in the dark pool during the entire trade execution, the trader will follow the dotted line until time T . In the figures, it can be observed how the trading speed is slowed down by the introduction of a dark pool. The dashed line denotes the expected asset position over time if a dark pool is used.

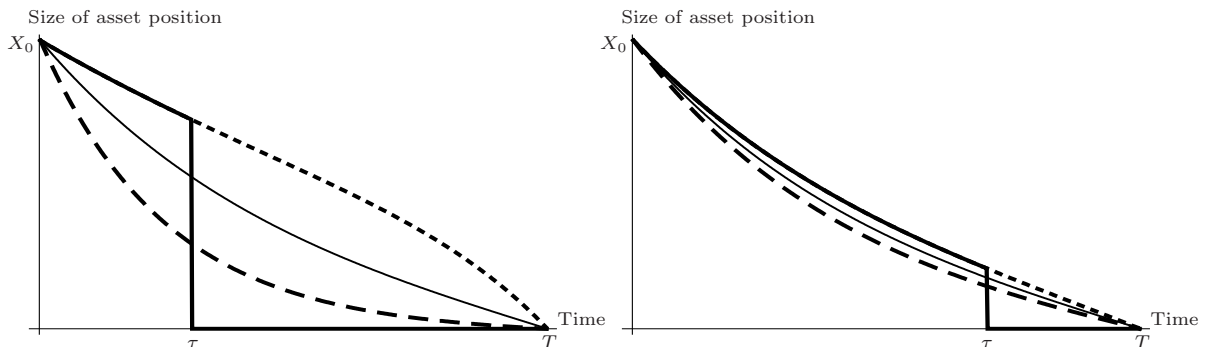


Figure 2: The same liquidation problems as in Figure 1, but for a risk-averse trader ($\alpha = 4$).

It is also clear that a higher probability of execution reduces the costs. Figure 3 illustrates the costs of the strategies depending on $N \cdot p$, i.e., the expected number of executions in the dark pool within the trading horizon $[0, T]$, if the probability of execution in each trading interval is p . The dotted line denotes the costs of not using a dark pool (which is obviously independent of p). The solid line represents the costs of the optimal strategy and the dashed line represents the costs of the following naïve strategy used in a similar form by many traders in practice:

Use the optimal strategy without dark pools for the primary venue and place the remainder of the position in the dark pool!

This strategy is obviously cheaper than not using dark pools, as both impact costs and risk costs are at most as large and smaller if an order in the dark pool is executed before time T . However, it is significantly more expensive than the optimal strategy, as illustrated by Figure 3.

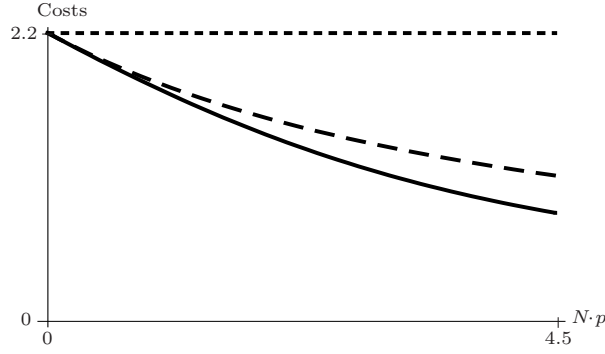


Figure 3: Costs of the optimal strategy (solid line), the optimal strategy without dark pool (dotted line) and the naïve strategy (dashed line) dependent on $N \cdot p$, i.e., the expected number of executions in the dark pool. $N = 15, \Lambda = 15, \Sigma = \frac{1}{15}, \alpha = 4$.

The costs of a liquidation strategy are composed of impact costs of trading in the primary venue

$$\Lambda \cdot \mathbb{E} \left[\sum_{i=0}^N x_i^2 \right] \quad (35)$$

and the risk costs

$$\alpha \cdot \Sigma \cdot \mathbb{E} \left[\sum_{i=0}^N X_i^2 \right]. \quad (36)$$

Applying the optimal strategy we obtain

$$\mathbb{E} [X_i^2] = (1-p)^i X_i^{\text{ne}}(p)^2 = \left(\frac{\sinh(\kappa_p(N+1-i))}{\sinh(\kappa_p(N+1))} \right)^2 X_0^2 \quad (37)$$

by Equation (29). This term is increasing for $p \in (0, \frac{\alpha\Sigma}{\Lambda+\alpha\Sigma})$ and decreasing for $p \in (\frac{\alpha\Sigma}{\Lambda+\alpha\Sigma}, 1)$. Thus the risk costs are in general not decreasing in p , in particular it is generally not true that the risk costs of using a dark pool are less than the risk costs of not using a dark pool.

On the other hand, we obtain

$$\mathbb{E}[x_i^2] = (1-p)^i x_i^{\text{ne}}(p)^2 = \frac{1}{1-p} \left(\frac{\sqrt{1-p} \sinh(\kappa_p(N+1-i)) - \sinh(\kappa_p(N-i))}{\sinh(\kappa_p(N+1))} \right)^2 X_0^2 \quad (38)$$

by (30). Equation (38) strongly suggests that the impact costs in Equation (35) are strictly decreasing in p . We leave a proof of this for future research. Figure 4 illustrates the dependence of risk and impact costs on p .

Let us consider the limit of an infinite trading time horizon $T \rightarrow \infty$ (respectively $N \rightarrow \infty$). This is equivalent to dropping the requirement that the liquidation has to be finished within a finite time horizon. If a trader can only use the primary exchange, then she can liquidate positions at arbitrarily small costs if she is risk-neutral, but not if she is risk-averse. In both cases, the liquidation requires an infinite time to complete (see Almgren (2003)). The following proposition shows that a risk-averse investor with an infinite trading time horizon cannot liquidate arbitrarily cheaply even if a dark pool is available, but the average time required for liquidation is finite.

Proposition 4.5. *For a trader with an infinite time horizon ($N \rightarrow \infty$), the cost function is $J(X_0) = CX_0^2$ with*

$$C := \lim_{N \rightarrow \infty} C_0(p) = \frac{\Lambda}{1-p} \left(\sqrt{1-p} \exp(\kappa_p) - 1 \right). \quad (39)$$

C is strictly positive for risk averse traders ($\alpha > 0$) and zero for risk-neutral traders.

Let \tilde{T} be the time taken for liquidation. The average liquidation time is

$$\mathbb{E}_p[\tilde{T}] = \frac{1-p}{p} < \infty. \quad (40)$$

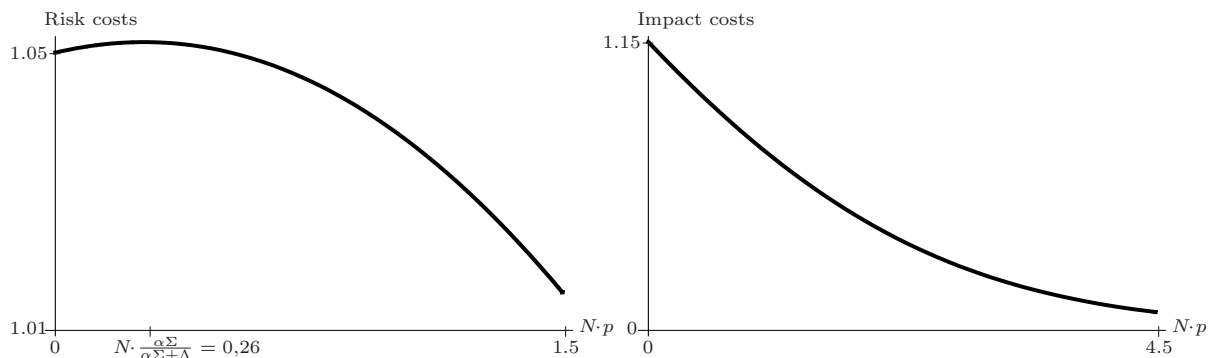


Figure 4: Risk costs (left graph) and impact costs (right graph) of the optimal strategy dependent on the expected number of executions in the dark pool. For $(0, N \cdot \frac{\alpha \Sigma}{\alpha \Sigma + \Lambda})$, the risk costs are increasing, whereas the impact costs are strictly decreasing in the whole interval $(0, N \cdot p)$. The parameters are the same as in Figure 3.

Obviously, for the performance of the strategies it is essential to estimate the parameters Λ, Σ , and p appropriately. Especially for the probability of execution this seems to be difficult in practice: As orders are not reported openly in dark pools, it is hard to obtain useful data. Therefore it is important to know the effects of an imprecise estimate of p .

Let us assume that we have estimated the average number of executions in $[0, T]$ to be $N \cdot q$. We have seen already that applying the optimal strategy $(x_i(q), y_i(q))_i$ reduces liquidation costs significantly - provided that q equals the real-world probability of execution p .

If we have underestimated p , i.e. $q < p$, the strategy $(x_i(q), y_i(q))_i$ is still cheaper than the optimal strategy without using dark pools. On the other hand, overestimating p significantly can possibly make the strategy $(x_i(q), y_i(q))_i$ more expensive than the optimal strategy without using dark pools.

Using Equations (29) and (30) we can easily compute the range in which the real-world probability p needs to be in order to ensure that the costs of the strategy $(x_i(q), y_i(q))_i$ are less than the costs without dark pool: If p is the real-world probability of execution and q the estimate of a trader, the strategy $(x_i(q), y_i(q))_i$ yields the costs

$$C_0^q(p) := \Lambda \cdot \sum_{i=0}^N (1-p)^i x_i^{\text{ne}}(q)^2 + \alpha \Sigma \cdot \sum_{i=0}^N (1-p)^i X_i^{\text{ne}}(q)^2. \quad (41)$$

Thus as long as $p > \tilde{p}$, where \tilde{p} can be computed via

$$C_0^q(\tilde{p}) = C_0(0), \quad (42)$$

using $(x_i(q), y_i(q))_i$ is cheaper than not using the dark pool. Figure 5 illustrates the costs of $(x_i(q), y_i(q))_i$ dependent on p .

4.4 Liquidating a portfolio of two assets

If a portfolio of multiple assets has to be liquidated ($n \geq 2$) for a risk-averse investor, then correlation between the assets comes into play. It might no longer be optimal to always submit the remaining portfolio into the dark pool. For example, a trader liquidating a well diversified portfolio consisting of two assets will most likely not want to risk losing her unrisky position by being executed in only one of the two assets. Intuitively, we expect the optimal order placement in the dark pools to have the following properties:

- If the portfolio is well diversified in the beginning, the orders in the dark pools should be much smaller than the current portfolio, as the trader does not want to risk entering an undiversified position. The trading speed in the primary venue should be almost constant, since the portfolio position bears little risk and a constant trading speed minimizes the price impact cost (cf. the thin lines in Figure 1).
- If the portfolio is poorly diversified, the orders should initially be comparatively large both in the primary venue and in the dark pool. They might even be larger than the current portfolio. In this case the execution of the dark pool order for one of the assets leads to a less risky position.

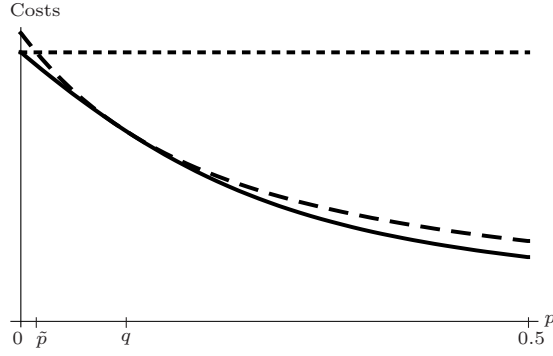


Figure 5: The dotted line represents the costs without dark pool, the solid line the costs of the optimal strategy dependent on p and the dashed line the costs of the strategy $(x_i(q), y_i(q))_i$ (dependent of the real world probability p); \tilde{p} denotes the minimal value of the real world probability p such that trading with dark pool does not become more expensive than without. The estimate is $q = 0.1$. The parameters are as in Figure 3.

It is possible to find explicit formulae for the recursions, both of the value function matrix and the optimal strategy by the same procedure as for the cases $n = 1$ for any fixed $n \in \mathbb{N}$. However, the calculation becomes extremely tedious and the formulae become rather unhandy even for $n = 2$. Therefore we abstain to state the formulae for any $n > 1$ (the formulae for the case $n = 2$ are available from the authors upon request).

We will now illustrate the above intuitions by some numeric examples. First, we want to compare the expected portfolio evolution using the optimal strategies with and without dark pools. To this end, we consider different portfolios of two highly correlated stocks with

$$\Sigma = \frac{1}{500} \cdot \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}. \quad (43)$$

We model the second stock as being more liquid. This is reflected in both a smaller price impact and a higher execution probability in the dark pool⁵ compared to the first asset:

$$\Lambda = 500 \cdot \begin{pmatrix} 3 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad (44)$$

$$\mathbb{P}[\text{No dark pool execution}] = \frac{993}{1000}, \quad (45)$$

$$\mathbb{P}[\text{Dark pool execution of first asset}] = \frac{1}{1000}, \quad (46)$$

$$\mathbb{P}[\text{Dark pool execution of second asset}] = \frac{6}{1000}, \quad (47)$$

$$\mathbb{P}[\text{Dark pool execution of both assets}] = 0. \quad (48)$$

We will consider two cases in more depth:

- Long positions in both stocks, i.e., a poorly diversified portfolio: $X_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- A long position in the first and a short position in the second one, i.e., a well diversified portfolio: $X_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Figure 6 shows the evolution of the two portfolios. The left picture corresponds to the first case, the right one to the second. In both pictures, thin lines are used for the first stock and thick lines for the second. Dotted lines correspond to trading without dark pools, dashed lines correspond to the expected position in the assets if dark pools are used and the solid lines correspond to a realization of the liquidation process using dark pools, where the orders in the second dark pool are executed at times τ_1, τ_2, τ_3 and in the first dark pool only at time τ_4 .

⁵Intuitively, we expect a close connection between liquidity costs in the primary venue and probability of execution in the dark pool. However, we are not aware of any empirical work supporting this.

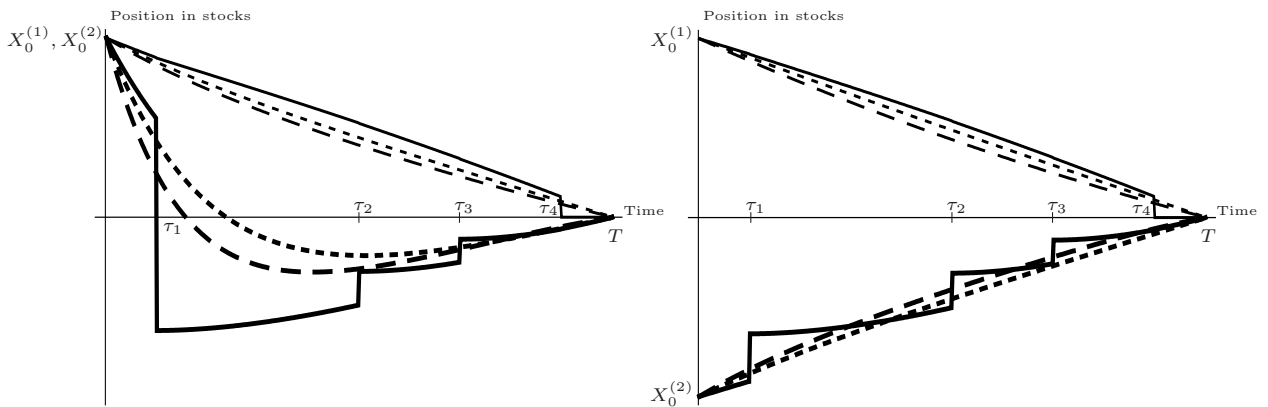


Figure 6: Evolution of a portfolio consisting of two highly correlated stocks over time. The left figure illustrates the poorly diversified portfolio, the right figure the well diversified portfolio. In the displayed scenario, orders are executed at times τ_1, τ_2, τ_3 and τ_4 .

For the poorly diversified portfolio, the trader tries to improve her risky position by trading out of the second stock. For this stock, trading in the primary venue is less expensive and being executed in the dark pool is more probable. If the trader uses dark pools, this process on average evolves significantly faster than without dark pools.

For the well diversified portfolio the portfolio position is decreasing almost linearly in time in all cases. We expect to trade only slightly faster if we use dark pools. Note that this corresponds to the intuition given at the beginning of the subsection: It is most profitable to trade out of the position almost evenly.

Additionally, orders in the dark pool are very large for the poorly diversified portfolio and comparatively small in the well diversified portfolio. The reason can be observed in Figure 7. The solid lines in the two pictures represent the evolution of portfolio risk $X^\top \Sigma X$ over time corresponding to the realized liquidation paths in Figure 6. The dotted lines represent the evolution of risk if the optimal strategy without dark pools is used and the dashed lined represent the expected evolution of risk.

As long as the portfolio is poorly diversified, the risk is relatively large and is decreased by a large execution in one of the dark pools significantly (left picture). However, if it is well diversified as in the right picture, each execution in the dark pools increases the risk. Therefore in this case dark pools save price impact costs, but potentially increase risk costs. Note also that in the case of an initially well diversified portfolio, the expected risk can be larger than the risk without using dark pools (right picture).

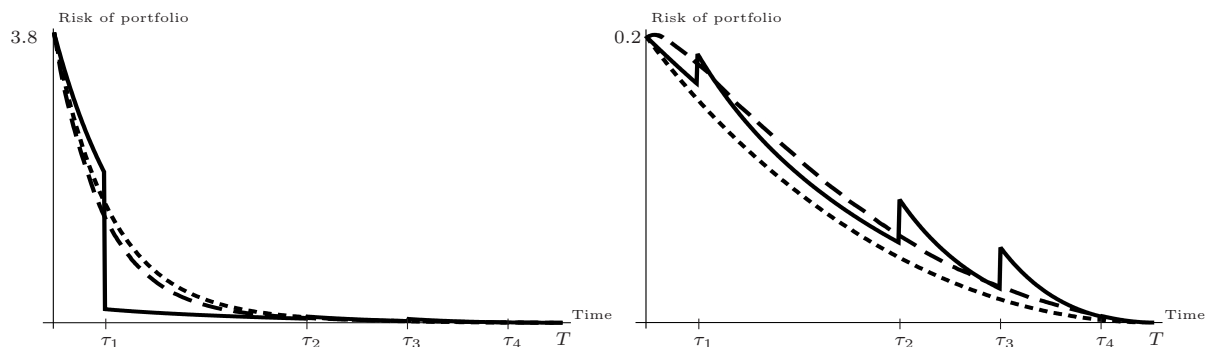


Figure 7: Evolution of risk $X_i^\top \Sigma X_i$ over time for the liquidation paths in Figure 6. The dashed line denotes the expected evolution of risk. Note the different scales in the left and the right graph.

5 Adverse selection

In the previous Section 4, we have assumed that the price increments ϵ_i are independent of the liquidity a_i, b_i in the dark pool. This assumption is not necessarily fulfilled in reality. In particular, other large traders might simultaneously seek liquidity in the dark pool and move prices at the primary trading venue. For a trader active both at the exchange and in the dark pool, this can have two consequences:

- Adverse selection: If price changes ϵ_i and liquidity a_i, b_i in the dark pool at time i are connected, liquidity seeking traders might find that their trades in the dark pool are usually executed just before a favorable price move, i.e., exactly when they *do not* want to be executed, since they miss out on the price improvement. In advance of adverse price movements, they might observe that they rarely find liquidity in the dark pool. We denote such a phenomenon as “adverse selection”.
- Dark pool ping: Any smart trader might try to obtain information about the level of liquidity a_i, b_i in the dark pool in order to predict future price movements ϵ_{i+1}, \dots at the exchange. This predictive information can then be exploited in a profit seeking trading strategy.

Since the focus of this paper is optimal trade execution, we limit the discussion to the effects of adverse selection for two reasons. First, most dark pools go to great length to prevent ping, since such practices obviously bring light into the dark pool and thus counteract its primary purpose. Hence, any investigation of dark pool ping needs to take into account the detailed legal and algorithmic measures introduced by dark pools, a topic which we do not want to touch upon. Second, the analysis of dark pool ping leaves the realm of optimal trade execution, since the optimal strategy will include trades even without any initial trading intention.

Note first that our model covers the case where the price increments ϵ_{i+1}, \dots are dependent on the dark pool variables $a_0, b_0, \dots, a_i, b_i$ (as long as the conditions in Section 2 are satisfied). Therefore we obtain a unique optimal trading strategy by Theorem 3.1.

As in Section 4, we assume the linear form of price impact f_i described in Equation (21) and the Poisson type execution in the dark pool ($a_i, b_i \in \{0, \infty\}$). Furthermore, we assume that \tilde{P} is a martingal with identically distributed increments ϵ_i . Thus Equation (18) becomes

$$J_i(X_i) = \min_{(x,y) \in \mathbf{A}_i(X_i)} \left\{ \mathbb{E} \left[\sum_{j=i}^N x_j^\top \Lambda x_j \right] + \mathbb{E} \left[\sum_{j=i}^N z_j^\top \epsilon_{j+1} \right] + \alpha \cdot \mathbb{E} \left[\sum_{j=i}^N X_j^\top \Sigma X_j \right] \right\}. \quad (49)$$

We obtain the Bellman Equation

$$J_i(X_i) = \min_{(x_i, y_i) \in \mathbb{R}^n \times \mathbb{R}^n} \left\{ x_i^\top \Lambda x_i + \alpha \cdot X_i^\top \Sigma X_i + \mathbb{E} [z_i^\top \epsilon_{i+1}] + \mathbb{E} [J_{i+1}(X_i - x_i - z_i)] \right\}. \quad (50)$$

For simplicity, we will restrict ourselves to the case $n = 1$ for the rest of the section. We assume that the dependence of the price movements on dark pool liquidity is of the following form (note that $\mathbb{E}[\epsilon_{i+1}] = 0$ by the assumption that \tilde{P} is a martingale):

$$\mathbb{E}[\epsilon_{i+1}|a_i = \infty] = -A < 0, \quad \mathbb{E}[\epsilon_{i+1}|b_i = \infty] = A > 0. \quad (51)$$

If p is the probability of execution in the dark pool, Equation (50) yields:

$$J_i(X_i) = \min_{(x_i, y_i) \in \mathbb{R} \times \mathbb{R}} \left\{ \Lambda x_i^2 + \alpha \Sigma X_i^2 + p|y_i|A + pJ_{i+1}(X_i - x_i - y_i) + (1-p)J_{i+1}(X_i - x_i) \right\}. \quad (52)$$

We will from now on use the following notation: the (unique) optimal strategy at time t_i for asset position X_i is denoted by $(x_i(X_i), y_i(X_i)) \in \mathbb{R} \times \mathbb{R}$. We obtain the following result:

Theorem 5.1. *For $i = 0, \dots, N$, the cost functional J_i is piecewise a quadratic polynomial and the optimal strategy $(x_i(X_i), y_i(X_i))$ is piecewise affine linear in X_i . More precisely:*

1. If $X_i > 0$, there exist

$$0 =: \bar{X}_{i,0} < \dots < \bar{X}_{i,N-i+1} := \infty, \quad (53)$$

such that for $i = 0, \dots, N$, $j = 0, \dots, N - i$ and $X_i \in (\bar{X}_{i,j}, \bar{X}_{i,j+1}]$ the optimal strategy and the cost functional are given by

$$x_i(X_i) = A_{i,j,1}X_i + A_{i,j,2}, \quad (54)$$

$$y_i(X_i) = B_{i,j,1}X_i + B_{i,j,2}, \quad (55)$$

$$J_i(X_i) = C_{i,j,1}X_i^2 + C_{i,j,2}X_i + C_{i,j,3}. \quad (56)$$

Thereby, for $i \in \{0, \dots, N-1\}$, $j \in \{1, \dots, N-i\}$

$$\begin{aligned} \bar{X}_{i,j} &= \frac{1}{2\alpha\Sigma(1-p)\sinh(\kappa_p)} \cdot \\ &\cdot \left(2\alpha\Sigma \left(d(i,j)\sinh(j\kappa_p)\sqrt{1-p}^{j+1} - e(i,j)\sinh((j-1)\kappa_p)\sqrt{1-p}^{j+2} \right) \right. \\ &\left. + Ap \left(\sqrt{1-p}^{j-1}\sinh(j\kappa_p) - \sqrt{1-p}^j\sinh((j-1)\kappa_p) - \sinh(\kappa_p) \right) \right), \end{aligned} \quad (57)$$

where

$$d(i,j) = \frac{A(C_{i+j}(0) + \Lambda)}{2C_{i+j}(0)\Lambda}, \quad (58)$$

$$e(i,j) = \frac{A}{2C_{i+j}(0)}, \quad (59)$$

in particular for $i = 0, \dots, N-1$

$$\bar{X}_{i,1} = \frac{A(C_{i+1}(0) + \Lambda)}{2C_{i+1}(0)\Lambda}. \quad (60)$$

Furthermore for $i = 0, \dots, N$

$$C_{i,0,1} = C_i(0), \quad C_{i,0,2} = C_{i,0,3} = 0, \quad (61)$$

$$A_{i,0,1} = A_i(0), \quad A_{i,0,2} = 0, \quad (62)$$

$$B_{i,0,1} = B_{i,0,2} = 0, \quad (63)$$

for $i = 0, \dots, N-1$, $j = 1, \dots, N-i-1$

$$A_{i,j,1} = 1 - \frac{\bar{X}_{i+1,j} - \bar{X}_{i+1,j-1}}{\bar{X}_{i,j+1} - \bar{X}_{i,j}}, \quad (64)$$

$$A_{i,j,2} = \frac{\bar{X}_{i+1,j}\bar{X}_{i,j} - \bar{X}_{i,j+1}\bar{X}_{i+1,j-1}}{\bar{X}_{i,j+1} - \bar{X}_{i,j}}, \quad (65)$$

$$B_{i,j,1} = 1 - A_{i,j,1}, \quad (66)$$

$$B_{i,j,2} = -A_{i,j,2} - \frac{A}{2C_{i+1}(0)}, \quad (67)$$

$$C_{i,j,1} = \frac{\Lambda}{1-p} \left(\frac{\bar{X}_{i-1,j+2} - \bar{X}_{i-1,j+1}}{\bar{X}_{i,j+1} - \bar{X}_{i,j}} - 1 \right), \quad (68)$$

$$C_{i,j,2} = \frac{\bar{X}_{i,j+1}(2\Lambda\bar{X}_{i-1,j+1} - Ap) - \bar{X}_{i,j}(2\Lambda\bar{X}_{i-1,j+2} - Ap)}{(1-p)(\bar{X}_{i,j+1} - \bar{X}_{i,j})} \quad (69)$$

and for $i = 0, \dots, N-1$

$$A_{i,N-i,1} = A_i(p), \quad (70)$$

$$A_{i,N-i,2} = \frac{Ap(\sqrt{1-p}\sinh(\kappa_p(N+1-i)) - \sinh(\kappa_p(N-i)) - \sqrt{1-p}^{N+1-i}\sinh(\kappa_p))}{2\alpha\Sigma(1-p)^{3/2}\sinh(\kappa_p(N+1-i))}, \quad (71)$$

$$B_{i,N-i,1} = B_i(p), \quad (72)$$

$$B_{i,N-i,2} = -A_{i,j,2} - \frac{A}{2C_{i+1}(0)}, \quad (73)$$

$$C_{i,N-i,1} = C_i(p), \quad (74)$$

$$C_{i,N-i,2} = \frac{Ap\Lambda(\sqrt{1-p}^{N+2-i}\sinh(\kappa_p) - \sqrt{1-p}\sinh(\kappa_p(N+2-i)))}{(1-p)^2\alpha\Sigma\sinh(\kappa_p(N+1-i))} - \frac{Ap\Lambda(\Lambda + (1-p)\alpha\Sigma)}{(1-p)^2\alpha\Sigma}. \quad (75)$$

Finally, for $i = 0, \dots, N-1$, $j = 1, \dots, N-i$, $C_{i,j,3}$ is given recursively by

$$C_{i,j,3} = \frac{-1}{4C_{i+1}(0)(C_{i+1,j-1,1}(1-p) + \Lambda)} \cdot \left\{ A^2 p (C_{i+1,j-1,1}(1-p) + \Lambda) + C_{i+1}(0) \left((C_{i+1,j-1,2}^2 - 4C_{i+1,j-1,1}C_{i+1,j-1,3})(1-p)^2 + 2AC_{i+1,j-1,2}(1-p)p + A^2 p^2 - 4C_{i+1,j-1,3}\Lambda(1-p) \right) \right\}. \quad (76)$$

2. For $i = 0, \dots, N$, $X_i \leq \bar{X}_{i,1}$ we have

$$y_i(X_i) = 0, \quad (77)$$

$$x_i(X_i) = A_i(0)X_i, \quad (78)$$

$$J_i(X_i) = C_i(0)X_i^2, \quad (79)$$

in particular $X_i - x_i(X_i) \leq \bar{X}_{i+1,1}$ for $i < N$.

Furthermore, for $i < N$, $j = 1, \dots, N-i$ and $X_i \in (\bar{X}_{i,j}, \bar{X}_{i,j+1})$, $x_i(X_i)$, $y_i(X_i)$ and $X_i - x_i(X_i)$ are increasing in X_i and

$$X_i - x_i(X_i) - y_i(X_i) = \frac{A}{2C_{i+1}(0)} < \bar{X}_{i+1,1}, \quad (80)$$

$$X_i - x_i(X_i) \in (\bar{X}_{i+1,j-1}, \bar{X}_{i+1,j}). \quad (81)$$

3. For $X_i < 0$ the optimal strategy and the cost functional are given by

$$x_i(X_i) = -x_i(-X_i), \quad (82)$$

$$y_i(X_i) = -y_i(-X_i), \quad (83)$$

$$J_i(X_i) = J_i(-X_i). \quad (84)$$

We can directly deduce the following properties of the optimal strategy.

Corollary 5.2. Let $i = 0, \dots, N$ and $X_i > 0$.

1. For $X_i < \bar{X}_{i,1}$ the dark pool is not used at times t_i, \dots, t_N and the optimal strategy and the cost functional are the same as without adverse selection and without dark pool.

2. For $i < N$, $j = 1, \dots, N-i$ and $X_i \in (\bar{X}_{i,j}, \bar{X}_{i,j+1})$ the dark pool is only used at times t_i, \dots, t_{i+j-1} (provided that no order in the dark pool is executed before). From time t_{i+j} respectively after execution in the dark pool it is not used anymore.

3. For $i = 0, \dots, N$, $j = 0, \dots, N-i$

$$C_{i,j,1}, C_{i,j,2} \geq 0, \quad C_{i,j,3} \leq 0 \quad (85)$$

and $C_{i,j,1}$ and $C_{i,j,3}$ are decreasing in j and $C_{i,j,2}$ is increasing in j .

Moreover

$$A_{i,j,1}, A_{i,j,2}, B_{i,j,1} \geq 0, \quad B_{i,j,2} \leq 0 \quad (86)$$

and $A_{i,j,1}$ and $B_{i,j,2}$ are decreasing in j and $A_{i,j,2}$ and $B_{i,j,1}$ are increasing in j , i.e., $x_i(X_i)$ is concave and $y_i(X_i)$ is convex in X_i .

According to Theorem 5.1 and Corollary 5.2 the optimal strategy has the following properties:

- As trading in the dark pool is not entirely free anymore (but inhabits the miss of a favourable price movement), it is not optimal to place the entire remainder of the position in the dark pool. Furthermore, if X_i is below the time-dependent boundary $\bar{X}_{i,1}$ it is optimal to place no order in the dark pool. The optimal strategy below this boundary is the same as the one without adverse selection and without dark pool.

- Only if $X_0 > \bar{X}_{0,N}$ and orders in the dark pool are never executed, the optimal asset position stays above this boundary throughout the whole time horizon $[0, T]$, i.e., it is optimal to use the dark pool at all trading times t_0, \dots, t_{N-1} .

If $X_0 \in (\bar{X}_{0,j}, \bar{X}_{0,j+1})$ for $j < N$, then - provided no order is executed in the dark pool before - $X_{j-1} > \bar{X}_{j-1,1}$ and $X_j < \bar{X}_{j,1}$, i.e., the dark pool is only used for the first j trades.

- If the order in the dark pool is executed, the boundary is crossed.
- As soon as the boundary is crossed (either by execution in the dark pool or by trading in the primary venue), the optimal asset position stays below the boundary for the rest of the trading horizon.
- By Equations (70), (72) and (74) adverse selection neither changes the optimal strategy nor the cost functional significantly for very large X_i , i.e., for $X_i \gg \bar{X}_{i,N-i}$.

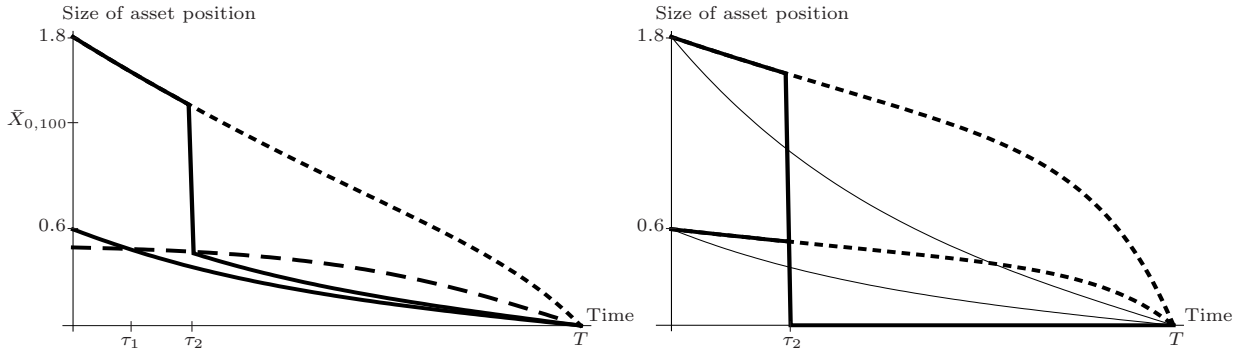


Figure 8: The left picture shows the optimal strategies for a smaller and a larger asset position, if adverse selection is expected. The right picture shows the respective strategies, if adverse selection is not expected. In all cases we consider the scenario where the order in the dark pool is executed at time τ_2 . The solid lines denote the optimal strategies for this scenario, the dotted those for the scenario where orders in the dark pool are never executed. The dashed line in the left picture reflects $\bar{X}_{i,1}$, i.e., the boundary below which the optimal order in the dark pool is zero. The thin lines in the right pictures denote the optimal strategies without dark pools. $N = 100$, $\Lambda = 100$, $\Sigma = \frac{1}{100}$, $p = \frac{6}{100}$, $\alpha = 4$, $A = 2$.

We illustrate these properties in Figure 8. The left picture shows the optimal strategies for two initial asset positions. A larger one ($X_0 = 1.8$) which lies above $\bar{X}_{0,N} = \bar{X}_{0,100} = 1.27$ and a smaller one ($X_0 = 0.6$) which lies between $\bar{X}_{0,12}$ and $\bar{X}_{0,13}$. Consequently, the larger asset position crosses the boundary $\bar{X}_{i,1}$ (dashed line) only if the order in the dark pool is executed (which happens at time τ_2 in the displayed scenario). The smaller one crosses the boundary after the twelfth trading period, i.e. at time τ_1 , if no order in the dark pool is executed before.

Compared to the optimal strategies, if no adverse selection is expected (right picture, solid lines) the trading speed in the primary venue is initially faster, but still slower than the trading speed of the optimal strategies without dark pool (thin lines). Additionally the order in the dark pool is smaller than the remainder of the position and after execution in the dark pool the trader trades out of the rest solely in the primary venue. The reason for both properties is the fact that trading in the dark pool is not entirely free anymore - and thus (in relation to it) trading is cheaper in the primary venue.

Figure 9 shows the dependence of the optimal order size in the primary venue (solid line) and in the dark pool (dashed line) on the asset position. As proven in Corollary 5.2, the order size in the primary venue is concave in X_0 and the order size in the dark pool is convex in X_0 .

In Subsection 4.3 we showed that a high probability of execution in the dark pool, slows down trading in the primary venue initially. Intuitively adverse selection should have the opposite effect: Higher adverse selection should speed up trading in the primary venue, as trading in the dark pool is more expensive and thus waiting for execution is less profitable. The following corollary confirms this intuition.

Corollary 5.3. *Let $X_0 > 0$ and define X_i^{ne} ($i = 0, \dots, N$) recursively by $X_0^{\text{ne}} = X_0$, $X_{i+1}^{\text{ne}} = X_i^{\text{ne}} - x_i(X_i^{\text{ne}})$, i.e., X_i^{ne} is the optimal asset position at time t_i provided no order has been executed before. We have that $x_i(X_i)$ is strictly increasing in A and X_i^{ne} is strictly decreasing in A . Furthermore, $y_i(X_i)$ is strictly decreasing in A .*

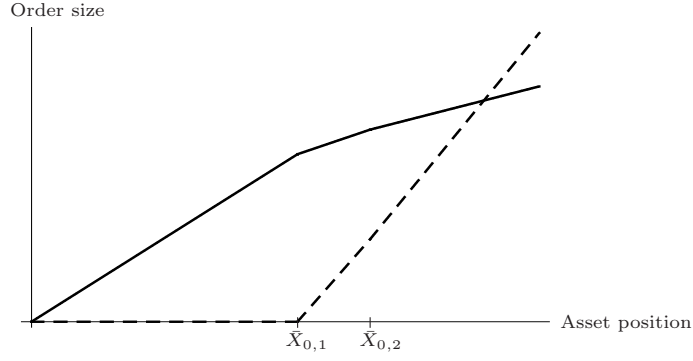


Figure 9: Optimal order sizes in the dark pool and the primary venue at time t_0 dependent on the asset position X_0 . The solid line denotes the optimal order size in the primary venue $x_0(X_0)$, the dashed line the optimal order size in the dark pool $y_0(X_0)$. $N = 2$, $\Lambda = 10$, $\Sigma = \frac{1}{2}$, $p = 0.6$, $\alpha = 4$, $A = 10$.

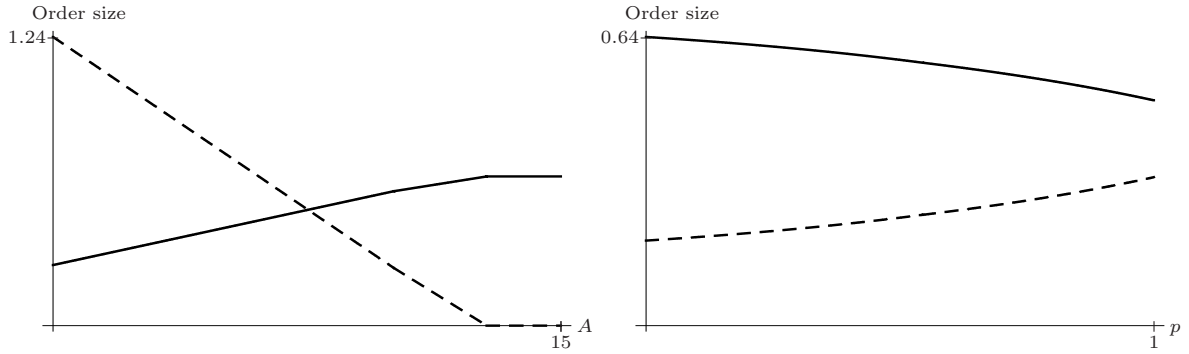


Figure 10: Dependence of the optimal order size in the primary venue (solid lines) and in the dark pool (dashed lines) on adverse selection (left picture) and on the probability of execution in the dark pool (right picture), respectively. $X_0 = 1.5$, $N = 2$, $\Lambda = 10$, $\Sigma = 1/2$, $\alpha = 4$; furthermore, $p = 0.6$ in the left picture, $A = 10$ in the right picture.

Figure 10 shows the dependence of the trading speed in the primary venue on adverse selection (left picture) and on the probability of execution (right picture). As we have shown in Corollary 5.3, large adverse selection speeds up trading in the primary venue and decreases the optimal order size in the dark pool. On the other hand, probable execution in the dark pools slows down trading in the primary venue and increases the optimal order size in the dark pool.

6 Trading restrictions

In practice, traders liquidating a portfolio for a client often face trading restrictions. For example, traders might not be allowed to short any of the stocks in the portfolio. Additionally, they are often bound to the trading direction intended by the client at all points in time: if the client wants to sell stock A and buy stock B, then the trader must not submit intermediate buy orders for stock A or sell orders for stock B, even if they might appear attractive from a risk mitigation perspective as described in Section 4.4. In particular, trading strategies as in the left picture of Figure 6 are not admissible.

In this section, we will allow for general trading restrictions on trades x_i, y_i and intermediate portfolio positions X_i . For each point in time t_i and asset k , we assume that there bounds $x_{i,\min}^{(k)}, x_{i,\max}^{(k)}, y_{i,\min}^{(k)}, y_{i,\max}^{(k)}$ for

trading and $X_{i,\min}^{(k)}, X_{i,\max}^{(k)}$ for the portfolio position. We then define the set of admissible strategies as

$$\mathbb{A}'_i(X_i) = \left\{ (x, y) \in \mathbb{A}_i(X_i) \mid x_{j,\min}^{(k)} \leq x_j^{(k)} \leq x_{j,\max}^{(k)}, y_{j,\min}^{(k)} \leq y_j^{(k)} \leq y_{j,\max}^{(k)}, X_{j,\min}^{(k)} \leq X_j^{(k)} \leq X_{j,\max}^{(k)}, \forall j \geq i, k \right\}. \quad (87)$$

Since the set of admissible strategies is convex, the proof of Theorem 3.1 also establishes the following theorem:

Theorem 6.1. *There is a unique optimal strategy under trading constraints, i.e., there exists a unique*

$$(x, y) \in \mathbb{A}'_i(X_i) \quad (88)$$

realizing the minimum in

$$J'_i(x_0, \dots, x_{i-1}, X_i) := \min_{(x,y) \in \mathbb{A}'_i(X_i)} \left\{ \mathbb{E}[\mathcal{R}_i] + \alpha \cdot \mathbb{E} \left[\sum_{j=i}^N X_j^\top \Sigma_{j+1} X_j \right] \right\}. \quad (89)$$

Let us turn back to the case of the trading direction restriction. This is equivalent to

$$x_{i,\min}^{(k)} := y_{i,\min}^{(k)} := \begin{cases} 0 & \text{if } X_0^{(k)} \geq 0 \\ -\infty & \text{otherwise} \end{cases} \quad x_{i,\max}^{(k)} := y_{i,\max}^{(k)} := \begin{cases} \infty & \text{if } X_0^{(k)} > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (90)$$

Under the assumptions of Section 4, the optimal strategy in the single asset case ($n = 1$) *without a trading constraint* does not violate condition (90) (see (129)) and is hence also optimal within \mathbb{A}' . As already discussed, the optimal strategy within \mathbb{A} in the multiple asset case ($n \geq 2$) might violate condition (90). Unfortunately, the value function J' is not of a quadratic form like J and is thus harder to analyze analytically.

We illustrate the effects of restriction (90) by a simple 1-period example. The left graph of Figure 11 shows the unrestricted optimal strategy for the liquidation of a poorly diversified portfolio by a risk averse trader ($\alpha = 4$), i.e., long positions in two strongly correlated stocks:

$$\Sigma = \frac{1}{2} \cdot \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (91)$$

We consider the second stock to be more liquid than the first one with probabilities of execution in the dark pools 1/4 and 1/20, respectively (and probability 0 for simultaneous execution) and price impact matrix

$$\Lambda = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}. \quad (92)$$

The three lines represent the three possible scenarios: Either none of the orders in the dark pools is executed (solid line) or exactly one of the orders in the dark pools is executed (dashed line for the first stock and dotted line for the second stock).

If we restrict short-selling the picture differs significantly. In the specific 1-period example outlined above it is easy to see that the optimal strategy has to fulfil $X_0 = x_0 + y_0$, i.e., the whole portfolio is split into orders in the primary venue and orders in the dark pools. Using this restriction it is easy to compute x_0 and y_0 . While the orders in the primary venue are very similar to the ones of the unrestricted optimal strategy, the orders in the dark pools are significantly smaller.

7 Trading prices in the dark pool

So far we have assumed that trades in the dark pool are executed at the unaffected price \tilde{P} . Within this section, we will instead assume that dark pool orders are executed at the exchange quoted price P which includes the temporary market impact of the orders x_i . As discussed in Section 2.2, this might be a more appropriate assumption for some dark pools. As we will see in this section, this results in profitable market manipulating strategies unless the model parameters are chosen with great care. In this section, we will for simplicity assume the tractable single asset model described in Section 4.3 (in particular zero drift of the fundamental price, linear temporary impact and no adverse selection) and furthermore assume that the investor is risk neutral ($\alpha = 0$).

Market manipulation is a concern in all market models where a large trader's orders have a feedback effect on the execution price of her own orders. Huberman and Stanzl (2004) and Gatheral (2008) derive necessary

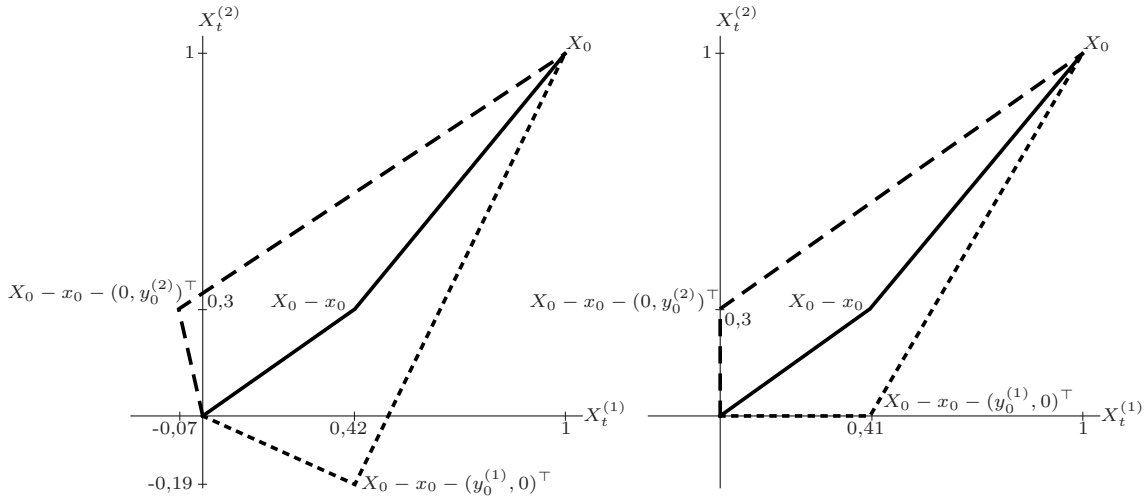


Figure 11: Change of the optimal strategy for the liquidation of a portfolio in a 1-period model, if short-selling is not allowed (left graph).

conditions for market models that exclude profitable market manipulation at a primary exchange. Both papers disregard trading opportunities in dark pools. Our market model for the primary exchange fulfills the requirements established in these papers, i.e., it is not possible to generate profits from market manipulation by trading *only at the primary exchange*. However it might be possible to generate profits from market manipulation if orders are placed cleverly in parallel in the dark pool. It is unclear whether such profitable market manipulation strategies exist in reality; given that such strategies were used and had to be forbidden⁶, such opportunities seem to be available at least sometimes. Nevertheless we agree with Huberman and Stanzl (2004) and Gatheral (2008) that an appropriate mathematical market model should exclude profitable market manipulation. For the purposes of this chapter, we define market manipulation as any order (whether in the dark pool or at the primary exchange) that is opposed to the underlying trading target, i.e., a buy order ($x_i < 0$ or $y_i < 0$) if a long position is to be liquidated ($X_i \geq 0$) or conversely a sell order if a short position is to be liquidated⁷.

In the following, we will consider in particular a market manipulation strategy similar to the classical ‘pump and dump’ strategy⁸. In our market model, selling the stock at the primary exchange after artificially elevating its price (‘pumping’) cannot generate profits due to the associated price reaction. A liquidation in the dark pool however does not face such a price penalty. Consider the following strategy: Assume that the initial asset position is 0 and that the number of trading time points $N + 1$ is divisible by four. From t_0 until $t_{(N+1)/4}$, the investor buys a stock quantity X at each point in time at the primary exchange. Simultaneously she seeks to dump shares by placing a sell order for $(N + 1)X/2$ in the dark pool until the order gets executed in the dark pool (if at all). At time $t_{(N+1)/4}$, the investor either holds a long or short position of $((N + 1)/4)X$ in the risky asset, which she liquidates at a constant rate over the remaining time points $t_{(N+1)/4}, \dots, t_N$. The expected trading proceeds are then

$$\mathbb{E} \left[\sum_{i=0}^N (x_i + z_i) P_i \right] = \underbrace{\mathbb{E} \left[\sum_{i=0}^N (x_i + z_i) \tilde{P}_i \right]}_{=0} - \mathbb{E} \left[\sum_{i=0}^N (x_i + z_i) \Lambda x_i \right] \quad (93)$$

$$= \Lambda \left(-(N + 1)(1/4 + 3/4 \cdot 1/9) X^2 + (1 - p^{(N+1)/4})(N + 1) X^2 / 2 \right) \quad (94)$$

$$= \Lambda(N + 1) \left(1/6 - p^{(N+1)/4} / 2 \right) X^2 \quad (95)$$

⁶See Gatheral (2008) for an exposition

⁷As we saw in Section 4.4, such orders can be attractive as risk mitigation tools in a multi asset setting. In the single asset setting of this section this justification does not apply, and we saw in Section 4.3 that if trades are executed in the dark pool at fundamental prices then market manipulation as defined here is never optimal.

⁸“‘Pump and dump’ schemes, also known as ‘hype and dump manipulation’, involve the touting of a company’s stock [...] . After pumping the stock, fraudsters make huge profits by selling their cheap stock into the market.” (From <http://www.sec.gov/answers/pumpedump.htm>)

The last expression is positive if the number of trading time points $N + 1$ is large enough. Furthermore, the expected proceeds grow in the position sizing factor X : the larger the bets, the larger the expected proceeds. The following proposition summarizes the issues we found.

Proposition 7.1. *Assume that trades in the dark pool are executed at the market price P . If*

$$N + 1 > 2 \log(1/6) / \log(p), \quad (96)$$

i.e., the number of trading time points $N + 1$ is large enough, then profitable market manipulation strategies exist and optimal strategies do not exist.

In Section 4.3, we assumed both infinite liquidity in the dark pool if trading is possible ($a_i, b_i \in \{0, \infty\}$) and no adverse selection (ϵ_{i+1} independent of a_i, b_i). Let us consider two alternative assumptions:

- Liquidity in the dark pool could be limited even if trading in the dark pool is possible: $a_i, b_i \in \{0, L\}$ with $0 < L < \infty$
- There might be adverse selection: $\mathbb{E}[\epsilon_{i+1}] = 0$, $\mathbb{E}[\epsilon_{i+1}|a_i = L] = -A < 0$, $\mathbb{E}[\epsilon_{i+1}|b_i = L] = A > 0$

By limiting dark pool liquidity, market manipulating strategies with very large trades cannot be profitable. On the other hand, adverse selection makes market manipulation by small trades unprofitable. The following proposition shows that if the original assumptions of Section 4.3 are replaced by the two previous assumptions and adverse selection is strong enough, then the undesirable properties outlined in Proposition 7.1 disappear.

Proposition 7.2. *Assume that trades in the dark pool are executed at the market price P . Furthermore assume that liquidity in the dark pool is limited even at points in time when dark pool trading is possible and that there is adverse selection. If*

$$A > 2\Lambda L, \quad (97)$$

i.e., adverse selection is strong enough, then optimal strategies exist and these are not market manipulating.

Note that neither limited dark pool liquidity nor adverse selection *alone* are sufficient to establish the previous proposition; only the *combination* of the two ensures the desired property. The assumptions of Proposition 7.2 are strong; we leave it for future research to determine tighter necessary and sufficient conditions for the exclusion of profitable market manipulation in markets with dark pools. We only want to remark that our assumptions are not too restrictive for dark pool usage in general: for large initial asset positions X_0 , the optimal strategy places orders in the dark pool in a non-market manipulating fashion.

A Appendix

A.1 Proofs

Proof of Proposition 3.2. Instead of describing a strategy (x, y) as a predictable function, we can alternatively describe it as a vector

$$v = \underbrace{(x_i(\omega_1), x_0(\omega_2), \dots, x_i(\omega_M), x_1(\omega_1), \dots, x_N(\omega_M))}_{=:v_x}, \underbrace{y_i(\omega_1), \dots, y_N(\omega_M)}_{=:v_y})^\top \in \mathbb{R}^{2 \times M \times (N+1-i)} \quad (98)$$

where $\Omega = \{\omega_1, \dots, \omega_M\}$. The objective function

$$J(v) = \mathbb{E}[\mathcal{R}_i] + \alpha \cdot \mathbb{E} \left[\sum_{j=i}^N X_j^\top \Sigma_{j+1} X_j \right] \quad (99)$$

is continuous in the strategy $v \in \mathbb{R}^{2 \times M \times (N+1-i)}$. We will now show that

$$\lim_{\|v\| \rightarrow \infty} J(v) = \infty. \quad (100)$$

The existence of an optimal strategy is a direct consequence of the previous property since the set of admissible strategies corresponds to a closed subset of $\mathbb{R}^{2 \times M \times (N+1-i)}$.

First we note that it is sufficient to prove Equation (100) for $\alpha = 0$, for which we obtain

$$J(v) = \mathbb{E}[\mathcal{R}_i] = \underbrace{\mathbb{E}\left[\sum_{j=i}^N x_j^\top f_j(x_0, \dots, x_j)\right]}_{=:A} + \underbrace{\mathbb{E}\left[\sum_{j=i}^N x_j^\top (\tilde{P}_i - \tilde{P}_j)\right]}_{=:B} + \underbrace{\mathbb{E}\left[\sum_{j=i}^N z_j^\top (\tilde{P}_i - \tilde{P}_j)\right]}_{=:C}. \quad (101)$$

It follows directly from the superlinear growth of the price impact cost of trading (Assumption (5)) that

$$\lim_{\|v_x\| \rightarrow \infty} \frac{A}{\|v_x\|} = \infty. \quad (102)$$

Since Ω is finite, $\tilde{P}_i - \tilde{P}_j$ is bounded, and thus

$$\lim_{\|v_x\| \rightarrow \infty} \frac{|B|}{\|v_x\|} < \text{const}_B < \infty \quad (103)$$

and

$$\lim_{\|v_y\| \rightarrow \infty} \frac{|C|}{\|v_y\|} < \text{const}_C < \infty. \quad (104)$$

By Assumption (13), we have

$$\lim_{\|v_y\| \rightarrow \infty} \frac{\|v_x\|}{\|v_y\|} > \text{const}_{xy} > 0. \quad (105)$$

Equation (100) now follows directly from Equations (102), (103), (104) and (105). \square

Proof of Proposition 3.3. We proceed by backwards induction over i . For $i = N$, the validity of the proposition follows since the only admissible strategy is $x_N = X_N, y_N = 0$ due to Assumption (13), and the convexity of Equation (19) follows directly from the convexity of the price impact cost of trading (Assumption (4)).

We now prove that if the proposition holds for $i + 1$, then it also holds for i . Consider two points $(x_0, \dots, x_{i-1}, X_i)$ and $(\tilde{x}_0, \dots, \tilde{x}_{i-1}, \tilde{X}_i)$. For these points, an optimal order x_i, y_i respectively \tilde{x}_i, \tilde{y}_i exists by Proposition 3.2. We now define

$$x_j(s) := (1-s)x_j + s\tilde{x}_j \quad \text{for all } 0 \leq j \leq i, \quad 0 \leq s \leq 1, \quad (106)$$

$$X_i(s) := (1-s)X_i + s\tilde{X}_i \quad \text{for all } 0 \leq s \leq 1. \quad (107)$$

Let us assume that we have a continuous function $y_i(s)$ such that $y_i(0) = y_i$ and $y_i(1) = \tilde{y}_i$. By the dynamic programming principle, we then have

$$\begin{aligned} J_i(x_0(s), \dots, x_{i-1}(s), X_i(s)) &\leq x_i(s)^\top f_i(x_0(s), \dots, x_i(s)) + \alpha X_i(s)^\top \Sigma_{i+1} X_i(s) \\ &\quad + (X_i(s) - x_i(s))^\top \mathbb{E}[\tilde{P}_i - \tilde{P}_{i+1}] - \mathbb{E}[z_i(s)^\top (\tilde{P}_i - \tilde{P}_{i+1})] \\ &\quad + \mathbb{E}[J_{i+1}(x_0(s), \dots, x_i(s), X_i(s) - x_i(s) - z_i(s))] \end{aligned} \quad (108)$$

$$=: h(s) \quad (109)$$

where the first inequality is an equality for $s = 0$ and $s = 1$. Note that we used the shorthand notation $z_i(s) := z_i(y_i(s))$. We will now show that for a suitable choice of the function $y_i(s)$, the expression

$$\sum_{j=0}^{i-1} x_j(s)^\top f_j(x_0(s), \dots, x_j(s)) + h(s) \quad (110)$$

is strictly convex in s . Equation (20) and thus the convexity of Expression (19) as well as the uniqueness of the optimal strategy at time t_i are then a direct consequence.

The key step in the proof is the definition of $y_i(s)$. We set it such that $\mathbb{E}[z_i(s)]$ is linear in s . More formally, we define a function

$$g(y) := \mathbb{E}[z(y)] \quad (111)$$

and set

$$y_i(s) := g^{-1}((1-s)\mathbb{E}[z_i] + s\mathbb{E}[\bar{z}_i]). \quad (112)$$

By $0 < s_1 < \dots < s_M < 1$ we denote the points at which there is a coordinate k such that $\mathbb{P}[-y_i^{(k)}(s_j) = a_i^{(k)}] > 0$ or $\mathbb{P}[y_i^{(k)}(s_j) = b_i^{(k)}] > 0$. On $]s_j, s_{j+1}[$, the convexity of Expression (110) is clear since the map

$$s \mapsto (x_0(s), \dots, x_i(s), z_i(s), X_i(s)) \quad (113)$$

is pathwise linear and the Expression (110) is pathwise strictly convex by the induction hypothesis. The only points where convexity of Expression (110) can break down are thus the points s_j . Pick one point s_j of these and define

$$\bar{z}_i^{(k)}(s) := \begin{cases} \mathbb{E} \left[z_i^{(k)}(s) \mid -a_i^{(k)} \leq y_i^{(k)}(s_j) \leq b_i^{(k)} \right] & \text{if } -a_i^{(k)} \leq y_i^{(k)}(s_j) \leq b_i^{(k)} \\ z_i^{(k)}(s) & \text{otherwise,} \end{cases} \quad (114)$$

$$\begin{aligned} \bar{h}(s) &:= x_i(s)^\top f_i(x_0(s), \dots, x_i(s)) + \alpha X_i(s)^\top \Sigma_{i+1} X_i(s) \\ &\quad + (X_i(s) - x_i(s))^\top \mathbb{E}[\tilde{P}_i - \tilde{P}_{i+1}] - \mathbb{E}[\bar{z}_i(s)^\top (\tilde{P}_i - \tilde{P}_{i+1})] \\ &\quad + \mathbb{E}[J_{i+1}(x_0(s), \dots, x_i(s), X_i(s) - x_i(s) - \bar{z}_i(s))]. \end{aligned} \quad (115)$$

Since $\bar{z}_i(s)$ is linear in s on $]s_{j-1}, s_{j+1}[$, we obtain that

$$\sum_{j=0}^{i-1} x_j(s)^\top f_j(x_0(s), \dots, x_j(s)) + \bar{h}(s) \quad (116)$$

is convex in s on $]s_{j-1}, s_{j+1}[$. Furthermore, it is clear that $\bar{h}(s_j) = h(s_j)$. Let us assume that $\bar{h}(s) \leq h(s)$ for $s \in]s_{j-1}, s_{j+1}[$. Then the convexity of Expression (110) follows at the point s_j and thus we have convexity of Expression (110) at all points s_j and on all intervals $]s_j, s_{j+1}[$, i.e., on all $[0, 1]$, completing the proof.

To show that $\bar{h}(s) \leq h(s)$ for $s \in]s_{j-1}, s_{j+1}[$, we observe

$$\begin{aligned} h(s) - \bar{h}(s) &= - \underbrace{\mathbb{E} \left[(z_i(s) - \bar{z}_i(s))(\tilde{P}_i - \tilde{P}_{i+1}); \exists k : -a_i^{(k)} \leq y_i^{(k)}(s_j) \leq b_i^{(k)} \right]}_{=: A'} \\ &\quad + \underbrace{\mathbb{E} \left[J_{i+1}(x_0(s), \dots, x_i(s), X_i(s) - x_i(s) - z_i(s)); \exists k : -a_i^{(k)} \leq y_i^{(k)}(s_j) \leq b_i^{(k)} \right]}_{=: B'} \\ &\quad - \underbrace{\mathbb{E} \left[J_{i+1}(x_0(s), \dots, x_i(s), X_i(s) - x_i(s) - \bar{z}_i(s)); \exists k : -a_i^{(k)} \leq y_i^{(k)}(s_j) \leq b_i^{(k)} \right]}_{=: C'}. \end{aligned} \quad (117)$$

It is clear that $B' > C'$ by convexity of $J_{i+1}(x_0, \dots, x_i, X_i)$ in X_i (by the induction hypothesis) and

$$\bar{z}_i(s) = \mathbb{E}[z_i(s) \mid \exists k : -a_i^{(k)} \leq y_i^{(k)}(s_j) \leq b_i^{(k)}]. \quad (118)$$

Finally, because of the rank correlation of dark pool liquidity and expected market price moves (Assumptions (11) and (12)), we know that $A' \leq 0$. \square

Proof of Theorem 4.1. For the proof of the theorem, we need to introduce the following notation: In each time-interval $[t_i, t_{i+1})$, there are 2^n possible scenarios with respect to execution and non-execution of the order $y_i \in \mathbb{R}^n$ in the dark pool. Each of these scenarios occurs with a fixed probability, which we denote by p_l for the l^{th} scenario, determined by the distribution of the random variables a_i and b_i . We denote the amount executed in the dark pool at time i in scenario l by $z_{i,l}$. Due to the symmetry condition (22), we have

$$z_{i,l}^{(k)} := \begin{cases} y_i^{(k)}, & \text{if in the } l^{\text{th}} \text{ scenario the order in the } k^{\text{th}} \text{ asset in the dark pool is executed,} \\ 0 & \text{otherwise} \end{cases} \quad (119)$$

independent of the sign of y_i . There exists a diagonal matrix $Z_l \in \mathbb{R}^{n \times n}$ (with 1's and 0's on the diagonal) such that

$$z_{i,l} = Z_l y_i. \quad (120)$$

We prove the theorem by backward induction. For $k = N$ the claim is trivial, as $x_N = IX_N$, $y_N = 0X_N$ and $J_N(X_N) = X_N^\top(\Lambda + \alpha \cdot \Sigma)X_N$. Note that $C_N = \Lambda + \alpha \cdot \Sigma$ is positive definite, as Λ is positive definite and Σ (as a covariance matrix) is positive semi-definite.

For $i < N$ we consider the Bellman Equation

$$J_i(X_i) = \min_{(x_i, y_i) \in \mathbb{R}^n \times \mathbb{R}^n} \left\{ x_i^\top \Lambda x_i + \alpha \cdot X_i^\top \Sigma X_i + \sum_l p_l J_{i+1}(X_i - x_i - Z_l y_i) \right\} \quad (121)$$

and use the induction hypothesis:

$$J_i(X_i) = \min_{(x_i, y_i) \in \mathbb{R}^n \times \mathbb{R}^n} \left\{ \underbrace{x_i^\top \Lambda x_i + \alpha \cdot X_i^\top \Sigma X_i + \sum_l p_l (X_i - x_i - Z_l y_i)^\top C_{i+1} (X_i - x_i - Z_l y_i)}_{=: \tilde{J}_i(X_i, x_i^{(1)}, \dots, x_i^{(n)}, y_i^{(1)}, \dots, y_i^{(n)})} \right\}. \quad (122)$$

$\tilde{J}(X_i, \cdot)$ is strictly convex (as C_{i+1} is positive definite) and thus any solution $(x_i, y_i) \in \mathbb{R}^n \times \mathbb{R}^n$ of

$$\frac{\partial \tilde{J}_i}{\partial x_i^{(1)}}(X_i, x_i, y_i) = 0, \dots, \frac{\partial \tilde{J}_i}{\partial x_i^{(n)}}(X_i, x_i, y_i) = 0, \frac{\partial \tilde{J}_i}{\partial y_i^{(1)}}(X_i, x_i, y_i) = 0, \dots, \frac{\partial \tilde{J}_i}{\partial y_i^{(n)}}(X_i, x_i, y_i) = 0 \quad (123)$$

is a global minimum of $\tilde{J}(X_i, \cdot)$. Therefore, by existence and uniqueness of the optimal strategy (Theorem 3.1), (123) has a unique solution (x_i, y_i) dependent on the portfolio X_i . It is easy to see that (123) is of the form

$$F(x_i, y_i) = G(X_i, X_i) \quad (124)$$

for matrices $F, G \in \mathbb{R}^{2n \times 2n}$. Thus the solution of (123) is linear in X_i and there exist matrices A_i, B_i such that $x_i = A_i X_i$, $y_i = B_i X_i$ and functions $g_A, g_B : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ such that

$$A_i = g_A(C_{i+1}), B_i = g_B(C_{i+1}). \quad (125)$$

Combining these observations, we obtain

$$J_i(X_i) = (A_i X_i)^\top \Lambda (A_i X_i) + \alpha \cdot X_i^\top \Sigma X_i + \sum_l p_l \cdot (X_i - A_i X_i - Z_l B_i X_i)^\top C_{i+1} (X_i - A_i X_i - Z_l B_i X_i) \quad (126)$$

$$= X_i^\top \left(\underbrace{A_i^\top \Lambda A_i + \alpha \Sigma + \sum_l p_l \cdot (I - A_i - Z_l B_i)^\top C_{i+1} (I - A_i - Z_l B_i)}_{=: C_i} \right) X_i, \quad (127)$$

and C_i is of the required form by (125). Clearly, C_i is positive semi-definite by induction hypothesis. To see that C_i is indeed positive definite, we note that for each $x \in \mathbb{R}^n$, $x \neq 0$ either $A_i x \neq 0$ or there exists an l such that $(I - A_i - Z_l B_i)x \neq 0$ by (13). \square

Proof of Proposition 4.2. Proof by backward induction: For $i = N$, (28) becomes

$$C_N(p) = \Lambda + \alpha \Sigma \quad (128)$$

as required. For $i > N$, we precede as in the proof of Theorem 4.1 and obtain

$$A_i(p) = \frac{(1-p)C_{i+1}(p)}{(1-p)C_{i+1}(p) + \Lambda}, \quad B_i(p) = \frac{\Lambda}{(1-p)C_{i+1}(p) + \Lambda} \quad (129)$$

and

$$C_i(p) = \frac{\alpha \Sigma \Lambda + (1-p)C_{i+1}(p)(\Lambda + \alpha \Sigma)}{\Lambda + (1-p)C_{i+1}(p)}. \quad (130)$$

Plugging the induction hypothesis into (129) and (130), we obtain (26), (27) and (28).

Finally, $B_i(p) < 1$ follows directly from Equation (129) and the fact that $C_i(p) > 0$. \square

Proof of Corollary 4.3. The proof by forward induction is a straightforward calculation. \square

Proof of Corollary 4.4. 1. If p is the real-world probability of execution in the dark pool and $q > p$, applying the strategy $((x_j(q), y_j(q)))_{j \geq i}$ yields strictly less costs than $C_i(q)$ and strictly more costs than $C_i(p)$. Thus $C_i(p) < C_i(q)$.

2. Note first that

$$\kappa'_p = \frac{p(\alpha\Sigma + \Lambda) - \alpha\Sigma}{4\Lambda\sqrt{1-p}^3 \sqrt{\left(\frac{\sqrt{1-p}}{2} \left(\frac{\alpha\Sigma}{\Lambda} + 1 + \frac{1}{1-p}\right)\right)^2 - 1}}. \quad (131)$$

Therefore, κ_p is strictly decreasing in p in $(0, \frac{\alpha\Sigma}{\alpha\Sigma+\Lambda})$ and strictly increasing in $(\frac{\alpha\Sigma}{\alpha\Sigma+\Lambda}, 1)$ and $\frac{\sinh(\kappa_p(N+1-i))}{\sinh(\kappa_p(N+1))}$ is strictly increasing in p in $(0, \frac{\alpha\Sigma}{\alpha\Sigma+\Lambda})$ and strictly decreasing in $(\frac{\alpha\Sigma}{\alpha\Sigma+\Lambda}, 1)$. Thus $X_i^{\text{ne}}(p)$ is strictly increasing for $p \in (0, \frac{\alpha\Sigma}{\alpha\Sigma+\Lambda})$ and we assume $p > \frac{\alpha\Sigma}{\alpha\Sigma+\Lambda}$ from now on. We show that $(X_i^{\text{ne}})'(p) > 0$. To this end we compute

$$\begin{aligned} \frac{(X_i^{\text{ne}})'(p)}{X_0} &= \frac{1}{(1-p)^i \sinh^2(\kappa_p(N+1))} \\ &\cdot \left((N+1-i)\sqrt{1-p}^i \kappa'_p \sinh(\kappa_p(N+1)) \cosh(\kappa_p(N+1-i)) \right. \\ &\quad - (N+1)\sqrt{1-p}^i \kappa'_p \sinh(\kappa_p(N+1-i)) \cosh(\kappa_p(N+1)) \\ &\quad \left. - \frac{-i}{2}\sqrt{1-p}^{i-2} \sinh(\kappa_p(N+1-i)) \sinh(\kappa_p(N+1)) \right) \end{aligned} \quad (132)$$

by Equation (29). As $\kappa'_p > 0$ for $p > \frac{\alpha\Sigma}{\alpha\Sigma+\Lambda}$ we have that $(X_i^{\text{ne}})'(p) > 0$ if and only if

$$\begin{aligned} &(N+1) \sinh(\kappa_p(N+1-i)) \cosh(\kappa_p(N+1)) \\ &\quad - (N+1-i) \sinh(\kappa_p(N+1)) \cosh(\kappa_p(N+1-i)) \\ &\quad < ic(p) \sinh(\kappa_p(N+1-i)) \sinh(\kappa_p(N+1)), \end{aligned} \quad (133)$$

where

$$c(p) := \frac{1}{2(1-p)\kappa'_p} = \frac{\sqrt{((1-p)\alpha\Sigma + (2-p)\Lambda)^2 - 4\Lambda^2(1-p)}}{p(\alpha\Sigma + \Lambda) - \alpha\Sigma}. \quad (134)$$

Note that $c(p) \geq 1$ for $p \in (\frac{\alpha\Sigma}{\alpha\Sigma+\Lambda}, 1)$ and thus it is sufficient to show

$$\beta \sinh(\alpha x) \cosh(\beta x) - \alpha \sinh(\beta x) \cosh(\alpha x) < (\beta - \alpha) \sinh(\alpha x) \sinh(\beta x) \quad (135)$$

for $0 < \alpha < \beta$, $x > 0$. Inequality (135) is equivalent to

$$f(x) := (\beta - \alpha) \exp(-(\alpha + \beta)x) - \beta \exp((\alpha - \beta)x) + \alpha \exp((\beta - \alpha)x) > 0. \quad (136)$$

It is easy to see that $f(0) = 0$ and $f'(x) > 0$ for $x > 0$ which finishes the proof.

3. As κ_p is strictly decreasing in p in $(0, \frac{\alpha\Sigma}{\alpha\Sigma+\Lambda})$ and strictly increasing in $(\frac{\alpha\Sigma}{\alpha\Sigma+\Lambda}, 1)$, it is clear that $\mathbb{E}_p[X_i]$ is strictly decreasing in p for $p \in (\frac{\alpha\Sigma}{\alpha\Sigma+\Lambda}, 1)$. Therefore we assume $p < \frac{\alpha\Sigma}{\alpha\Sigma+\Lambda}$ from now on and show that $\mathbb{E}_p[X_i]' < 0$. To this end we compute

$$\begin{aligned} \frac{\mathbb{E}_p[X_i]'}{X_0} &= \frac{1}{\sinh^2(\kappa_p(N+1))} \\ &\cdot \left((N+1-i)\sqrt{1-p}^i \kappa'_p \sinh(\kappa_p(N+1)) \cosh(\kappa_p(N+1-i)) \right. \\ &\quad - (N+1)\sqrt{1-p}^i \kappa'_p \sinh(\kappa_p(N+1-i)) \cosh(\kappa_p(N+1)) \\ &\quad \left. + \frac{-i}{2}\sqrt{1-p}^{i-2} \sinh(\kappa_p(N+1-i)) \sinh(\kappa_p(N+1)) \right) \end{aligned} \quad (137)$$

by Equation (33). As $\kappa'_p < 0$ for $p < \frac{\alpha\Sigma}{\alpha\Sigma+\Lambda}$ we have that $\mathbb{E}_p[X_i]' < 0$ if and only if

$$\begin{aligned} &(N+1) \sinh(\kappa_p(N+1-i)) \cosh(\kappa_p(N+1)) \\ &\quad - (N+1-i) \sinh(\kappa_p(N+1)) \cosh(\kappa_p(N+1-i)) \\ &\quad < -ic(p) \sinh(\kappa_p(N+1-i)) \sinh(\kappa_p(N+1)), \end{aligned} \quad (138)$$

for $c(p)$ as in Equation (134). Note that $-c(p) \geq 1$ for $p \in (0, \frac{\alpha\Sigma}{\alpha\Sigma+\Lambda})$ and thus the claim follows as before from Inequality (135). \square

Proof of Proposition 4.5. The calculation of C is straightforward. For $\alpha = 0$, we have

$$\kappa_p = \log\left(\frac{1}{\sqrt{1-p}}\right) \quad (139)$$

and thus $C = 0$. Furthermore, for fixed p , κ_p is strictly increasing in alpha and thus $C > 0$ for risk averse traders.

For the expected time of liquidation we compute

$$\mathbb{E}_p[\tilde{T}] = \lim_{N \rightarrow \infty} \left(p \sum_{i=1}^{N-1} i(1-p)^{i-1} + N(1-p)^{N-1} \right) \quad (140)$$

$$= p \sum_{i=1}^{\infty} i(1-p)^{i-1} = \frac{1-p}{p}. \quad (141)$$

\square

For the proof of Theorem 5.1 we need the following lemma.

Lemma A.1. $A_{i,j,1}, \dots, C_{i,j,3}$ and $\bar{X}_{i,j}$ are given recursively as follows: For $i = 0, \dots, N$

$$C_{i,0,1} = C_i(0), \quad C_{i,0,2} = C_{i,0,3} = 0, \quad (142)$$

$$A_{i,0,1} = A_i(0), \quad A_{i,0,2} = 0, \quad (143)$$

$$B_{i,0,1} = B_{i,0,2} = 0 \quad (144)$$

and for $i = 0, \dots, N-1$

$$\bar{X}_{i,1} = \frac{A(C_{i+1}(0) + \Lambda)}{2C_{i+1}(0)\Lambda}, \quad (145)$$

Furthermore, for $i = 0, \dots, N-1$, $j = 1, \dots, N-i$

$$C_{i,j,1} = \frac{\alpha\Sigma\Lambda + C_{i+1,j-1,1}(1-p)(\Lambda + \alpha\Sigma)}{C_{i+1,j-1,1}(1-p) + \Lambda}, \quad (146)$$

$$C_{i,j,2} = \frac{(C_{i+1,j-1,2}(1-p) + Ap)\Lambda}{C_{i+1,j-1,1}(1-p) + \Lambda}, \quad (147)$$

$$C_{i,j,3} = \frac{-1}{4C_{i+1}(0)(C_{i+1,j-1,1}(1-p) + \Lambda)} \cdot \left\{ A^2 p (C_{i+1,j-1,1}(1-p) + \Lambda) + C_{i+1}(0) \left((C_{i+1,j-1,2}^2 - 4C_{i+1,j-1,1}C_{i+1,j-1,3})(1-p)^2 + 2AC_{i+1,j-1,2}(1-p)p + A^2p^2 - 4C_{i+1,j-1,3}\Lambda(1-p) \right) \right\}, \quad (148)$$

$$A_{i,j,1} = \frac{C_{i+1,j-1,1}(1-p)}{C_{i+1,j-1,1}(1-p) + \Lambda}, \quad (149)$$

$$A_{i,j,2} = \frac{C_{i+1,j-1,2}(1-p) + Ap}{2(C_{i+1,j-1,1}(1-p) + \Lambda)}, \quad (150)$$

$$B_{i,j,1} = \frac{\Lambda}{C_{i+1,j-1,1}(1-p) + \Lambda}, \quad (151)$$

$$B_{i,j,2} = -\frac{C_{i+1}(0)C_{i+1,j-1,2}(1-p) + A(C_{i+1,j-1,1}(1-p) + C_{i+1}(0)p + \Lambda)}{2C_{i+1}(0)(C_{i+1,j-1,1}(1-p) + \Lambda)} \quad (152)$$

and for $i = \dots, N-2$ and $j = 2, \dots, N-i$

$$\bar{X}_{i,j} = \min\{X > \bar{X}_{i,j-1} \mid X - x_i(X) = \bar{X}_{i+1,j-1}\} \quad (153)$$

$$= \frac{C_{i+1,j-2,2}(1-p) + Ap + 2(C_{i+1,j-2,1}(1-p) + \Lambda)\bar{X}_{i+1,j-1}}{2\Lambda}. \quad (154)$$

Proof of Lemma A.1. Observing (52), a straightforward backward induction shows that J_i is monotonic in the following sense:

$$|X_i| < |Y_i| \Rightarrow J_i(X_i) < J_i(Y_i) \quad \text{and} \quad J_i(-X_i) = J_i(X_i). \quad (155)$$

Using (155), it is easy to see that for $X_i \geq 0$ we have

$$x_i(X_i), y_i(X_i) \geq 0 \quad \text{and} \quad x_i(X_i) + y_i(X_i) \leq X_i \quad (156)$$

(and therefore we can assume without loss of generality that $X_i \geq 0 \forall i$).

Furthermore, J_i is strictly convex by Proposition 3.3 and therefore continuous. Using (156) again, we can easily deduce that also $x_i(X_i)$ and $y_i(X_i)$ (as functions of X_i) are continuous.

We now prove the lemma by backward induction. For $i = N$, there is only one admissible strategy and all the properties are trivially satisfied.

For the induction step, we assume that all properties are valid for time t_{i+1} ($i < N$). By Equation (50) and Inequalities (156) we obtain:

$$J_i(X_i) = \min_{x_i, y_i \geq 0} \{ \Lambda x_i^2 + \alpha \Sigma X_i^2 + p y_i A + p J_{i+1}(X_i - x_i - y_i) + (1-p) J_{i+1}(X_i - x_i) \}. \quad (157)$$

For finishing the proof, we will show the following two statements:

- i) For $j \in \{1, \dots, N-i\}$ and $\bar{X}_{i,j} < X_i < \bar{X}_{i,j+1}$ (for $\bar{X}_{i,j}, \bar{X}_{i,j+1}$ as in Equation (154)), $J_i(X_i), x_i(X_i), y_i(X_i)$ are given by Equations (54)- (56).
- ii) For $X_i \leq \bar{X}_{i,1}$, $y_i(X_i) = 0$, $x_i(X_i) = A_i(0)X_i$, $J_i(X_i) = C_i(0)X_i^2$.

We will show i) by induction on j .

By (156) and (backward) induction hypothesis we know that

$$J_i(X_i) = \min_{x_i, y_i \geq 0} \underbrace{\{ \Lambda x_i^2 + \alpha \Sigma X_i^2 + p y_i A + p C_{i+1}(0)(X_i - x_i - y_i)^2 + (1-p) C_{i+1}(0)(X_i - x_i)^2 \}}_{=: \tilde{J}_i(x_i, y_i)}. \quad (158)$$

for small X_i (at least for $X_i < \bar{X}_{i+1,1}$ as this implies $0 \leq X_i - x_i(X_i) - y_i(X_i) \leq X_i - x_i(X_i) \leq \bar{X}_{i+1,1}$).

The system of linear equations $\frac{\partial \tilde{J}_i}{\partial x_i}(x_i, y_i) = 0$, $\frac{\partial \tilde{J}_i}{\partial y_i}(x_i, y_i) = 0$ has the unique solution $(\tilde{x}_i, \tilde{y}_i)$ given by

$$\tilde{x}_i = \frac{C_{i+1}(0)(1-p)}{C_{i+1}(0)(1-p) + \Lambda} X_i + \frac{Ap}{2(C_{i+1}(0)(1-p) + \Lambda)} = A_{i,1,1}X_i + A_{i,1,2}, \quad (159)$$

$$\tilde{y}_i = \frac{\Lambda}{C_{i+1}(0)(1-p) + \Lambda} X_i - \frac{A(C_{i+1}(0) + \Lambda)}{2C_{i+1}(0)(C_{i+1}(0)(1-p) + \Lambda)} = B_{i,1,1}X_i + B_{i,1,2}. \quad (160)$$

By existence and uniqueness of the optimal strategy (Theorem 3.1), the positive definiteness of the Hessian of \tilde{J} and the (backward) induction hypothesis $(\tilde{x}_i, \tilde{y}_i)$ is the optimal strategy as long as

$$\tilde{y}_i > 0 \quad \text{and} \quad X_i - \tilde{x}_i < \bar{X}_{i+1,1}. \quad (161)$$

An easy calculation shows that (161) is equivalent to

$$X_i > \frac{A(C_{i+1}(0) + \Lambda)}{2C_{i+1}(0)\Lambda} = \bar{X}_{i,1} \quad \text{and} \quad X_i < \bar{X}_{i,2}. \quad (162)$$

Consequently $(x_i(X_i), y_i(X_i))$ is the optimal strategy for $X_i \in (\bar{X}_{i,1}, \bar{X}_{i,2})$. Plugging this into \tilde{J} we obtain

$$J_i(X_i) = C_{i,1,1}X_i^2 + C_{i,1,2}X_i + C_{i,1,3}, \quad (163)$$

which finishes the proof of the induction basis for i).

For the induction step, we assume that the claims are true for some $j \in \{1, \dots, N-i-1\}$. Therefore, $\bar{X}_{i,j+1} - x_i(\bar{X}_{i,j+1}) = \bar{X}_{i+1,j}$ by continuity of $x_i(X_i)$. Furthermore, by continuity of $x_i(X_i)$ and $y_i(X_i)$ there exists an $\epsilon > 0$ such that

$$y_i(X_i) > 0, \quad (164)$$

$$X_i - x_i(X_i) < \bar{X}_{i+1,j+1}, \quad (165)$$

$$X_i - x_i(X_i) - y_i(X_i) < \bar{X}_{i+1,1} \quad (166)$$

for $X_i \in (\bar{X}_{i,j+1}, \bar{X}_{i,j+1} + \epsilon)$.

Assume that $X_i - x_i(X_i) < \bar{X}_{i+1,j}$. The backward induction hypothesis implies

$$J_i(X_i) = \min_{x_i, y_i > 0} \left\{ \underbrace{\Lambda x_i^2 + \alpha \Sigma X_i^2 + p y_i A + p C_{i+1}(0)(X_i - x_i - y_i)^2}_{=: \hat{J}_i(x_i, y_i)} + (1-p) (C_{i+1,j-1,1}(X_i - x_i)^2 + C_{i+1,j-1,2}(X_i - x_i) + C_{i+1,j-1,3}) \right\}. \quad (167)$$

A simple calculation shows that this minimization problem has the unique solution

$$\bar{x}_i = A_{i,j,1} X_i + A_{i,j,2}, \quad (168)$$

$$\bar{y}_i = B_{i,j,1} X_i + B_{i,j,2}. \quad (169)$$

This implies $X_i - \bar{x}_i > \bar{X}_{i+1,j}$, a contradiction.

Therefore $X_i - x_i(X_i) \geq \bar{X}_{i+1,j}$ for $X_i \in (\bar{X}_{i,j+1}, \bar{X}_{i,j+1} + \epsilon)$. Thus, by (backward) induction hypothesis

$$J_i(X_i) = \min_{x_i, y_i > 0} \left\{ \underbrace{\Lambda x_i^2 + \alpha \Sigma X_i^2 + p y_i A + p C_{i+1}(0)(X_i - x_i - y_i)^2}_{=: \hat{J}_i(x_i, y_i)} + (1-p) (C_{i+1,j,1}(X_i - x_i)^2 + C_{i+1,j,2}(X_i - x_i) + C_{i+1,j,3}) \right\}. \quad (170)$$

Again, the Hessian of \hat{J} is positive definite and the gradient equals zero for (\hat{x}, \hat{y}) such that

$$\hat{x}_i = \frac{C_{i+1,j,1}(1-p)}{C_{i+1,j,1}(1-p) + \Lambda} X_i + \frac{C_{i+1,j,2}(1-p) + Ap}{2(C_{i+1,j,1}(1-p) + \Lambda)} \quad (171)$$

$$= A_{i,j+1,1} X_i + A_{i,j+1,2}, \quad (172)$$

$$\hat{y}_i = \frac{\Lambda}{C_{i+1,j,1}(1-p) + \Lambda} X_i - \frac{C_{i+1}(0)C_{i+1,j,2}(1-p) + A(C_{i+1,j,1}(1-p) + C_{i+1}(0)p + \Lambda)}{2C_{i+1}(0)(C_{i+1,j,1}(1-p) + \Lambda)} \quad (173)$$

$$= B_{i,j+1,1} X_i + B_{i,j+1,2}. \quad (174)$$

Note that $X_i - \hat{x}_i - \hat{y}_i = \frac{A}{2C_{i+1}(0)}$ and $C_{i+1,j,1} > 0$ by strict convexity of J_{i+1} and (backward) induction hypothesis. Therefore \hat{x}_i , \hat{y}_i and $X_i - \hat{x}_i$ are strictly increasing in X_i . (\hat{x}_i, \hat{y}_i) defines the optimal strategy as long as $X_i - \hat{x}_i < \bar{X}_{i+1,j+1}$ and $\hat{y}_i > 0$, i.e., as long as $X_i < \bar{X}_{i,j+2}$. Plugging (\hat{x}_i, \hat{y}_i) into \hat{J}_i we obtain

$$J_i(X_i) = C_{i,j+1,1} X_i^2 + C_{i,j+1,2} X_i + C_{i,j+1,3} \quad (175)$$

which finishes the proof of i).

For ii) let $X_i < \bar{X}_{i,1}$. Note that $\bar{X}_{i,1} - x_i(\bar{X}_{i,1}) < \bar{X}_{i+1,1}$ by i) and therefore $X_i - x_i(X_i) < \bar{X}_{i+1,1}$ for some ϵ and $X_i \in (\bar{X}_{i,1}, \bar{X}_{i,1} - \epsilon)$ by continuity of $x_i(X_i)$. Similarly as before, $y_i(X_i) > 0$ yields a contradiction (as long as $X_i - x_i(X_i) < \bar{X}_{i+1,1}$) as it implies $y_i(X_i) = \tilde{y}_i < 0$. Therefore $y_i(X_i) = 0$ and $x_i(X_i)$ can be obtained by minimizing

$$\Lambda x_i^2 + \alpha \Sigma X_i^2 + C_{i+1}(0)(X_i - x_i)^2, \quad (176)$$

i.e.,

$$x_i(X_i) = \frac{C_{i+1}(0)}{C_{i+1}(0) + \Lambda} X_i = A_i(0) X_i. \quad (177)$$

Plugging this into (176), we obtain

$$J_i(X) = \frac{\alpha \Sigma \Lambda + C_{i+1}(0)(\Lambda + \alpha \Sigma)}{C_{i+1}(0) + \Lambda} X_i^2 = C_i(0) X_i^2. \quad (178)$$

□

Proof of Theorem 5.1. Let $h \in \{1, \dots, N\}$ and $X_0 = \bar{X}_{0,h}$. Furthermore, let X_i^{ne} be the asset position at time t_i provided no order has been executed before, i.e., X_i^{ne} is defined recursively by $X_0^{\text{ne}} = X_0$, $X_i^{\text{ne}} = X_{i-1}^{\text{ne}} - x_i(X_{i-1}^{\text{ne}})$ and for $i > 0$. By Equation (153) we have

$$X_i^{\text{ne}} = \bar{X}_{i,h-i} \quad (179)$$

for $i = 0, \dots, h-1$, in particular

$$X_{h-1}^{\text{ne}} = \frac{A(C_h(0) + \Lambda)}{2C_h(0)\Lambda}. \quad (180)$$

Moreover, we obtain

$$X_h^{\text{ne}} = X_{h-1}^{\text{ne}} - x_{h-1}(X_{h-1}^{\text{ne}}) = (1 - A_{h-1}(0))X_{h-1}^{\text{ne}} = \frac{A}{2C_h(0)} \quad (181)$$

by Equations (143) and (129) and for $i = 0, \dots, h-2$.

$$X_i^{\text{ne}} - x_i(X_i^{\text{ne}}) - y_i(X_i^{\text{ne}}) = \frac{A}{2C_{i+1}(0)} \quad (182)$$

by Equation (80).

Let now (x'_i, y'_i) be any trading strategy such that for $i = 0, \dots, h-2$

$$X_0 - \sum_{i=0}^{h-2} x'_i = \bar{X}_{h-1,1}, \quad (183)$$

$$X'_i - x'_i - y'_i = \frac{A}{2C_{i+1}(0)}, \quad (184)$$

where $X'_0 = X_0$ and $X'_i = X'_{i-1} - x'_{i-1}$ for $i > 0$ (in particular $X'_{h-1} = \bar{X}_{h-1,1}$) and (x'_i, y'_i) is the optimal trading strategy for $i \geq h-1$ and after execution in the dark pool, respectively (in particular $X'_h = \frac{A}{2C_h(0)}$ and the optimal strategy satisfies these criteria).

This strategy yields the costs

$$\begin{aligned} U(X'_0, \dots, X'_N) &= \Lambda \sum_{i=0}^{h-1} (1-p)^i (X'_i - X'_{i+1})^2 + \alpha \Sigma \sum_{i=0}^{h-1} (1-p)^i (X'_i)^2 + A \sum_{i=0}^{h-2} p(1-p)^i \left(X'_{i+1} - \frac{A}{2C_{i+1}(0)}\right) \\ &\quad + \sum_{i=0}^{h-2} (1-p)^i p C_{i+1}(0) \left(\frac{A}{2C_{i+1}(0)}\right)^2 + (1-p)^h C_h(0) \left(\frac{A}{2C_h(0)}\right)^2. \end{aligned} \quad (185)$$

By Lemma A.1 $(X_0^{\text{ne}}, \dots, X_N^{\text{ne}})$ minimizes U uniquely. Therefore we know that $(X_0^{\text{ne}}, \dots, X_N^{\text{ne}})$ solves the system of linear equations

$$\frac{\partial U}{\partial X_i}(X'_0, \dots, X'_N) = 0 \quad (i = 0, \dots, h-1). \quad (186)$$

We obtain the inhomogeneous linear difference equation

$$X'_i \left(1 + \frac{1}{1-p} + \frac{\alpha \Sigma}{\Lambda}\right) + \frac{Ap}{2(1-p)\Lambda} = X'_{i-1} + \frac{1}{1-p} X'_{i-1}. \quad (187)$$

By standard methods we compute a solution of Equation (187)

$$X'_i = \frac{-Ap}{2(1-p)\alpha \Sigma} \quad (188)$$

and two linearly independent solutions of the corresponding homogeneous linear difference equation

$$X'_i = \frac{\exp(\kappa_p(h-1-i))}{\sqrt{1-p}^i}, \quad X'_i = \frac{\exp(-\kappa_p(h-1-i))}{\sqrt{1-p}^i}. \quad (189)$$

Consequently, the solutions of Equation (187) are given by

$$\frac{-Ap}{2(1-p)\alpha \Sigma} + a \cdot \frac{\exp(\kappa_p(h-1-i))}{\sqrt{1-p}^i} + b \cdot \frac{\exp(-\kappa_p(h-1-i))}{\sqrt{1-p}^i} \quad (190)$$

for $a, b \in \mathbb{R}$. $\bar{X}_{i,h-i}$ ($i = 0, \dots, h-1$) is the unique solution of Equation (187) satisfying the boundary conditions from Equations (180) and (181). Some tedious algebraic manipulation (and substituting h by $j+i$) confirms Equation.

Moreover, by Lemma A.1 we have

$$\bar{X}_{i,j} - A_{i,j,1}\bar{X}_{i,j} - A_{i,j,2} = \bar{X}_{i+1,j-1} \quad (191)$$

$$\bar{X}_{i,j+1} - A_{i,j,1}\bar{X}_{i,j+1} - A_{i,j,2} = \bar{X}_{i+1,j} \quad (192)$$

for $i = 0, \dots, N-1$, $j = 1, \dots, N-i-1$. Solving this system of linear equations in $A_{i,j,1}$ and $A_{i,j,2}$ yields Equations (64) and (65). Equations (66) - (69) follow from that directly with the recursions from Lemma A.1.

Finally, the formulae for $j = N-i$ can be proven by backward induction using the recursions from Lemma A.1, which completes the proof of 1. 2. and 3. follow directly. \square

Proof of Corollary 5.2. 1. and 2. are direct consequences of Theorem 5.1. 3. is a straightforward backward induction using the recursions from Lemma A.1. \square

Proof of Corollary 5.3. It is easy to see by backward induction that for $i = 0, \dots, N-1$, $j = 1, \dots, N-i$, $C_{i,j,2}$ and thus $A_{i,j,2}$ and $\bar{X}_{i,j}$ are strictly increasing in A .

Before we proceed, we introduce the following notation: For a setting with adverse selection A , we denote the optimal strategy at time t_i by adding A as a superscript: $(x_i^A(X_i), y_i^A(X_i))$. In a similar fashion we characterize optimal trajectories etc.

Let now $A > A'$, $i \in \{0, \dots, N-1\}$ and $X_i \in [\bar{X}_{i,j}^A, \bar{X}_{i,j+1}^A]$ for $j \in \{1, \dots, N-i\}$. As $\bar{X}_{i,j}$ is increasing in A , we have $X_i \in [\bar{X}_{i,h}^{A'}, \bar{X}_{i,h+1}^{A'}]$ for $h \geq j$. Therefore

$$x_i^A(X_i) = A_{i,j,1}^A X_i + A_{i,j,2}^A \quad (193)$$

$$\geq A_{i,h,1}^A X_i + A_{i,h,2}^A \quad (194)$$

$$> A_{i,h,1}^{A'} X_i + A_{i,h,2}^{A'} \quad (195)$$

$$= x_i^{A'}(X_i), \quad (196)$$

where (194) follows from Corollary 5.2 2. and (195) follows from the fact that $A_{i,j,1}$ is independent of A and $A_{i,j,2}$ is strictly increasing in A . Consequently $x_i(X_i)$ is increasing in A .

Furthermore, by Equation (80), $x_i(X_i) + y_i(X_i)$ is decreasing in A and therefore - as $x_i(X_i)$ is increasing in A - $y_i(X_i)$ is decreasing in A .

By forward induction we deduce $(X_i^{\text{ne}})^A < (X_i^{\text{ne}})^{A'}$ for $i > 0$, $X_0 > \bar{X}_{0,1}^{A'}$ (note that always $((X_0^{\text{ne}})^A = (X_i^{\text{ne}})^{A'})$ and $((X_0^{\text{ne}})^A = (X_i^{\text{ne}})^{A'})$ for all i , if $X_0 \leq \bar{X}_{0,1}^{A'}$):

$$(X_i^{\text{ne}})^A = (X_{i-1}^{\text{ne}})^A - x_i^A((X_{i-1}^{\text{ne}})^A) \quad (197)$$

$$< (X_{i-1}^{\text{ne}})^{A'} - x_i^A((X_{i-1}^{\text{ne}})^{A'}) \quad (198)$$

$$\leq (X_{i-1}^{\text{ne}})^{A'} - x_i^{A'}((X_{i-1}^{\text{ne}})^{A'}) \quad (199)$$

$$= (X_i^{\text{ne}})^{A'}, \quad (200)$$

where (198) follows by induction hypothesis and from the fact that $X_i - x_i(X_i)$ is strictly increasing in X_i and (199) follows from the fact that $x_i(X_i)$ is increasing in A . \square

Proof of Theorem 7.1. Direct consequence of the preceding example of a market manipulating strategy. \square

Proof of Proposition 7.2. The same line of argument as in the proof of Proposition 3.2 establishes the existence of optimal strategies since dark pool liquidity is limited.

Consider the optimal strategy for an initial asset position of X_0 . Assume that at any time t_i an asset position of X_i is being held and orders of x_i and y_i are optimal. By our requirements for admissible strategies, we know that $|y_i| \leq L$. We first assume that $\text{sign}(x_i) \neq \text{sign}(y_i)$. The expected cost of trading at time t_i and thereafter are

$$\Lambda x_i^2 + pJ_{i+1}(X_i - x_i) + (1-p)(J_{i+1}(X_i - x_i - y_i) + (\Lambda x_i + A \text{sign}(y_i))y_i) \quad (201)$$

A direct calculation shows that if $J_{i+1}(X_i - x_i) < J_{i+1}(X_i - x_i - y_i) + (\Lambda x_i + A \text{sign}(y_i))y_i$ then orders of $\tilde{x}_i = x_i$ and $\tilde{y}_i = 0$ result in lower costs; otherwise, $\tilde{x}_i = x_i + y_i$ and $\tilde{y}_i = 0$ result in lower costs due to Condition 97. In both cases a contradiction is established. Hence the optimal strategy satisfies $\text{sign}(x_i) = \text{sign}(y_i)$ at all times t_i .

Given that $\text{sign}(x_i) = \text{sign}(y_i)$, it is obvious that an optimal strategy cannot have $\text{sign}(x_i) = \text{sign}(y_i) \neq \text{sign}(X_i)$, i.e., cannot be market manipulating. \square

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