

# The price of interest rate variance risk and optimal investments in interest rate derivatives

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## Abstract

Recent research on unspanned stochastic variance raises the possibility that interest rate derivatives constitute an important component of optimal fixed income portfolios. In this paper, I estimate a flexible dynamic term structure model that allows for unspanned stochastic variance on an extensive data set of swaps and swaptions. I find that variance risk is predominantly unspanned by bonds, and that the price of risk on the unspanned variance factor is significantly larger in absolute value than the prices of risk on the term structure factors. Consequently, Sharpe ratios on variance sensitive derivatives are about three times larger than Sharpe ratios on bonds or short-term bond futures. These findings are corroborated by an analysis of the Treasury futures market, where the variance risk premium is estimated with a model independent approach. I then solve the dynamic portfolio choice problem for a long-term fixed income investor with and without access to interest rate derivatives and find substantial utility gains from participating in the derivatives market.

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# 1 Introduction

The market for interest rate derivatives has grown rapidly over the last decade. For instance, the notional amount of outstanding over-the-counter interest rate options has increased 691 percent from USD 7.6 trillion in June 1998 to USD 62.1 trillion in June 2008, see BIS (2008). In addition, many standard fixed income securities such as mortgage-backed securities and agency securities imbed interest rate options. While there is an enormous amount of literature on the pricing, hedging, and risk management of interest rate derivatives, few papers view interest rate derivatives from a portfolio perspective, despite the fact that this issue is obviously important for many fixed income investors. This paper attempts to fill this gap in the literature.

One reason that the portfolio choice literature has ignored interest rate derivatives is that standard term structure models assume that the fixed income market is complete in the sense that all risk, including variance risk, is completely spanned by bonds. In these models, interest rate derivatives are redundant securities that can be perfectly replicated by trading in the underlying bonds. However, this assumption about market completeness has been challenged in recent years with a number of papers showing that a component of variance risk is not spanned by bonds and, therefore, that interest rate derivatives are not redundant securities. Unspanned stochastic variance was first discussed by Collin-Dufresne and Goldstein (2002) and further evidence in support for it has been provided by Heidari and Wu (2003), Andersen and Benzoni (2005), Li and Zhao (2006, 2008), Trolle and Schwartz (2008a), and Collin-Dufresne, Goldstein, and Jones (2008), among others.

Unspanned stochastic variance raises the possibility that interest rate derivatives constitute an important component of optimal fixed income portfolios. First, to the extent that the unspanned component of variance risk is priced, derivatives improve investment opportunities and second, to the extent that that investment opportunities depend on variance, derivatives improve the ability to hedge against adverse changes in the investment opportunity set.

The first goal of the paper is to analyze the extent to which interest rate variance risk – in particular, the unspanned component of variance risk – is priced. I use two approaches; first, I estimate the variance risk premium in the Treasury futures market without using a particular pricing model, and, second, I estimate the variance risk premium in the interest rate swap market using a dynamic term structure model. The reason for using different data in the two approaches is the following: the model independent analysis is predicated upon the existence of a liquid market for options across a wide range of strikes. While such a market has long

existed in the case of options on Treasury futures, it has only recently emerged in the case of options on interest rate swaps, i.e. swaptions, where only ATM options were actively quoted until four or five years ago. On the other hand, estimating a dynamic term structure model simultaneously on Treasury futures and their associated options is rather involved, whereas it is relatively straightforward to estimate such a model simultaneously on swaps and their associated swaptions.

The two approaches complement each other. The model independent analysis provides robust estimates of the variance risk premium with which the model dependent estimates may be compared, and the model independent analysis also gives insights into the dynamics of the variance risk premium which forms the basis for parameterizing the market price of variance risk in the dynamic term structure model.

To estimate the interest rate variance risk premium without a particular pricing model, I rely on synthetic variance swaps which pay the difference between the realized variance of a Treasury futures contract over the life of the swap and a fixed variance swap rate, which can be inferred from a cross-section of options on the Treasury futures contract. The average payoff or return on a sequence of such variance swaps provides a model independent estimate of the Treasury futures variance risk premium. I use daily data on 5, 10, and 30 year Treasury futures and their associated options from January 3, 1995 until March 5, 2008.

I find that 1) the Treasury futures variance risk premium is significantly negative, 2) shorting variance swaps generate Sharpe ratios that are about two to three times larger than the Sharpe ratios of the underlying Treasury futures, 3) the variance risk premium is not a compensation for exposure to bond or equity market risks, suggesting that there is an unspanned variance factor with a significant risk premium, and 4) the variance risk premium varies over time and becomes more negative when variance increases, particularly when the premium is measured in dollar terms.

To estimate the interest rate variance risk premium within a dynamic term structure model, I develop a model that shares many features with that in Trolle and Schwartz (2008a) but has a more parsimonious structure. It has  $N$  term structure factors and one additional unspanned variance factor. Innovations to variance and the term structure may be correlated so that variance can contain both a spanned and an unspanned component. Inspired by the model independent analysis, I parameterize the variance risk premium such that it is linear in variance itself. I estimate the model using daily data on LIBOR and swap rates and short-term ATM swaptions on 2, 5, 10, and 30 year swaps from January 23, 1997 to April 30, 2008. For the last

four years of the sample period, I also use unique data on swaption “smiles”, i.e. swaptions with a wide range of strikes, obtained from the largest broker in the interest rate derivatives market. This “smile” information is important for estimating the extent to which variance is unspanned, and, to my knowledge, it is the first time that this data has been used in the empirical term structure literature. I consider model specifications with one, two, and three term structure factors and estimate with quasi-maximum likelihood in conjunction with the extended Kalman filter. In a methodological contribution, I derive a fast and accurate Fourier-based swaption pricing formula that enables me to estimate the model on a large set of swaptions.

I find that 1) innovations to variance are only weakly related to innovations to the term structure, i.e. variance risk is predominantly unspanned, 2) the estimated market price of risk on the unspanned variance factor is strongly negative – much more negative than the prices of risk on the term structure factors – and statistically significant, and 3) the model-implied Sharpe ratio on a derivative exposed solely to variance is about three times larger in absolute value than the model-implied Sharpe ratios on bonds or short-term bond futures, consistent with the findings in the model-independent analysis. These findings hold true regardless of the number of term structure factors.

The second goal of the paper is to analyze the benefits of including interest rate derivatives in fixed income portfolios. I assume that investment opportunities evolve according to the term structure model estimated in this paper. I then derive the optimal portfolio strategy for a long-term fixed income investor with CRRA utility over terminal wealth, who either does or does not participate in the interest rate derivatives market, and I compute the utility gains from optimally adding interest rate derivatives to fixed income portfolios.<sup>1</sup>

Interest rate derivatives are attractive for two reasons. First, derivatives provide the investor with exposure to the unspanned variance factor which, because it carries a market price of risk that is significantly larger (in absolute value) than the term structure factors, substantially increases the Sharpe ratio of the mean-variance tangency portfolio. Second, because the Sharpe ratio of the tangency portfolio depends on variance, derivatives improve the ability of the investor to hedge adverse changes in investment opportunities, which is a concern for long-term investors that are more risk averse than log-utility investors.

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<sup>1</sup>To focus the discussion, I only consider pure fixed income portfolios. It would be straightforward to extend the analysis to allow for investments in an equity index. Moreover, in line with much of the literature I also abstract from portfolio constraints and transaction costs.

I find substantial utility gains from participating in the interest rate derivatives market. For instance, an investor with an investment horizon of five years and a relative risk aversion of three, would be willing to give up between 15 and 20 percent of his wealth, depending on the specification of the term structure model, to be able to optimally invest in the interest rate derivatives market.

The paper is related to Duarte, Longstaff, and Yu (2007) who analyze risk and return in fixed income arbitrage strategies, among these a strategy of selling interest rate volatility through delta-hedged caps, which they find generate annualized Sharpe ratios between -0.08 and 0.82 depending on the cap maturity. However, a crucial difference between their study and mine is that their results are model-dependent (caps are delta-hedged using a particular model to compute hedge ratios), whereas I provide both model independent and model dependent results.

A number of papers have estimated dynamic term structure models simultaneously on interest rates and interest rate derivatives, see Bikbov and Chernov (2004), Almeida, Grave-line, and Joslin (2006), Joslin (2007), and Trolle and Schwartz (2008a). However, except for Joslin (2007), none of these papers focus directly on the variance risk premium. He studies a model that is very restrictive in its ability to generate unspanned stochastic variance and, consequently, he formally rejects the parameter restrictions necessary to generate this feature. Nevertheless, he finds that a component of variance is only very weakly related to the term structure and that this approximately unspanned component carries a sizable risk premium. In contrast to his paper, the model framework used here is much more flexible in terms of its ability to generate unspanned stochastic variance.

Many papers have studied the price of variance risk in equity indices. For instance, Carr and Wu (2008), Driessen, Maenhout, and Vilkov (2008), and Bondarenko (2007) use a model independent approach similar to the one applied in this paper, and find a large negative variance risk premium, larger than the interest rate variance risk premium estimated here.<sup>2</sup>

The paper is also related to a literature that analyzes optimal positioning in derivatives – primarily equity derivatives. Closest to my paper are Liu and Pan (2003) and Egloff, Leippold,

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<sup>2</sup>See also Chernov and Ghysels (2000), Coval and Shumway (2001), Pan (2002), Bakshi and Kapadia (2003), and Jones (2003, 2006), among others. Trolle and Schwartz (2008c) use the model independent approach to study variance risk premia in energy markets. The magnitude of these premia are comparable to the interest rate variance risk premium. Hence negative variance risk premia seem to be a characteristic feature across different markets.

and Wu (2007) who analyze dynamic equity derivative strategies in Heston (1993) type models and report large gains from participating in the equity derivatives market.<sup>3,4</sup>

The paper is structured as follows. Section 2 provides model-free estimates of the variance risk premium. Section 3 sets up a dynamic term structure model featuring unspanned stochastic variance and estimates the variance risk premium within this model. Section 4 derives optimal dynamic portfolio strategies for a long-term investor, with and without access to interest rate derivatives, and estimates the utility gains from participating in the interest rate derivatives market. Section 5 concludes. Four appendices contain technical details.

## 2 Estimating the variance risk premium without a model

I first estimate the interest rate variance risk premium without using a particular pricing model. For this purpose, I use synthetic variance swap contracts, which allow investors to trade future realized variance of a given asset against current implied variance. At maturity, a variance swap pays off the difference between the realized variance of the reference asset over the life of the swap and the fixed variance swap rate. Since a variance swap has zero net market value at initiation, absence of arbitrage implies that the fixed variance swap rate equals the conditional risk-neutral expectation of the realized variance over the life of the swap. Therefore, the time-series average of the payoff or excess return on a variance swap is a measure of the variance risk premium on the reference asset. A similar methodology is used by Carr and Wu (2008), Driessen, Maenhout, and Vilkov (2008), and Bondarenko (2007) to study variance risk premia in equities and by Trolle and Schwartz (2008c) to study variance risk premia in energy markets.

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<sup>3</sup>Other papers on optimal positioning in derivatives include classic papers such as Brennan and Solanki (1981) and Leland (1980) on the use of portfolio insurance, Carr and Madan (2001) and Buraschi and Jiltsov (2007) on the effect of differences in beliefs, Franke, Stapleton, and Subrahmanyam (1998) on the impact of non-hedgeable background risk, and Back (1993), Biais and Hillion (1994), Brennan and Cao (1996), Easley, O'Hara, and Srinivas (1998), and John, Koticha, Narayanan, and Subramanyam (2003) on the effect of asymmetric information.

<sup>4</sup>More broadly the paper is related to a growing literature, starting with Brennan, Schwartz, and Lagnado (1997), which analyzes dynamic portfolio strategies for long-term investors when investment opportunities are stochastic. In the fixed income space, the effects of stochastic interest rates are by now fairly well understood, see e.g. Sørensen (1999), Brennan and Xia (2000), Campbell and Viceira (2001), Munk and Sørensen (2004) and Sangvinatsos and Wachter (2005). For that reason my focus is exclusively on the effects of unspanned stochastic variance.

Variance swaps are rarely traded in fixed income markets, in contrast to equity markets where they are very popular. However, it is possible to construct synthetic variance swap contracts from a cross-section of options on a given reference asset.

I consider the 5 year and 10 year Treasury note futures and the 30 year Treasury bond future as reference assets. Since there is a very liquid market for options on these futures, I can compute synthetic variance swap rates at three points on the yield curve. I also report results for the S&P 500 index to facilitate comparison with the equity market.<sup>5</sup>

## 2.1 Methodology

The payoff at time  $T$  of a variance swap for the period  $t$  to  $T$  is given by

$$(V(t, T) - K(t, T))L, \quad (1)$$

where  $V(t, T)$  denotes the realized annualized return variance between time  $t$  and  $T$ ,  $K(t, T)$  denotes the fixed variance swap rate, determined at time  $t$ , and  $L$  denotes the notional of the swap. At initiation, the variance swap has zero net market value. Assuming that short-term interest rates are uncorrelated with realized variance,<sup>6</sup> absence of arbitrage implies that the fixed variance swap rate is given by

$$K(t, T) = E_t^Q[V(t, T)]. \quad (2)$$

That is, the fixed variance swap rate equals the conditional risk-neutral expectation of the realized variance over the life of the swap.

Let  $F(t, T_1)$  denote the time- $t$  price of a Treasury futures contract expiring at time  $T_1$  and suppose that  $V(t, T)$  is given by the realized annualized continuously sampled futures return variance (i.e. the realized quadratic variation) over the period  $[t, T]$ ,  $T \leq T_1$ . Then, following Carr and Madan (1998), Demeterfi, Derman, Kamal, and Zou (1999), Britten-Jones and Neuberger (2000), Jiang and Tian (2005), Carr and Wu (2008), and others, one can show that under very general circumstances,  $K(t, T)$  may be inferred from a continuum of European out-of-the-money (OTM) options. In particular

$$K(t, T) = \frac{2}{P(t, T)(T - t)} \left( \int_0^{F(t, T_1)} \frac{\mathcal{P}(t, T, T_1, X)}{X^2} dX + \int_{F(t, T_1)}^{\infty} \frac{\mathcal{C}(t, T, T_1, X)}{X^2} dX \right), \quad (3)$$

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<sup>5</sup>There is also a 2 year Treasury note future. However, its associated options have, until recently, been very illiquid.

<sup>6</sup>Several papers argue that interest rate variance is largely unspanned by the yield curve, see the discussion and results in Section 3.

where  $P(t, T)$  is the time- $t$  price of a zero-coupon bond maturing at time  $T$ , and  $\mathcal{P}(t, T, T_1, X)$  and  $\mathcal{C}(t, T, T_1, X)$  denote the time- $t$  price of a European put and call option, respectively, expiring at time  $T$  with strike  $X$  on a futures contract expiring at time  $T_1$ . This relation is model-free in the sense that no assumptions are made about the price process of the reference asset. In particular, the price process may contain jumps.<sup>7</sup>

In actual variance swap contracts,  $V(t, T)$  is the realized annualized discretely sampled return variance. Typically, the asset price is sampled each business day at the official close or settlement, and the mean of daily asset returns is assumed to be zero. For a variance swap with  $N$  business days to expiry, I define a set of dates  $t = t_0 < t_1 < \dots < t_N = T$  with  $\Delta t = t_i - t_{i-1} = 1/252$ .  $V(t, T)$  is then computed as

$$V(t, T) = \frac{1}{N\Delta t} \sum_{i=1}^N R(t_i)^2, \quad (4)$$

where  $R(t_i) = \log(F(t_i, T_1)/F(t_{i-1}, T_1))$ .

Now, for each business day in the sample, I compute the synthetic variance swap rate,  $K(t, T)$ , using (3) and the realized futures return variance over the life of the swap using (4). I then compute the dollar payoff of a long position in a variance swap contract with a notional amount of  $L = 100$  USD held to expiration,

$$(V(t, T) - K(t, T))100. \quad (5)$$

I also compute the log excess return given by

$$\log(V(t, T)/K(t, T)), \quad (6)$$

since  $K(t, T)$  is the forward cost of a variance swap.<sup>8</sup> The sample mean of (5) is an estimate of

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<sup>7</sup>The derivations of (3) relies on the assumption that short-term interest rates used for discounting the option payoffs are uncorrelated with Treasury futures prices. Although this assumption may seem restrictive, in fact several papers have shown that very short-term interest rates exhibit low correlation with longer-term interest rates, see e.g. Duffee (1996). To check this finding, I compute correlations between daily changes in the three month Treasury bill rate and returns on the 5 year, 10 year and 30 year Treasury futures. The correlations are -0.28, -0.23, and -0.17, respectively. Even if the correlations were not low, the bias in (3) would be small, since Treasury futures prices are much more volatile than the option discount factor, since I only use short-term options with maturities between 11 and 35 business days.

<sup>8</sup>As in Carr and Wu (2008), I report the log excess return rather than the discrete excess return,  $V(t, T)/K(t, T) - 1$ , in order to facilitate comparison with their results and because the former is closer to normally distributed.



the average variance risk premium in dollar terms, while the sample mean of (6) is an estimate of the average variance risk premium in log return terms.

## 2.2 Data and implementation details

I use an extensive data set of 5 year and 10 year Treasury note futures and 30 year Treasury bond futures and their associated options trading on the Chicago Board of Trade (CBOT) exchange. I use daily settlement prices from January 3, 1995 until March 28, 2008 – a total of 3334 business days.<sup>9,10</sup> For each maturity, CBOT lists futures contracts with expiration in the first four months in the quarterly cycle (March, June, September, and December). It also lists options for the first three consecutive contract months (two serial expirations and one quarterly expiration) plus the next four months in the quarterly cycle. Serial options exercise into the first nearby quarterly futures contract and quarterly options exercise into futures contracts for the same month.<sup>11</sup> On each business day, among the option contract months where expiration is more than 10 business days away, I select the one with the shortest time to expiration. For this options contract month, I select all OTM puts and calls that have open interest in excess of 100 contracts and have prices larger than 3/64 USD. I only use options (and, hence, variance swap contracts) with short maturities in order to minimize the overlap in variance swap returns. However, I set a lower bound on the option maturities in order to avoid market microstructure related issues. The reason for requiring option prices to exceed the given thresholds is that options are quoted with a precision of 1/64 USD.<sup>12</sup> From these options, I compute a synthetic variance swap rate (details are given in Appendix A). The maturity of these synthetic variance swaps varies between 11 and 35 business days.<sup>13</sup>

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<sup>9</sup>Actual sample sizes are shorter than 3334 business days due to missing or insufficient data.

<sup>10</sup>Settlement prices for all contracts are determined by a “Settlement Price Committee” at the end of regular trading hours and represent a very accurate measure of the true market prices at the time of close. Settlement prices are widely scrutinized by all market participants since they are used for marking to market all account balances.

<sup>11</sup>Options expire on the last Friday which precedes by at least two business days, the last business day of the month preceding the option contract month. Last trading day of the underlying futures contract is the last business day of the futures expiration month.

<sup>12</sup>For the interest rate, I use the three month Treasury bill rate.

<sup>13</sup>A synthetic 30 calendar day variance swap rate for the S&P 500 index (SPX) is easily obtained by squaring the CBOE volatility index (VIX). This is because the VIX squared approximates the conditional risk-neutral expectation of the realized 30 calendar day S&P 500 index variance. It is constructed along the lines of (3), using

## 2.3 Results

Table 1 shows summary statistics of the variance swap rates and realized variances. For the Treasury futures, the mean variance swap rate is larger than the mean realized variance, reflecting a negative variance risk premium in dollar terms on average. As expected, both the average variance and the volatility of the variance increase with the note/bond maturity. This holds true for both the variance swap rate and realized variance. Furthermore, variance swap rates and realized variances display positive skewness and excess kurtosis. The variance swap rate and realized variance for the S&P 500 index display similar characteristics.

Figure 1 displays the time-series of the variance swap rates and the payoffs on long positions in variance swaps. Clearly, variances display high volatility and increase around episodes such as the LTCM crisis, the September 11, 2001 terrorist attacks, the sharp increase in interest rates in late July, 2003, which caused massive convexity hedging of MBS portfolios, and the escalation of the credit crisis (all marked with vertical dotted lines). The correlations between the variance swap rates for the different Treasury futures are very high (between 0.90 and 0.96) but the correlations with the variance swap rate for the S&P 500 index are much lower (between 0.35 and 0.44). Similarly, the correlations between the variance swap payoffs for the different Treasury futures are high (between 0.72 and 0.93) while the correlations with the variance swap payoff for the S&P 500 index are again much lower (between 0.05 and 0.24).

Table 2 shows summary statistics of the dollar payoffs and the log excess returns on long positions in variance swaps. The  $T$ -statistics are adjusted for the autocorrelation induced by the overlap in observations. The mean payoffs and log excess returns are negative and statistically significant for all the Treasury futures as well as for the S&P 500 index. For the Treasury futures variance swaps, the distributions of payoffs exhibit fat tails and positive skewness. In contrast, the distributions of log excess returns are much closer to normal.

The table also reports the annualized Sharpe ratios (computed from standard deviations adjusted for the autocorrelation induced by the overlap in observations) of shorting variance swaps. These are 0.56, 0.56, and 0.34 for the 5 year, 10 year and 30 year Treasury futures, respectively. Although sizable, they are less than the annualized Sharpe ratio of 1.02 for the S&P 500 index. The table also reports the annualized Sharpe ratios of investing in the underlying

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OTM S&P 500 index options along with a particular discretization scheme as well as interpolation between two option maturities to obtain a constant 30 calendar day maturity (the CBOE website contains the details of the construction). Daily data on the VIX and SPX indices was downloaded from the CBOE website.

Treasury futures or S&P 500 index, which are much lower. The ratios between the annualized Sharpe ratios of shorting variance swaps and of going long the underlying futures are 3.00, 2.98, and 2.25 for the 5 year, 10 year and 30 year Treasury futures, respectively, suggesting that variance sensitive derivatives, such as variance swaps, can significantly enhance the performance of fixed income portfolios. In fact, these ratios may to some extent underestimate the true difference in Sharpe ratios, since the downward trend in interest rates over the sample period has boosted the return on Treasury futures.

The Sharpe ratios reported in Table 2 are broadly consistent with Duarte, Longstaff, and Yu (2007), who find annualized Sharpe ratios between -0.08 and 0.82 from shorting interest rate (specifically cap) volatility. However, a crucial difference between the analysis in this section and that of Duarte, Longstaff, and Yu (2007) is that their result are model-dependent (caps are delta-hedged using a particular model to compute hedge ratios), whereas my results are model independent.

In Table 3, I investigate if the variance risk premium represents a compensation for exposure to bond or equity market risks. In particular, I regress the log excess returns on a variance swaps on the the log excess return on the S&P 500 index and the log excess return on three portfolios of Treasury bonds with maturities 1–3 years, 5–7 years, and greater than 10 years.<sup>14</sup> For the Treasury futures variance swaps, the loadings on the equity market portfolio are insignificant, and although some of the loadings on the bond market portfolios are significant, the  $R^2$ s are small and the intercepts, or alphas, remain significant and close to the average excess returns reported in Table 2.<sup>15</sup> This suggests the existence of an unspanned variance factor with a significant risk premium.

Finally, as in Carr and Wu (2008) I investigate if the risk premium is related to the level of variance by running the following two regressions:

$$V(t, T) = a + bK(t, T) + \epsilon \tag{7}$$

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<sup>14</sup>The source for the returns on the Treasury bond portfolios is the Merrill Lynch U.S. Treasury bond index. This index also has returns on portfolios of Treasury bonds with maturities 3–5 years and 7–10 years. Including these portfolios in the regressions has virtually no impact except to generate a high degree of multi-collinearity. Therefore, I have reported the results without these portfolios.

<sup>15</sup>Other risk factors, such as the Fama-French size and value factors, also come out insignificant. For brevity, these results are not reported. For the S&P 500 index variance swap, the  $R^2$  is larger and the loading on the equity market factor is significant and negative (consistent with the well documented “leverage effect”), while the loading on the bond market factor is insignificant. However, the alpha remains significant and strongly negative as reported by Carr and Wu (2008), and Bondarenko (2007).

and

$$\log V(t, T) = a + b \log K(t, T) + \epsilon. \quad (8)$$

Under the null hypothesis of a constant variance risk premium in dollar terms, the slope in (7) is one. Absence of a variance risk premium in dollar terms would further imply that the intercept in (7) is zero. Similarly, under the null hypothesis of constant a variance risk premium in log return terms, the slope in (8) is one. Zero variance risk premium in log return terms would further imply that the intercept in (8) is zero.

Table 4 displays estimates of both regressions. The regressions are estimated by OLS with the  $T$ -statistics under the null hypotheses of  $a = 0$  and  $b = 1$  adjusted for the autocorrelation induced by the overlap in observations. For the Treasury futures, the slope estimates in (7) are significantly less than one and are of similar magnitude. This indicates that the variance risk premium in dollar terms becomes more negative when the variance swap rate increases. The slope estimates in (8) are closer to one, although still significantly less than one. Hence, the variance risk premium in log return terms, also becomes more negative when the variance swap rate increases, although the sensitivity is lower than for the variance risk premium in dollar terms.<sup>16</sup>

In summary, the Treasury futures variance risk premium is significantly negative, shorting variance swaps generate Sharpe ratios that are substantially higher than the Sharpe ratios of the underlying Treasury futures, the variance risk premium is not a compensation for exposure to bond or equity market risks, suggesting that there is an unspanned variance factor with a significant risk premium, and the variance risk premium becomes more negative when the variance swap rate increases, particularly when the premium is measured in dollar terms.

### **3 Estimating the variance risk premium within a dynamic term structure model**

I now estimate the variance risk premium within a flexible dynamic term structure model using a panel data set of interest rates and derivatives. The model shares many features with that in Trolle and Schwartz (2008a), but has a more parsimonious structure. It has  $N$  factors, which drive the term structure, and one additional unspanned variance factor. Innovations

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<sup>16</sup>For the S&P 500 index the variance risk premium depends significantly on the variance swap rate when measured in dollar terms but not when measured in log return terms as previous found by Carr and Wu (2008).

to the term structure and variance may be correlated so that variance may contain both a spanned and an unspanned component. Furthermore, the model accommodates a wide range of shocks to the term structure including hump-shaped shocks. I parameterize the variance risk premium such that it is proportional to variance. While this specification is convenient both for estimation and for solving the dynamic portfolio choice problem in Section 4, it is also supported by the model-free analysis in Section 2. This is important, because the estimate of the variance risk premium is clearly conditional upon its parametrization.

The model is estimated on a different data set than that used in Section 2. Rather than using Treasury futures and options, I use LIBOR and swap rates and swaptions (i.e. options on swaps), since within a dynamic term structure model it is much easier to price these instruments than Treasury futures and options.<sup>17</sup> An advantage of using a different data set is that it allows me to compare the variance risk premium in the Treasury futures market with that in the interest rate swap market, which is an over-the-counter market.

### 3.1 The dynamic term structure model

#### The risk-neutral dynamics

Let  $f(t, T)$  denote the time- $t$  instantaneous forward interest rate for risk-free borrowing and lending at time  $T$ . I model the risk-neutral dynamics of forward rates as

$$df(t, T) = \mu_f(t, T)dt + \sqrt{v(t)} \sum_{i=1}^N \sigma_{f,i}(t, T) dW_i^Q(t) \quad (9)$$

$$dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)} \left( \sum_{i=1}^N \rho_i dW_i^Q(t) + \sqrt{1 - \sum_{i=1}^N \rho_i^2} dW_{N+1}^Q(t) \right), \quad (10)$$

where  $W_i^Q(t)$ ,  $i = 1, \dots, N + 1$  denote independent standard Wiener processes under the risk-neutral measure  $Q$ . Absence of arbitrage implies that the drift term in (9) is given by

$$\mu_f(t, T) = v(t) \sum_{i=1}^N \sigma_{f,i}(t, T) \int_t^T \sigma_{f,i}(t, u) du. \quad (11)$$

Forward rates are driven by  $N$  factors, while forward rate volatilities, and hence interest rate derivatives, may be driven by an additional unspanned factor. The model in Trolle and

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<sup>17</sup>Swaptions are European-style options with constant maturities, whereas Treasury futures options are American-style options with fixed expiration dates and therefore varying maturities. Furthermore, the Treasury futures contracts themselves are fairly complex instruments, since they embed delivery options, where the short side of a futures contract can decide which (deliverable) bond to deliver and when delivery occurs.

Schwartz (2008a) allows for  $N$  unspanned variance factors but to keep the model relatively parsimonious, here I only allow for a single unspanned variance factor. Innovations to variance may be correlated with innovations to the term structure, so the extent to which variance is unspanned is an empirical question. As in Trolle and Schwartz (2008a), I use the following flexible specification

$$\sigma_{f,i}(t, T) = (\alpha_{0,i} + \alpha_{1,i}(T - t))e^{-\gamma_i(T-t)}, \quad (12)$$

which allows for a wide range of shocks to the forward rate curve. In particular it allows for hump-shaped shocks.

### An affine representation

Although the model is based on the Heath, Jarrow, and Morton (1992) framework, it may be represented as an affine model with a finite-dimensional state vector. In particular, the time- $t$  short rate,  $r(t)$ , is given by

$$r(t) = f(0, t) + \sum_{i=1}^N \mathcal{A}_{x_i} x_i(t) + \sum_{i=1}^N \sum_{j=1}^6 \mathcal{A}_{\phi_{j,i}} \phi_{j,i}(t), \quad (13)$$

where

$$\begin{aligned} \mathcal{A}_{x_i} &= \alpha_{0i} & \mathcal{A}_{\phi_{1,i}} &= \alpha_{1i} & \mathcal{A}_{\phi_{2,i}} &= \frac{\alpha_{0i}}{\gamma_i} \left( \frac{\alpha_{1i}}{\gamma_i} + \alpha_{0i} \right) \\ \mathcal{A}_{\phi_{3,i}} &= -\frac{\alpha_{0i}}{\gamma_i} \left( \frac{\alpha_{1i}}{\gamma_i} + \alpha_{0i} \right) & \mathcal{A}_{\phi_{4,i}} &= \frac{\alpha_{1i}}{\gamma_i} \left( \frac{\alpha_{1i}}{\gamma_i} + \alpha_{0i} \right) & \mathcal{A}_{\phi_{5,i}} &= -\frac{\alpha_{1i}}{\gamma_i} \left( \frac{\alpha_{1i}}{\gamma_i} + 2\alpha_{0i} \right) \\ \mathcal{A}_{\phi_{6,i}} &= -\frac{\alpha_{1i}^2}{\gamma_i}, \end{aligned}$$

and  $f(0, t)$  is the initial forward curve. The state variables evolve according to

$$dx_i(t) = -\gamma_i x_i(t) dt + \sqrt{v(t)} dW_i^Q(t) \quad (14)$$

$$d\phi_{1,i}(t) = (x_i(t) - \gamma_i \phi_{1,i}(t)) dt \quad (15)$$

$$d\phi_{2,i}(t) = (v(t) - \gamma_i \phi_{2,i}(t)) dt \quad (16)$$

$$d\phi_{3,i}(t) = (v(t) - 2\gamma_i \phi_{3,i}(t)) dt \quad (17)$$

$$d\phi_{4,i}(t) = (\phi_{2,i}(t) - \gamma_i \phi_{4,i}(t)) dt \quad (18)$$

$$d\phi_{5,i}(t) = (\phi_{3,i}(t) - 2\gamma_i \phi_{5,i}(t)) dt \quad (19)$$

$$d\phi_{6,i}(t) = (2\phi_{5,i}(t) - 2\gamma_i \phi_{6,i}(t)) dt, \quad (20)$$

subject to  $x_i(0) = \phi_{1,i}(0) = \dots = \phi_{6,i}(0) = 0$ . Furthermore, the time- $t$  price of a zero-coupon bond maturing at time  $T$ ,  $P(t, T)$ , is given by

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left\{ \sum_{i=1}^N B_{x_i}(T-t)x_i(t) + \sum_{i=1}^N \sum_{j=1}^6 B_{\phi_{j,i}}(T-t)\phi_{j,i}(t) \right\}, \quad (21)$$

where

$$B_{x_i}(\tau) = \frac{\alpha_{1i}}{\gamma_i} \left( \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) (e^{-\gamma_i \tau} - 1) + \tau e^{-\gamma_i \tau} \right) \quad (22)$$

$$B_{\phi_{1,i}}(\tau) = \frac{\alpha_{1i}}{\gamma_i} (e^{-\gamma_i \tau} - 1) \quad (23)$$

$$B_{\phi_{2,i}}(\tau) = \left( \frac{\alpha_{1i}}{\gamma_i} \right)^2 \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) \left( \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) (e^{-\gamma_i \tau} - 1) + \tau e^{-\gamma_i \tau} \right) \quad (24)$$

$$B_{\phi_{3,i}}(\tau) = -\frac{\alpha_{1i}}{\gamma_i^2} \left( \left( \frac{\alpha_{1i}}{2\gamma^2} + \frac{\alpha_{0i}}{\gamma} + \frac{\alpha_{0i}^2}{2\alpha_{1i}} \right) (e^{-2\gamma\tau} - 1) + \left( \frac{\alpha_{1i}}{\gamma} + \alpha_{0i} \right) \tau e^{-2\gamma\tau} + \frac{\alpha_{1i}}{2} \tau^2 e^{-2\gamma\tau} \right) \quad (25)$$

$$B_{\phi_{4,i}}(\tau) = \left( \frac{\alpha_{1i}}{\gamma_i} \right)^2 \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) (e^{-\gamma_i \tau} - 1) \quad (26)$$

$$B_{\phi_{5,i}}(\tau) = -\frac{\alpha_{1i}}{\gamma_i^2} \left( \left( \frac{\alpha_{1i}}{\gamma} + \alpha_{0i} \right) (e^{-2\gamma\tau} - 1) + \alpha_{1i} \tau e^{-2\gamma\tau} \right) \quad (27)$$

$$B_{\phi_{6,i}}(\tau) = -\frac{1}{2} \left( \frac{\alpha_{1i}}{\gamma_i} \right)^2 (e^{-2\gamma_i \tau} - 1). \quad (28)$$

From the zero-coupon bonds I can compute swap rates, and in Appendix B, I develop a fast and accurate Fourier-based swaption pricing formula.

Despite the large number of state variables, the model is actually quite parsimonious.<sup>18</sup> In a model with  $N$  term structure factors, there are  $N \times 4 + 3$  parameters that are identified under  $Q$ .<sup>19</sup> For instance, in a model with two term structure factors (i.e. a total of three factors), there are 11 identifiable parameters. In contrast, in the maximal  $A_1(3)$  model of Dai and Singleton (2000), which also has two conditionally Gaussian factors and one square-root factor, there are 14 identifiable parameters.

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<sup>18</sup>Note that there are no stochastic terms in the  $\phi_{1,i}(t), \dots, \phi_{6,i}(t)$  processes, which are “auxiliary”, locally deterministic, state variables that reflect the path information of  $x_i(t)$  and  $v_i(t)$ .

<sup>19</sup>When estimating the model, I reduce it to its time-homogeneous counterpart by replacing  $f(0, t)$  with  $\varphi$  in (13) and  $\frac{P(0, T)}{P(0, t)}$  with  $\exp\{-\varphi(T-t)\}$  in (21). This adds one additional parameter under  $Q$ . On the other hand, for the model to be identified, I set  $\sigma = 1$ .

## Market prices of risk

The market prices of risk,  $\Lambda_i$ , link the Wiener processes under  $Q$  and  $P$  through

$$dW_i(t) = dW_i^Q(t) - \Lambda_i(t)dt, \quad (29)$$

$i = 1, \dots, N + 1$ . I specify the market prices of risk as

$$\Lambda_i(t) = \lambda_i \sqrt{v(t)}, \quad (30)$$

implying that the variance risk premium is linear in variance, which, qualitatively at least, is consistent with the model-free evidence in Section 2.<sup>20</sup> With this specification, the dynamics of  $x_i(t)$  and  $v(t)$  under  $P$  are given by

$$dx_i(t) = (-\gamma_i x_i(t) + \lambda_i v(t))dt + \sqrt{v(t)}dW_i(t) \quad (31)$$

$$dv(t) = \bar{\kappa}(\bar{\theta} - v(t))dt + \sigma \sqrt{v(t)} \left( \sum_{i=1}^N \rho_i dW_i(t) + \sqrt{1 - \sum_{i=1}^N \rho_i^2} dW_{N+1}(t) \right), \quad (32)$$

where  $\bar{\kappa} = \kappa - \sigma \left( \sum_{i=1}^N \lambda_i \rho_i + \lambda_{N+1} \sqrt{1 - \sum_{i=1}^N \rho_i^2} \right)$  and  $\bar{\theta} = \frac{\kappa \theta}{\bar{\kappa}}$ . Obviously the dynamics of  $\phi_{1,i}(t), \dots, \phi_{6,i}(t)$  do not change since these contain no stochastic terms.

## 3.2 Data and estimation approach

I estimate the model on an extensive panel data set of LIBOR and swap rates and swaptions.<sup>21</sup> The data is daily from January 23, 1997 to April 30, 2008.

The LIBOR/swap term structures consist of LIBOR rates with maturities of 6 and 12 months and swap rates with maturities of 2, 3, 5, 7, 10, 15, and 30 years, which were obtained from Bloomberg.

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<sup>20</sup>Trolle and Schwartz (2008a) use the “extended affine” market price of risk specification suggested by Cheredito, Filipovic, and Kimmel (2007) and Collin-Dufresne, Goldstein, and Jones (2008). However, the “completely affine” specification has several advantages. First, it allows me to solve the dynamic portfolio choice problem in Section 4 in quasi-closed form, which is not possible with the “extended affine” specification. Second, it seems more intuitive than the “extended affine” specification, where market prices of risk can become arbitrarily large as variance approaches the zero boundary. Third, it is parsimonious and one avoids having to impose the Feller restriction on the process for  $v(t)$ .

<sup>21</sup>I am implicitly assuming homogeneous credit quality across the LIBOR, swap, and swaption markets since all cash-flows are discounted using the same discount factors.



The swaptions have option maturities of 1 month and underlying swap maturities of 2, 5, 10, and 30 years. From January 23, 1997 until April 30, 2004 I have data on at-the-money-forward (ATMF) swaptions, where the strikes are equal to the forward rates on the underlying swaps. This data was also obtained from Bloomberg.

From May 1, 2004 until April 30, 2008 I have data on the entire swaption smiles, where swaption strikes are -100bp, -50bp, -25bp, 0bp, 25bp, 50bp, and 100bp away from the ATMF strike. To my knowledge, it is the first time that such data has been used in the empirical term structure literature.<sup>22</sup> This data was obtained from ICAP, which is the largest broker in the interest rate derivatives market.

Although swaptions with longer option maturities are also available, I only use 1 month options to make the data comparable to that used in Section 2, where variance swaps have an average maturity of 21 business days. Furthermore, as shown by Trolle and Schwartz (2008a), once swaptions with a wide range of option maturities are introduced, multiple variance state variables are needed to match the data.

Time series of swap rates and ATMF swaption volatilities are given in Figure 2. The model is estimated by quasi-maximum likelihood in conjunction with the extended Kalman filter along the lines of Trolle and Schwartz (2008a). In the interest of brevity, the details are omitted here.

### 3.3 Results

Parameter estimates are given in Table 5.<sup>23</sup> For all the model specifications, the estimates of  $\alpha_0$ ,  $\alpha_1$ , and  $\gamma$  imply that the forward rate volatility functions are hump shaped. In the specification with  $N = 3$ , the first factor affects forward rates of all maturities, the second factor affects forward rates with maturities up to about 10 years, while the third factor affects forward rates with maturities up to about 3 years.

As  $N$  increases,  $\sigma_{rates}$ , the standard deviation of LIBOR and swap rate pricing errors,

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<sup>22</sup>A number of papers use data on cap smiles, see e.g. Jarrow, Li, and Zhao (2007) and Trolle and Schwartz (2008a). However, the shortest cap maturity is one year, whereas here I want to use options with short maturities.

<sup>23</sup>The asymptotic covariance matrix of the estimated parameters is computed from the outer-product of the first derivatives of the likelihood function. Theoretically, it would be more appropriate to compute the asymptotic covariance matrix from both the first and second derivatives of the likelihood function. In reality, however, the second derivatives of the likelihood function are somewhat numerically unstable.

decreases from 50 basis points (bp) to 10 bp to 3 bp. Hence, with three term structure factors, the model is able to capture virtually all the variation in interest rates, consistent with much existing term structure literature.  $\sigma_{swaptions}$ , the standard deviation of swaption pricing errors, also decreases as  $N$  increases. Figure 3 displays, for the specification with  $N = 3$ , the time series of the root-mean-squared pricing errors (RMSEs) for interest rates and swaptions. The RMSEs for interest rates fluctuate around 3 bp throughout the entire sample period. The RMSEs for swaptions is around 1 percent for much of the sample period but is higher and more volatile during the LTCM crisis, during the period from 2002 until 2004, when interest rate volatility was high and volatile, and during the credit crisis.

The estimates of  $\rho$  indicate that innovations to variance are only weakly related to innovations to the term structure. This finding holds true regardless of the number of term structure factors. Indeed, it is straightforward to show that in the three model specification, the fraction of variation in variance that is spanned by the term structure is 0.0004, 0.1432, and 0.1216, respectively. Evidence for unspanned stochastic variance in fixed income markets has previously been reported by Collin-Dufresne and Goldstein (2002), Heidari and Wu (2003), Andersen and Benzoni (2005), Li and Zhao (2006, 2008), Trolle and Schwartz (2008a), and Collin-Dufresne, Goldstein, and Jones (2008), among others.

For the purpose of this paper, the most interesting issue is the market price of risk estimates. The estimated market prices of risk on the term structure factors are generally moderately negative, although in many cases not statistically significant, which implies that bonds of all maturities have positive risk premia. In contrast, the estimated market price of risk on the unspanned variance factor is strongly negative – much more negative than the prices of risk on the term structure factors – and statistically significant, in all the model specifications.

To better interpret the market price of risk estimates and to link them with the findings in Section 2, Table 6 displays, for each model specification, the model implied unconditional instantaneous Sharpe ratios on zero-coupon bonds, short term futures contracts on zero-coupon bonds, a derivative exposed solely to variance, and a derivative exposed solely to the unspanned variance factor.

The instantaneous Sharpe ratio on a zero-coupon bond with maturity  $\tau$  is given by

$$SR_{ZCB} = \frac{\sum_{i=1}^N B_{x_i}(\tau)\lambda_i}{\sqrt{\sum_{i=1}^N B_{x_i}(\tau)^2}} \sqrt{v(t)}. \quad (33)$$

Table 6 displays the unconditional  $SR_{ZCB}$  for maturities of 2, 5, 10, and 30 years. For  $N = 1$ ,

it equals 0.13 for all maturities. For  $N > 1$ , it varies with bond maturity but is generally of the same magnitude.

The price of a futures contract on a zero-coupon bond is given in Appendix C. In contrast to the zero-coupon bond itself, the futures contract depends on the unspanned variance factor. However, this dependence is weak, particularly for short term futures contracts. The instantaneous Sharpe ratio on a futures contract with maturity  $\tau_f$  on a zero-coupon bond with maturity  $\tau$  is given by

$$SR_{FUT} = \frac{\sum_{i=1}^N \left( \tilde{B}_{x_i}(\tau_f) + \tilde{B}_v(\tau_f)\sigma\rho_i \right) \lambda_i + \tilde{B}_v(\tau_f)\sigma\sqrt{1 - \sum_{i=1}^N \rho_i^2} \lambda_{N+1}}{\sqrt{\sum_{i=1}^N \left( \tilde{B}_{x_i}(\tau_f)^2 + 2\sigma\rho_i\tilde{B}_{x_i}(\tau_f)\tilde{B}_v(\tau_f) \right) + \sigma^2\tilde{B}_v(\tau_f)^2}} \sqrt{v(t)}, \quad (34)$$

where  $\tilde{B}_{x_i}(\tau_f)$  and  $\tilde{B}_v(\tau_f)$  are given in Appendix C. Table 6 shows the unconditional  $SR_{FUT}$  for 1 month futures contracts on zero-coupon bonds with maturities of 2, 5, 10, and 30 years. Since short term futures contracts have only weak exposure to the unspanned variance factor, the Sharpe ratios on the futures contracts are virtually identical to the Sharpe ratios on the underlying zero-coupon bonds.

The instantaneous Sharpe ratio on a derivative exposed solely to variance is given by

$$SR_{VAR} = \left( \sum_{i=1}^N \lambda_i \rho_i + \lambda_{N+1} \sqrt{1 - \sum_{i=1}^N \rho_i^2} \right) \sqrt{v(t)} \quad (35)$$

with unconditional values of -0.41, -0.40, and -0.43 in the three model specifications. Since variance is mostly unspanned by the term structure, this derivative is predominantly exposed to the unspanned variance factor and, therefore, has a significantly higher Sharpe ratio than an instrument exposed exclusively (in the case of bonds) or predominantly (in the case of short term futures on bonds) to the term structure factors. In fact,  $SR_{VAR}$  is generally about three times larger in absolute value than the model-implied Sharpe ratios on bonds or short-term bond futures, which is broadly consistent with the findings in the model-independent analysis in Section 2.

Finally, the instantaneous Sharpe ratio on a derivative exposed solely to the unspanned variance factor is given by

$$SR_{USV} = \lambda_{N+1} \sqrt{v(t)} \quad (36)$$

with unconditional values of -0.41, -0.42, and -0.46 in the three model specifications.

## 4 Optimal portfolio choice with interest rate derivatives

I now investigate the benefits of including interest rate derivatives in fixed income portfolios. I assume that investment opportunities are driven by the term structure model derived and estimated in the previous section. Based on this model, I derive the optimal portfolio strategy for a long-term investor, with and without access to interest rate derivatives, and compute two measures for the utility gains from optimally adding interest rate derivatives to fixed income portfolios.

### 4.1 Optimal dynamic portfolios

I assume that the investor is endowed with initial wealth,  $W(t)$ , and invests to maximize expected power utility at time  $T$  of the form

$$E_t[U(W(T))] \quad , \quad U(W) = \begin{cases} \frac{1}{1-\eta}W^{1-\eta}, & \eta > 1 \\ \log W, & \eta = 1, \end{cases} \quad (37)$$

where  $\eta$  is the parameter of relative risk aversion.<sup>24</sup>

Suppose that the investor invests in  $M$  securities that span the first  $M$  risk factors. To use a compact notation, let  $P(t) = (P_1(t), \dots, P_M(t))'$  denote the vector of asset prices. The dynamics of  $P(t)$  is assumed to be given by

$$dP(t) = \text{diag}(P(t))[(r(t)\mathbf{1} + \Sigma\lambda v(t))dt + \Sigma\sqrt{v(t)}dW(t)], \quad (38)$$

where  $\lambda = (\lambda_1, \dots, \lambda_M)'$ ,  $W(t) = (W_1(t), \dots, W_M(t))'$  and  $\Sigma$  is an  $M \times M$  invertible matrix of factor exposures.<sup>25</sup>

The investor chooses a portfolio process  $\pi = (\pi(s))_{s \in [t, T]}$ , where  $\pi(t)$  is an  $M$ -dimensional vector denoting the fractions of wealth allocated to the  $M$  risky assets. The remaining fraction  $1 - \pi(t)'\mathbf{1}$  is allocated to the money market account. Throughout, I assume that the investor is unconstrained; hence,  $\pi(t)$  can take any value. For a given  $\pi$ -process, the wealth  $W(t)$  of the investor then evolves according to

$$dW(t) = W(t)[(r(t) + \pi(t)'\Sigma\lambda v(t))dt + \pi(t)'\Sigma\sqrt{v(t)}dW(t)]. \quad (39)$$

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<sup>24</sup>Papers that work with power utility often assume  $\eta > 0$ . However, as discussed by Korn and Kraft (2004), in the case of stochastic investment opportunities and  $0 < \eta < 1$ , one may encounter problems with infinite expected utility. Therefore, I assume  $\eta \geq 1$ .

<sup>25</sup> $\text{diag}(P(t))$  denotes an  $M \times M$  matrix with the vector  $P(t)$  along the diagonal and zeros off the diagonal.

An investor without access to interest rate derivatives can obtain any desired exposure to the first  $N$  risk factors by trading in  $N$  bonds of different maturities. In this case,  $M = N$ . An investor with access to interest rate derivatives can obtain any desired exposure to all  $N + 1$  risk factors by trading in  $N + 1$  securities, at least one of which is an interest rate derivative. In this case,  $M = N + 1$ .

I solve the portfolio choice problem by dynamic programming. The indirect utility function is given by

$$J(t) = \max_{(\pi_s)_{s \in [t, T]}} \mathbb{E}_t[U(W(T))] \quad (40)$$

subject to (39). The optimal portfolio strategy for the investor, with and without access to interest rate derivatives, is given in the following Proposition:

**Proposition 1** *Consider the dynamic optimization problem of an investor with power utility over terminal wealth who faces investment opportunities that evolve according to the model described in Section 3.*

*i) When the investor can trade interest rate derivatives, the optimal portfolio strategy and the indirect utility function is given by*

$$\begin{aligned} \pi(t) &= \frac{1}{\eta} (\Sigma')^{-1} (\lambda_1, \dots, \lambda_{N+1})' + \\ &\quad \left(1 - \frac{1}{\eta}\right) \left( (\Sigma')^{-1} (B_{x_1}(T-t), \dots, B_{x_N}(T-t), 0)' - \right. \\ &\quad \left. (\Sigma')^{-1} \left( \rho_1, \dots, \rho_N, \sqrt{1 - \sum_{i=1}^N \rho_i^2} \right)' \sigma D_v(T-t) \right) \end{aligned} \quad (41)$$

and

$$J(W(t), P(t, T), v(t), t) = \begin{cases} \frac{1}{1-\eta} \left( \frac{W(t)}{P(t, T)} e^{C_v(T-t) + D_v(T-t)v(t)} \right)^{1-\eta}, & \eta > 1 \\ \log W(t) - \log P(t, T) + C_v(T-t) + D_v(T-t)v(t), & \eta = 1, \end{cases} \quad (42)$$

where  $D_v(\tau)$  solves the following ODE

$$\begin{aligned} \frac{dD_v(\tau)}{d\tau} &= \frac{1}{2\eta} \sum_{i=1}^{N+1} \lambda_i^2 - \sum_{i=1}^N (\lambda_i B_{x_i}(\tau) + B_{\phi_{2,i}}(\tau) + B_{\phi_{3,i}}(\tau)) - \bar{\kappa} D_v(\tau) + \\ &\quad \frac{1-\eta}{\eta} \left( - \sum_{i=1}^N \lambda_i B_{x_i}(\tau) + \sigma \left( \sum_{i=1}^N \lambda_i \rho_i + \lambda_{N+1} \sqrt{1 - \sum_{i=1}^N \rho_i^2} \right) D_v(\tau) \right) + \\ &\quad \frac{1-\eta}{2\eta} \left( \sum_{i=1}^N (B_{x_i}(\tau)^2 - 2\rho_i \sigma B_{x_i}(\tau) D_v(\tau)) + \sigma^2 D_v(\tau)^2 \right), \end{aligned} \quad (43)$$

subject to the boundary conditions  $D_v(0) = 0$ , and  $C_v(\tau)$  is given by

$$C_v(\tau) = \bar{\kappa}\bar{\theta} \int_0^\tau D_v(u)du. \quad (44)$$

ii) When the investor can only trade bonds the optimal portfolio strategy is given by

$$\begin{aligned} \pi(t) &= \frac{1}{\eta}(\Sigma')^{-1}(\lambda_1, \dots, \lambda_N)' + \\ &\quad \left(1 - \frac{1}{\eta}\right) \left( (\Sigma')^{-1} (B_{x_1}(T-t), \dots, B_{x_N}(T-t))' - \right. \\ &\quad \left. (\Sigma')^{-1} (\rho_1, \dots, \rho_N)' \sigma D_v(T-t) \right) \end{aligned} \quad (45)$$

while the indirect utility function still has the form (42). In this case,  $D_v(\tau)$  solves the following ODE

$$\begin{aligned} \frac{dD_v(\tau)}{d\tau} &= \frac{1}{2\eta} \sum_{i=1}^N \lambda_i^2 - \sum_{i=1}^N (\lambda_i B_{x_i}(\tau) + B_{\phi_{2,i}}(\tau) + B_{\phi_{3,i}}(\tau)) - \bar{\kappa} D_v(\tau) + \\ &\quad \frac{1-\eta}{\eta} \left( - \sum_{i=1}^N \lambda_i B_{x_i}(\tau) + \sigma \left( \sum_{i=1}^N \lambda_i \rho_i \right) D_v(\tau) \right) + \\ &\quad \frac{1-\eta}{2\eta} \left( \sum_{i=1}^N (B_{x_i}(\tau)^2 - 2\rho_i \sigma B_{x_i}(\tau) D_v(\tau)) + \sigma^2 D_v(\tau)^2 - \right. \\ &\quad \left. (1-\eta) \left( 1 - \sum_{i=1}^N \rho_i^2 \right) \sigma^2 D_v(\tau)^2 \right) \end{aligned} \quad (46)$$

subject to the boundary conditions  $D_v(0) = 0$ , and  $C_v(\tau)$  is given by (44).

**Proof:** See Appendix D. ■

The optimal portfolio strategy has a number of intuitive properties. First, it is a weighted average of the mean-variance tangency portfolio and a hedge portfolio, with the weights depending on the risk aversion parameter  $\eta$ . As usual, the hedge term disappears for log-utility investors and myopic investors (since  $B_{x_1}(0) = \dots = B_{x_N}(0) = D_v(0) = 0$ ).

Second, the Sharpe ratio of the tangency portfolio – in other words, the slope of the Capital Market Line – is given by

$$SR_{bonds}^{tan} = \sqrt{\sum_{i=1}^N \lambda_i^2 v(t)}, \quad (47)$$

when the investment universe only consists of bonds, and

$$SR_{deriv}^{tan} = \sqrt{\sum_{i=1}^{N+1} \lambda_i^2 v(t)}, \quad (48)$$

when the investment universe is expanded to include interest rate derivatives.<sup>26</sup> Therefore, to the extent that the unspanned variance factor is priced, i.e.  $\lambda_{N+1} \neq 0$ , interest rate derivatives improves the investment opportunity set.

Third, the hedge portfolio consists of two components, the first component hedging variations in  $r(t)$  and the second component hedging variations in  $v(t)$ . This is quite natural. As shown by Nielsen and Vassalou (2002) under general conditions, long-term investors that are more risk averse than log-utility investors will hedge variations in the state variables to the extent that these affect investment opportunities. Since investment opportunities can be summarized by the short term interest rate and Sharpe ratio of the tangency portfolio, which is a function of  $v(t)$  regardless of whether the investor can trade derivatives, the investor wants to hedge  $r(t)$  and  $v(t)$ . Note that the investor hedges  $v(t)$  not because it drives the volatility of the tangency portfolio but because it drives its Sharpe ratio.

Fourth, if variance contains a component that is spanned by the term structure, i.e. if  $\rho_1, \dots, \rho_N$  are not all zero, an investor, who can only trade bonds, is able to partially hedge variations in  $v(t)$  and the second component in (45) is non-zero. If variance is completely unspanned by the term structure, the second component in (45) is zero. In contrast, an investor who can trade interest rate derivatives is able to hedge variations in  $v(t)$  regardless of the extent to which it is spanned by the term structure.<sup>27</sup>

In summary, an investor finds interest rate derivatives attractive for two reasons; they improve investment opportunities by increasing the Sharpe ratio of the tangency portfolio and they improve the ability of the investor to hedge adverse changes in the Sharpe ratio.

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<sup>26</sup>To see this, let  $\pi^{tan}(t)$  denote the tangency portfolio. The expected excess return of the tangency portfolio is given by

$$\pi^{tan}(t)' \Sigma \lambda v(t) = \lambda' \lambda v(t), \quad (49)$$

while its volatility is given by

$$\sqrt{\Sigma' \pi^{tan}(t) \pi^{tan}(t)' \Sigma v(t)} = \sqrt{\lambda' \lambda v(t)}, \quad (50)$$

and, therefore, the Sharpe ratio is given by  $\sqrt{\lambda' \lambda v(t)}$ .

<sup>27</sup>I note in passing that, in the limit, as  $\eta \rightarrow \infty$ ,  $D_v(\tau) = 0$ ,  $\tau \geq 0$  (to see this, note that  $\frac{1}{2}B_{x_i}(\tau)^2 + B_{\phi_{2,i}}(\tau) + B_{\phi_{3,i}}(\tau) = 0$ ). Hence, the optimal portfolio for an infinitely risk averse investor is simply the first component of the hedge portfolio. Moreover, it is straightforward to show that if the investment universe includes a zero-coupon bond maturing at the end of the investment horizon, the infinitely risk averse investor will allocate his entire wealth to this bond. This result is quite intuitive and has been obtained in various settings by Sørensen (1999), Brennan and Xia (2000), Wachter (2003), and Munk and Sørensen (2004), among others.

The second effect only applies to long-term investors that are more risk averse than log-utility investors. Consequently, even if the market price of the unspanned variance factor were zero, in which case investment opportunities would not improve with the introduction of derivatives, a long term investor with  $\eta > 1$  would still find it optimal to hold a non-zero derivative position.

## 4.2 Portfolio improvements

Certainty equivalent wealth (CEW) is defined as the amount of wealth at time  $T$  which leaves the investor indifferent between receiving it and investing current wealth optimally according to either (41) or (45), depending on the ability to trade interest rate derivatives. That is, the CEW solves  $U(CEW) = J(W(t), P(t, T), v(t), t)$ , which implies that

$$CEW = \frac{W(t)}{P(t, T)} e^{C_v(T-t) + D_v(T-t)v(t)}. \quad (51)$$

I consider two measures of the utility gains from participating in the interest rate derivatives market. The first measure is the gain in CEW in terms of continuously compounded annualized returns. I denote this measure  $\mathcal{R}_{CEW}$  and it is given by

$$\begin{aligned} \mathcal{R}_{CEW} &= \frac{1}{T} \log \left( \frac{CEW^{deriv}}{CEW^{bonds}} \right) \\ &= \frac{1}{T} \left( C_v^{deriv}(T-t) - C_v^{bond}(T-t) + \left( D_v^{deriv}(T-t) - D_v^{bond}(T-t) \right) v(t) \right). \end{aligned} \quad (52)$$

The same measure is used by Liu and Pan (2003) to determine the value of participating in the equity derivatives market, making a direct comparison with their results possible.

The second measure is the fraction of wealth that an investor restricted to trading bonds would be willing to give up to be allowed to trade interest rate derivatives. I denote this measure  $\mathcal{X}_W$  and it solves  $J(W(t), P(t, T), v(t), t)^{bonds} = J((1 - \mathcal{X}_W)W(t), P(t, T), v(t), t)^{deriv}$ . Straightforward calculations show that

$$\mathcal{X}_W = 1 - \frac{CEW^{bonds}}{CEW^{deriv}} = 1 - e^{C_v^{bond}(T-t) - C_v^{deriv}(T-t) + (D_v^{bond}(T-t) - D_v^{deriv}(T-t))v(t)}. \quad (53)$$

## 4.3 Results

I first compute, for each of the three model specifications, the unconditional instantaneous Sharpe ratio of the tangency portfolio, with and without investments in interest rate derivatives. These are displayed in Table 6. When the investment universe only consists of bonds,



the Sharpe ratio lies between 0.13 and 0.17, whereas when the investment universe also includes interest rate derivatives, the Sharpe ratio lies between 0.43 and 0.48. Hence, interest rate derivatives provide a substantial improvement in investment opportunities.

Figure 4 shows, for each of the three model specifications, the two measures of the utility gain from participating in the interest rate derivatives market. I consider investors with investment horizons from 1 day to 10 years and relative risk aversion coefficients of 1, 3, 5, and 10, and assume that  $v(t)$  and the term structure equal their unconditional means. For a given investment horizon,  $\mathcal{R}_{CEW}$  decreases with the risk aversion since more risk averse investors will exploit the improved risk-return tradeoff to a lesser extent. This also means that more risk averse investors will give up a smaller fraction of wealth to obtain access to the interest rate derivatives market. For a given risk aversion level,  $\mathcal{R}_{CEW}$  increases with the investment horizon since longer term investors will exploit the hedging ability of interest rate derivatives to a greater extent.<sup>28</sup> The utility gains are substantial and fairly similar across model specifications. For instance, assuming an investment horizon of five years and  $\eta = 3$ ,  $\mathcal{R}_{CEW}$  equals 0.036, 0.038, and 0.043, for  $N = 1, 2,$  and  $3$ , respectively. That is, the utility gain from participating in the interest rate derivatives market corresponds to an additional return of about four percent per annum on certainty equivalent wealth. The fraction of wealth that this investor would be willing to give up to be able to trade interest rate derivatives is 0.166, 0.171, and 0.194, for  $N = 1, 2,$  and  $3$ , respectively.

Liu and Pan (2003) compute the utility gain for equity investors from participating in the equity derivatives market. For the same investment horizon and risk aversion, they report an  $\mathcal{R}_{CEW}$  of 0.142.<sup>29</sup> Hence, it appears that the utility gains from participating in the equity derivatives market are larger than from participating in the interest rate derivatives market. This is consistent with the finding in Section 2 that the Sharpe ratio on S&P 500 index variance swaps are larger than the Sharpe ratio on Treasury futures variance swaps, both in absolute terms and relative to the Sharpe ratios of the underlying assets.

Figure 5 shows the time series of  $\mathcal{R}_{CEW}$  for the  $N = 3$  specification, based on the Kalman filtered state variables and assuming  $\eta = 3$  and investment horizons of 1 day, 2 years, and 5

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<sup>28</sup>For a log-utility investor,  $\mathcal{R}_{CEW}$  is constant across investment horizons since this investor does not to hedge variations in investment opportunities.

<sup>29</sup>This is for their specification, where investment opportunities are driven by the Heston (1993) model, which is most comparable to my setup.

years.<sup>30</sup> The utility gain from participating in the interest rate derivatives market increases with variance, since all market prices of risk, including the one on the unspanned variance factor, increase with variance (in absolute terms). However, due to the mean reversion in variance, and hence mean reversion in the Sharpe ratio of the tangency portfolio, the volatility of  $\mathcal{R}_{CEW}$  is lower for longer investment horizons.

#### 4.4 Robustness checks

So far, the results are based on the parameter estimates from Section 3. To investigate the robustness of my conclusions, I analyze the sensitivity of  $\mathcal{R}_{CEW}$  to variations in the key parameters  $\bar{\kappa}$ ,  $\sigma$ , and  $\lambda_{N+1}$ , in case of the  $N = 3$  specification. I vary each of these parameters individually, holding all other  $P$ -parameters and market price of risk parameters constant. Figure 6 displays the results, assuming  $\eta = 3$  and investment horizons of 1 day, 2 years and 5 years.

When  $\kappa$  increases, shocks to variance, and hence shocks to the Sharpe ratio of the tangency portfolio, are less persistent. For long-term investors, the incentive to hedge variations in the Sharpe ratio is lower and, consequently, the benefits of including derivatives in the hedge component of the optimal portfolio strategy decrease and  $\mathcal{R}_{CEW}$  approaches that of short-term investors.

When  $\sigma$  increases, variance is more volatile making the Sharpe ratio of the tangency portfolio more volatile. For long-term investors, the benefits of including derivatives in the hedge component of the optimal portfolio strategy increase and  $\mathcal{R}_{CEW}$  increases relative to that of short-term investors.

When  $\lambda_{N+1}$  becomes less negative, for a given  $v(t)$ , the Sharpe ratio of the tangency portfolio in the presence of derivatives approaches the Sharpe ratio in the absence of derivatives, which reduces the benefits of including derivatives in the tangency portfolio. Furthermore, the Sharpe ratio of the tangency portfolio is less volatile, which reduces the benefits of including derivatives in the hedge component of the optimal portfolio strategy for long-term investors. Therefore, as  $\lambda_{N+1}$  becomes less negative,  $\mathcal{R}_{CEW}$  decreases for both short-term and long-term investors and the difference between the two decreases as well. When  $\lambda_{N+1} = 0$ , there are no benefits of including derivatives in the tangency portfolio and  $\mathcal{R}_{CEW} = 0$  for myopic investors. For long-term investors,  $\mathcal{R}_{CEW}$  is positive, but close to zero, as the benefits of including

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<sup>30</sup>The time series of  $v(t)$  is displayed in Figure 3, Panel C.

derivatives in the hedge component of the optimal portfolio strategy is small.

I conclude that the qualitative results are robust to large variations in the parameter estimates. The quantitative results, however, do display some sensitivity to the  $\lambda_{N+1}$ -estimate, in particular.

## 5 Conclusions

In this paper, I first estimate the variance risk premium in the Treasury futures market using a model independent approach. I find that the Treasury futures variance risk premium is significantly negative, shorting variance swaps generate Sharpe ratios that are about two to three times larger than the Sharpe ratios of the underlying Treasury futures, the variance risk premium is not a compensation for exposure to bond or equity market risks, suggesting that there is an unspanned variance factor with a significant risk premium, and the variance risk premium becomes more negative when variance increases.

I then estimate the variance risk premium in the interest rate swap market using a dynamic term structure model and an extensive panel data set of swap rates and swaptions. I find that variance risk is predominantly unspanned, and the estimated market price of risk on the unspanned variance factor is strongly negative – much more negative than the prices of risk on the term structure factors – and statistically significant. These findings hold true across different model specifications. The model-implied Sharpe ratio on a derivative exposed solely to variance is about three times larger in absolute value than the model-implied Sharpe ratios on bonds or short-term bond futures consistent with the model independent analysis.

Finally, assuming that investment opportunities evolve according to the estimated term structure model, I derive the optimal portfolio strategy for a long-term fixed income investor with CRRA utility over terminal wealth, who either does or does not participate in the interest rate derivatives market. Interest rate derivatives play a unique role in dynamic portfolio strategies. They expand the investment opportunity set and provide ways of hedging variations in investment opportunities, and I find substantial utility gains from participating in the interest rate derivatives market.

The analysis may be extended in a number of directions. First, the dynamic term structure model used in this paper is fairly parsimonious, since it only has one unspanned variance factor. In reality, there are multiple unspanned variance factors, see e.g. Trolle and Schwartz (2008a), and to the extent that all of these carry a risk premium, the benefits of adding interest rate

derivatives to fixed income portfolios will likely be higher than what is reported here.

Second, one could analyze the benefits of interest rate derivatives in a more general setting where the investor can invest also in equities and equity derivatives. However, since bonds and equities exhibit low correlation and since interest rate volatility and equity volatility is only moderately correlated,<sup>31</sup> I expect that there will still be substantial utility gains from participating in the interest rate derivatives market.

Third, it might be relevant to incorporate transaction costs since these are typically higher in the derivatives market. It is not clear, however, that transaction cost would markedly reduce the attractiveness of interest rate derivatives, since, in the over-the-counter market, one could presumably design products that minimize the need for frequent portfolio rebalancing.

Perhaps more important is to analyze the sources of unspanned stochastic variance and why the unspanned component carries a large negative market price of risk. I leave these issues for future research.

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<sup>31</sup>Over long sample periods, bonds and equities have a small positive correlation of about 0.20. However, the correlations between daily returns on the Treasury futures and the S&P 500 index for the sample period in Section 2 are actually negative in the range -0.14 to -0.08. As I report in Section 2, the correlations between the different Treasury futures variance swap rates and the S&P 500 index variance swap rate is between 0.35 and 0.44.

## Appendix A. Computing synthetic variance swap rates

This appendix contains details on the implementation of (3). The first issue is that the CBOT options are American-style while the synthetic variance swap formula utilizes European-style options. Therefore, it is necessary to convert the American option prices into European option prices by subtracting an estimate of the early exercise premium. This is done using the same approach as in Trolle and Schwartz (2008b) (see also Broadie, Chernov, and Johannes (2007)).<sup>32</sup> The estimated early exercise premium is always very small, since I only use short-maturity, OTM options.

The second issue is how to compute the integrals in (3) given that only a finite number of option prices are available. Suppose at time  $t$  we have a range of options expiring at time  $T$  on a futures contract maturing at time  $T_1$ , and let  $\sigma$  denote the Black (1976) implied volatility of the option that is closest to at the money (ATM). In a Black (1976) log-normal setting, for an option with strike  $X$ , moneyness defined as

$$d = \frac{\log(X/F(t, T_1))}{\sigma\sqrt{(T-t)}} \quad (54)$$

approximately gives the number of standard deviations that the log strike is away from the log futures price. I truncate the first integral in (3) at  $X_{min} = F(t, T_1)e^{-10\sigma\sqrt{(T-t)}}$ , corresponding to  $d = -10$ , and the second integral in (3) at  $X_{max} = F(t, T_1)e^{10\sigma\sqrt{(T-t)}}$ , corresponding to  $d = 10$ . The integrals are evaluated with ‘‘Simpson’s rule’’ using 999 integration points for each integral. On a given day, options prices corresponding to the required strikes in the integration rule are obtained by first linearly interpolating between the available Black (1976) implied volatilities and then converting from implied volatilities to prices. For strikes below the lowest available strike, I use the implied volatility at the lowest strike. Similarly, for strikes above the highest available strike, I use the implied volatility at the highest strike. This is basically the same interpolation/extrapolation approach as that used by Carr and Wu (2008). The approximation error caused by the extrapolation of implied volatilities is small, since option prices are very low in the regions of strikes where extrapolation is necessary, see Jiang and Tian (2005) for an extensive discussion.

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<sup>32</sup>The idea is, for each option, to assume that the price of the underlying futures contract follows a geometric Brownian motion. With this assumption, American options can be priced using the Barone-Adesi and Whaley (1987) formula. Inverting this formula for a given American option price yields an implied volatility, from which the associated European option can be priced with the Black (1976) formula.

## Appendix B. Quasi-analytical swaption prices

### Forward swap rate dynamics

Consider a discrete tenor structure

$$0 = T_0 < T_1 < \dots < T_K < T_{K+1} \quad (55)$$

and let

$$\tau_k = T_{k+1} - T_k, \quad k = 0, 1, \dots, K \quad (56)$$

denote the lengths between tenor dates. Let  $S(t, T_m, T_n)$  denote the time- $t$  forward swap rate for the period  $T_m$  to  $T_n$  with fixed-leg payments at  $T_{m+1}, \dots, T_n$ .  $S(t, T_m, T_n)$  is given by

$$S(t, T_m, T_n) = \frac{P(t, T_m) - P(t, T_n)}{A(t, T_m, T_n)}, \quad (57)$$

where

$$A(t, T_m, T_n) = \sum_{j=m+1}^n \tau_{j-1} P(t, T_j). \quad (58)$$

When pricing swaptions, it is convenient to work under the ‘‘forward swap measure’’ which is the measure associated with using  $A(t, T_m, T_n)$  as numeraire, see Jamshidian (1997). This measure is denoted  $Q^{T_m, T_n}$ . I can express  $Q^{T_m, T_n}$  in terms of  $Q$  through

$$\xi(t) = \frac{dQ^{T_m, T_n}}{dQ} = \frac{A(t, T_m, T_n)}{M(t)} \frac{1}{A(0, T_m, T_n)}, \quad (59)$$

where  $M(t) = \exp\left(\int_0^t r(u) du\right)$  denotes the time- $t$  value of the money market account. Since

$$\frac{d\xi(t)}{\xi(t)} = \sum_{j=m+1}^n \frac{\tau_{j-1} P(t, T_j)}{A(t, T_m, T_n)} \sum_{i=1}^N B_{x_i}(T_j - t) \sqrt{v(t)} dW_i^Q(t), \quad \xi(0) = 1, \quad (60)$$

it follows from Girsanov’s Theorem that

$$dW_i^{Q^{T_m, T_n}}(t) = dW_i^Q(t) - \sum_{j=m+1}^n \frac{\tau_{j-1} P(t, T_j)}{A(t, T_m, T_n)} B_{x_i}(T_j - t) \sqrt{v(t)} dt, \quad (61)$$

$i = 1, \dots, N$ , are Wiener processes under  $Q^{T_m, T_n}$ .

The dynamics of  $S(t, T_m, T_n)$  under  $Q^{T_m, T_n}$  (by construction,  $S(t, T_m, T_n)$  is a martingale under  $Q^{T_m, T_n}$  so I am only concerned with the diffusion term) is given by

$$dS(t, T_m, T_n) = \sum_{i=1}^N \sigma_{S,i}(t, T_m, T_n) \sqrt{v(t)} dW_i^{Q^{T_m, T_n}}(t), \quad (62)$$

where

$$\sigma_{S,i}(t, T_m, T_n) = \sum_{j=m}^n \zeta_j(t) B_{x_i}(T_j - t) \quad (63)$$

and  $\zeta_m(t) = \frac{P(t, T_m)}{A(t, T_m, T_n)}$ ,  $\zeta_j(t) = -\tau_{j-1} S(t, T_m, T_n) \frac{P(t, T_j)}{A(t, T_m, T_n)}$  for  $j = m+1, \dots, n-1$ , and  $\zeta_n(t) = -(1 + \tau_{n-1} S(t, T_m, T_n)) \frac{P(t, T_n)}{A(t, T_m, T_n)}$ . Furthermore, the dynamics of  $v(t)$  under  $Q^{T_m, T_n}$  is given by

$$dv(t) = (\kappa\theta - \tilde{\kappa}v(t))dt + \sigma\sqrt{v(t)}dW_{N+1}^Q(t), \quad (64)$$

where

$$\tilde{\kappa} = \kappa - \sigma \sum_{i=1}^N \rho_i \sum_{j=m+1}^n \xi_j(t) B_{x_i}(T_j - t) \quad (65)$$

and  $\xi_j(t) = \frac{\tau_{j-1} P(t, T_j)}{A(t, T_m, T_n)}$ .

The joint dynamics of  $S(t, T_m, T_n)$  and  $v(t)$  under  $Q^{T_m, T_n}$  is non-affine, since the  $\zeta_j(t)$  terms in (63) and the  $\xi_j(t)$  terms in (65) are stochastic. However, by “freezing” these terms at their initial expected values, I obtain an affine expression, which makes it possible to derive quasi-analytical prices of swaptions.<sup>33</sup>

## The pricing of swaptions

A payer swaption is an option to enter into a payer swap at a given fixed rate. Let  $\mathcal{P}(t, T_m, T_n, K)$  denote the time- $t$  value of a European payer swaption expiring at  $T_m$  with strike  $K$  on a swap for the period  $T_m$  to  $T_n$ . This is also denoted a  $(T_m - t)$ -into- $(T_n - T_m)$  payer swaption. At expiration, the swaption has a payoff of

$$\begin{aligned} V(T_m, T_m, T_n)^+ &= (1 - P(T_m, T_n) - KA(T_m, T_m, T_n))^+ \\ &= A(T_m, T_m, T_n) (S(T_m, T_m, T_n) - K)^+. \end{aligned} \quad (66)$$

At  $t < T_m$ , its price is given by

$$\begin{aligned} \mathcal{P}(t, T_m, T_n, K) &= E_t^Q \left[ e^{-\int_t^{T_m} r(s)ds} A(T_m, T_m, T_n) (S(T_m, T_m, T_n) - K)^+ \right] \\ &= A(t, T_m, T_n) E_t^{Q^{T_m, T_n}} [(S(T_m, T_m, T_n) - K)^+]. \end{aligned} \quad (67)$$

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<sup>33</sup>The time- $t$  expected values of  $\zeta_j(u)$  and  $\xi_j(u)$  are simply their time- $t$  values since these terms are martingales under  $Q^{T_m, T_n}$ . A similar approach is followed by Schrage and Pelsser (2006) in a general affine model. They argue that the approximation is very accurate since  $\zeta_j(u)$  and  $\xi_j(u)$  typically have low variances.

Given the (approximately) affine model for the dynamics of the forward swap rate, swaptions can be priced quasi-analytically. First, I find the characteristic function of  $S(T_m, T_m, T_n)$  given by

$$\psi(u, t, T_m, T_n) = E_t^{Q^{T_m, T_n}} \left[ e^{i u S(T_m, T_m, T_n)} \right], \quad (68)$$

where  $i = \sqrt{-1}$ . This has an exponentially affine solution as demonstrated in the following proposition.

**Proposition 2** (68) is given by

$$\psi(u, t, T_m, T_n) = e^{M(T_m - t) + N(T_m - t)v(t) + i u S(t, T_m, T_n)}, \quad (69)$$

where  $M(\tau)$  and  $N(\tau)$  solve the following system of ODEs

$$\begin{aligned} \frac{dM(\tau)}{d\tau} &= N(\tau)\kappa\theta \\ \frac{dN(\tau)}{d\tau} &= N(\tau) \left( -\tilde{\kappa} + i u \sigma \sum_{i=1}^N \rho_i \sigma_{S,i}(t, T_m, T_n) \right) + \frac{1}{2} N(\tau)^2 \sigma^2 - \frac{1}{2} u^2 \sum_{i=1}^N \sigma_{S,i}(t, T_m, T_n)^2 \end{aligned} \quad (70)$$

subject to the boundary conditions  $M(0) = 0$  and  $N(0) = 0$ .

**Proof:** Available upon request. ■

Next, I follow the general approach of Carr and Madan (1999) and Lee (2004) to price swaptions. The idea is that the Fourier transform of the modified swaption price

$$\widehat{\mathcal{P}}(t, T_m, T_n, K) = e^{\alpha K} \mathcal{P}(t, T_m, T_n, K) \quad (72)$$

can be expressed in terms of the characteristic function of  $S(T_m, T_m, T_n)$ .<sup>34</sup> The swaption price is then obtained by applying the Fourier inversion theorem. The result is given in the following proposition.

**Proposition 3** The time- $t$  price of a European payer swaption expiring at  $T_m$  with strike  $K$  on a swap for the period  $T_m$  to  $T_n$ ,  $\mathcal{P}(t, T_m, T_n, K)$ , is given by

$$\mathcal{P}(t, T_m, T_n, K) = A(t, T_m, T_n) \frac{e^{-\alpha K}}{\pi} \int_0^\infty \operatorname{Re} \left[ \frac{e^{-i u K} \psi(u - i \alpha, t, T_m, T_n)}{(\alpha + i u)^2} \right] du. \quad (73)$$

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<sup>34</sup>The control parameter  $\alpha$  must be chosen to ensure that the modified swaption price is  $L^2$  integrable, which is a sufficient condition for its Fourier transform to exist.



**Proof:** Available upon request. █

Simulations show that for pricing OTM swaptions, this pricing formula is more accurate than the stochastic duration approach of Munk (1999), which is used by Trolle and Schwartz (2008a) for estimating term structure models on ATM swaptions.

## Appendix C. Pricing futures contracts on zero-coupon bonds

Let  $F(t, T, T_1)$  denote the time- $t$  price of a futures contract expiring at time  $T$  on a zero-coupon bond maturing at time  $T_1$ . In the absence of arbitrage opportunities, the (continuously compounded) futures contract is a martingale under the risk-neutral measure, see e.g. Duffie (2001). Furthermore, at expiration

$$F(T, T, T_1) = P(T, T_1). \quad (74)$$

I conjecture that  $F(t, T, T_1)$  takes the form

$$F(t, T, T_1) = \exp \left\{ \tilde{B}_0(T-t) + \sum_{i=1}^N \tilde{B}_{x_i}(T-t)x_i(t) + \sum_{i=1}^N \sum_{j=1}^6 \tilde{B}_{\phi_{j,i}}(T-t)\phi_{j,i}(t) + \tilde{B}_v(T-t)v(t) \right\}. \quad (75)$$

It is straightforward to show that (75) is indeed a martingale and satisfies (74) provided that  $\tilde{B}_{x_i}(\tau)$ ,  $\tilde{B}_{\phi_{j,i}}(\tau)$ ,  $\tilde{B}_v(\tau)$ , and  $\tilde{B}_0(\tau)$  solve the following system of ODEs

$$\begin{aligned} \frac{d\tilde{B}_{x_i}(\tau)}{d\tau} &= -\gamma\tilde{B}_{x_i}(\tau) + \tilde{B}_{\phi_{1,i}}(\tau) \\ \frac{d\tilde{B}_{\phi_{1,i}}(\tau)}{d\tau} &= -\gamma\tilde{B}_{\phi_{1,i}}(\tau) \\ \frac{d\tilde{B}_{\phi_{2,i}}(\tau)}{d\tau} &= -\gamma\tilde{B}_{\phi_{2,i}}(\tau) + \tilde{B}_{\phi_{4,i}}(\tau) \\ \frac{d\tilde{B}_{\phi_{3,i}}(\tau)}{d\tau} &= -2\gamma\tilde{B}_{\phi_{3,i}}(\tau) + \tilde{B}_{\phi_{5,i}}(\tau) \\ \frac{d\tilde{B}_{\phi_{4,i}}(\tau)}{d\tau} &= -\gamma\tilde{B}_{\phi_{4,i}}(\tau) \\ \frac{d\tilde{B}_{\phi_{5,i}}(\tau)}{d\tau} &= -2\gamma\tilde{B}_{\phi_{5,i}}(\tau) + 2\tilde{B}_{\phi_{6,i}}(\tau) \\ \frac{d\tilde{B}_{\phi_{6,i}}(\tau)}{d\tau} &= -2\gamma\tilde{B}_{\phi_{6,i}}(\tau) \\ \frac{d\tilde{B}_v(\tau)}{d\tau} &= -\kappa\tilde{B}_v(\tau) + \sum_{i=1}^N \left( \tilde{B}_{\phi_{2,i}}(\tau) + \tilde{B}_{\phi_{3,i}}(\tau) \right) \\ &\quad + \frac{1}{2} \left( \sum_{i=1}^N \left( \tilde{B}_{x_i}(\tau)^2 + 2\rho_i\sigma\tilde{B}_{x_i}(\tau)\tilde{B}_v(\tau) \right) + \sigma^2\tilde{B}_v(\tau)^2 \right) \\ \frac{d\tilde{B}_0(\tau)}{d\tau} &= \kappa\theta\tilde{B}_v(\tau), \end{aligned} \quad (76)$$

subject to the boundary conditions  $\tilde{B}_{x_i}(0) = B_{x_i}(T_1 - T)$ ,  $\tilde{B}_{\phi_{j,i}}(0) = B_{\phi_{j,i}}(T_1 - T)$ ,  $\tilde{B}_v(0) = 0$ , and  $\tilde{B}_0(0) = \log(P(0, T_1)) - \log(P(0, T))$ .

## Appendix D. Proof of Proposition 1

Let  $X(t)$  denote the vector of the  $7N + 1$  state variables

$$X(t) = (x_1(t), \dots, x_N(t), \phi_{1,1}(t), \dots, \phi_{6,N}(t), v(t))', \quad (77)$$

which evolves according to

$$dX(t) = m(X(t))dt + \sqrt{v(t)}(\Sigma^X dW(t) + \widehat{\Sigma}^X d\widehat{W}(t)), \quad (78)$$

where  $W(t) = (W_1(t), \dots, W_M(t))'$ ,  $\widehat{W}(t) = (W_{M+1}(t), \dots, W_{N+1}(t))'$ , and  $\Sigma^X$  and  $\widehat{\Sigma}^X$  have dimensions  $(7N + 1) \times M$  and  $(7N + 1) \times (N + 1 - M)$ , respectively.

The Hamilton-Jacobi-Bellman equation is given by

$$\begin{aligned} 0 = & \max_{\pi_t} \left[ \frac{dJ}{dt} + J_W W_t (r(X_t) + \pi_t' \Sigma \lambda v_t) + \frac{1}{2} J_{WW} W_t^2 \pi_t' \Sigma \Sigma' \pi_t v_t + \right. \\ & \left. J_X' m(X_t) + \frac{1}{2} \text{tr} \left( J_{XX} \left( \Sigma^X (\Sigma^X)' + \widehat{\Sigma}^X (\widehat{\Sigma}^X)' \right) v_t \right) + W_t \pi_t' \Sigma (\Sigma^X)' J_{WX} v_t \right], \quad (79) \end{aligned}$$

where subscripts on  $J$  denote partial derivatives (the exception is the partial derivative of  $J$  w.r.t.  $t$ , which is denoted  $dJ/dt$ ). Maximization w.r.t.  $\pi_t$  gives the first order condition

$$\pi_t = -\frac{J_W}{W_t J_{WW}} (\Sigma')^{-1} \lambda - (\Sigma')^{-1} (\Sigma^X)' \frac{J_{WX}}{W_t J_{WW}}. \quad (80)$$

Substituting back into (79) gives the second-order PDE

$$\begin{aligned} 0 = & \frac{dJ}{dt} + r(X_t) W_t J_W - \frac{1}{2} \frac{J_W^2}{J_{WW}} \lambda' \lambda v_t + J_X' m(X_t) + \frac{1}{2} \text{tr} (J_{XX} (\Sigma^X (\Sigma^X)' + \\ & \widehat{\Sigma}^X (\widehat{\Sigma}^X)')) v_t - \frac{1}{2 J_{WW}} J_{WX}' \Sigma^X (\Sigma^X)' J_{WX} v_t - \lambda' (\Sigma^X)' \frac{J_W J_{WX}}{J_{WW}} v_t. \quad (81) \end{aligned}$$

If we can find a  $J$  that solves (81) subject to the boundary condition  $J(W, X, T) = U(W)$ , then we know from the Verification Theorem (see e.g. Fleming and Soner (1993) or Oksendal (1998)) that the portfolio strategy (80) is optimal. To this end, I conjecture that  $J$  has the form

$$J(W_t, X_t, t) = \begin{cases} \frac{1}{1-\eta} \left( e^{C(T-t)+D(T-t)'X_t} W_t \right)^{1-\eta}, & \eta > 1 \\ \log W_t + C(T-t) + D(T-t)'X_t, & \eta = 1, \end{cases} \quad (82)$$

in which case (81) reduces to

$$\begin{aligned} 0 = & -\frac{dC(\tau)}{d\tau} - \frac{dD(\tau)'}{d\tau} X_t + r(X_t) + \frac{1}{2\eta} \lambda' \lambda v_t + D(\tau)' m(X_t) + \\ & \frac{(1-\eta)}{2\eta} D(\tau)' \Sigma^X (\Sigma^X)' D(\tau) v_t + \frac{(1-\eta)}{2} D(\tau)' \widehat{\Sigma}^X (\widehat{\Sigma}^X)' D(\tau) v_t + \frac{(1-\eta)}{\eta} D(\tau)' \Sigma^X \lambda \quad (83) \end{aligned}$$

In the complete market case, where the investor can trade both bonds and derivatives, (83)

becomes

$$\begin{aligned}
0 = & \sum_{i=1}^N \left( \left[ -\frac{dD_{x_i}(\tau)}{d\tau} + A_{x_i} - \gamma_i D_{x_i}(\tau) + D_{\phi_{1,i}}(\tau) \right] x_i(t) + \right. \\
& \left[ -\frac{dD_{\phi_{1,i}}(\tau)}{d\tau} + A_{\phi_{1,i}} - \gamma_i D_{\phi_{1,i}}(\tau) \right] \phi_{1,i}(t) + \\
& \left[ -\frac{dD_{\phi_{2,i}}(\tau)}{d\tau} + A_{\phi_{2,i}} - \gamma_i D_{\phi_{2,i}}(\tau) + D_{\phi_{4,i}}(\tau) \right] \phi_{2,i}(t) + \\
& \left[ -\frac{dD_{\phi_{3,i}}(\tau)}{d\tau} + A_{\phi_{3,i}} - 2\gamma_i D_{\phi_{3,i}}(\tau) + D_{\phi_{5,i}}(\tau) \right] \phi_{3,i}(t) + \\
& \left[ -\frac{dD_{\phi_{4,i}}(\tau)}{d\tau} + A_{\phi_{4,i}} - \gamma_i D_{\phi_{4,i}}(\tau) \right] \phi_{4,i}(t) + \\
& \left[ -\frac{dD_{\phi_{5,i}}(\tau)}{d\tau} + A_{\phi_{5,i}} - 2\gamma_i D_{\phi_{5,i}}(\tau) + 2D_{\phi_{6,i}}(\tau) \right] \phi_{5,i}(t) + \\
& \left. \left[ -\frac{dD_{\phi_{6,i}}(\tau)}{d\tau} + A_{\phi_{6,i}} - 2\gamma_i D_{\phi_{6,i}}(\tau) \right] \phi_{6,i}(t) \right) + \\
& \left[ -\frac{dD_v(\tau)}{d\tau} + \frac{1}{2\eta} \sum_{i=1}^{N+1} \lambda_i^2 + \sum_{i=1}^N (\lambda_i D_{x_i}(\tau) + D_{\phi_{2,i}}(\tau) + D_{\phi_{3,i}}(\tau)) - \bar{\kappa} D_v(\tau) + \right. \\
& \left. \frac{1-\eta}{\eta} \left( \sum_{i=1}^N \lambda_i D_{x_i}(\tau) + \sigma \left( \sum_{i=1}^N \lambda_i \rho_i + \lambda_{N+1} \sqrt{1 - \sum_{i=1}^N \rho_i^2} \right) D_v(\tau) \right) + \right. \\
& \left. \frac{1-\eta}{2\eta} \left( \sum_{i=1}^N (D_{x_i}(\tau)^2 + 2\rho_i \sigma D_{x_i}(\tau) D_v(\tau)) + \sigma^2 D_v(\tau)^2 \right) \right] v(t) + \\
& \left[ -\frac{dC(\tau)}{d\tau} + f(0, T - \tau) + \bar{\kappa} \bar{\theta} D_v(\tau) \right]. \tag{84}
\end{aligned}$$

Therefore, (82) is indeed a solution to the PDE, provided that  $C(\tau)$ ,  $D_{x_i}(\tau)$ ,  $D_{\phi_{j,i}}(\tau)$ , and  $D_v(\tau)$  solve the system of ODEs given in brackets subject to the boundary conditions  $C(0) = D_{x_i}(0) = D_{\phi_{j,i}}(0) = D_v(0) = 0$ . One can show that  $D_{x_i}(\tau) = -B_{x_i}(\tau)$  and  $D_{\phi_{j,i}}(\tau) = -B_{\phi_{j,i}}(\tau)$ , and using that  $\exp(-\int_0^\tau f(0, T-u)du) = \frac{P(0,T)}{P(0,t)}$ , I obtain the result stated in the first part of Proposition 1.

In the incomplete market case, where the investor can trade only bonds, (83) becomes

$$\begin{aligned}
0 = & \sum_{i=1}^N \left( \left[ -\frac{dD_{x_i}(\tau)}{d\tau} + A_{x_i} - \gamma_i D_{x_i}(\tau) + D_{\phi_{1,i}}(\tau) \right] x_i(t) + \right. \\
& \left[ -\frac{dD_{\phi_{1,i}}(\tau)}{d\tau} + A_{\phi_{1,i}} - \gamma_i D_{\phi_{1,i}}(\tau) \right] \phi_{1,i}(t) + \\
& \left[ -\frac{dD_{\phi_{2,i}}(\tau)}{d\tau} + A_{\phi_{2,i}} - \gamma_i D_{\phi_{2,i}}(\tau) + D_{\phi_{4,i}}(\tau) \right] \phi_{2,i}(t) + \\
& \left[ -\frac{dD_{\phi_{3,i}}(\tau)}{d\tau} + A_{\phi_{3,i}} - 2\gamma_i D_{\phi_{3,i}}(\tau) + D_{\phi_{5,i}}(\tau) \right] \phi_{3,i}(t) + \\
& \left[ -\frac{dD_{\phi_{4,i}}(\tau)}{d\tau} + A_{\phi_{4,i}} - \gamma_i D_{\phi_{4,i}}(\tau) \right] \phi_{4,i}(t) + \\
& \left[ -\frac{dD_{\phi_{5,i}}(\tau)}{d\tau} + A_{\phi_{5,i}} - 2\gamma_i D_{\phi_{5,i}}(\tau) + 2D_{\phi_{6,i}}(\tau) \right] \phi_{5,i}(t) + \\
& \left. \left[ -\frac{dD_{\phi_{6,i}}(\tau)}{d\tau} + A_{\phi_{6,i}} - 2\gamma_i D_{\phi_{6,i}}(\tau) \right] \phi_{6,i}(t) \right) + \\
& \left[ -\frac{dD_v(\tau)}{d\tau} + \frac{1}{2\eta} \sum_{i=1}^N \lambda_i^2 + \sum_{i=1}^N (\lambda_i D_{x_i}(\tau) + D_{\phi_{2,i}}(\tau) + D_{\phi_{3,i}}(\tau)) - \bar{\kappa} D_v(\tau) + \right. \\
& \frac{1-\eta}{\eta} \left( \sum_{i=1}^N \lambda_i D_{x_i}(\tau) + \sigma \left( \sum_{i=1}^N \lambda_i \rho_i \right) D_v(\tau) \right) + \\
& \left. \frac{1-\eta}{2\eta} \left( \sum_{i=1}^N (D_{x_i}(\tau)^2 + 2\rho_i \sigma D_{x_i}(\tau) D_v(\tau)) + \sigma^2 D_v(\tau)^2 - (1-\eta) \left( 1 - \sum_{i=1}^N \rho_i^2 \right) \sigma^2 D_v(\tau)^2 \right) \right] v(t) + \\
& \left[ -\frac{dC(\tau)}{d\tau} + f(0, T - \tau) + \bar{\kappa} \bar{\theta} D_v(\tau) \right]. \tag{85}
\end{aligned}$$

Again, (82) is indeed a solution to the PDE, provided that  $C(\tau)$ ,  $D_{x_i}(\tau)$ ,  $D_{\phi_{j,i}}(\tau)$ , and  $D_v(\tau)$  solve the system of ODEs given in brackets subject to the boundary conditions  $C(0) = D_{x_i}(0) = D_{\phi_{j,i}}(0) = D_v(0) = 0$ . Note that only the ODE associated with  $v(t)$  changes relative to the complete market case and I obtain the result stated in the second part of Proposition 1.

	5 year		10 year		30 year		SPX	
	$K(t, T)$	$V(t, T)$	$K(t, T)$	$V(t, T)$	$K(t, T)$	$V(t, T)$	$K(t, T)$	$V(t, T)$
Mean	0.202	0.164	0.424	0.347	0.872	0.813	4.419	2.850
Median	0.177	0.129	0.371	0.274	0.761	0.683	3.791	1.976
Minimum	0.044	0.016	0.100	0.029	0.240	0.107	0.978	0.215
Maximum	0.655	1.158	1.629	1.680	3.033	3.263	20.921	20.452
Std. dev.	0.112	0.120	0.231	0.248	0.424	0.517	3.067	2.865
Skewness	1.010	1.968	1.180	1.737	1.415	1.667	1.656	2.565
Kurtosis	3.649	9.251	4.626	6.525	5.238	6.476	6.691	11.492
Number of obs.	3131	3131	3147	3147	3146	3146	3318	3318

Notes: Summary statistics of the variance swap rates,  $K(t, T)$ , and realized variances,  $V(t, T)$ , both multiplied by 100, for the 5, 10, and 30 year Treasury futures and the S&P 500 equity index ( $SPX$ ).

Table 1: Summary statistics of  $K(t, T)$  and  $V(t, T)$

	$(V(t, T) - K(t, T)) \times 100$				$\log(V(t, T)/K(t, T))$			
	5 year	10 year	30 year	SPX	5 year	10 year	30 year	SPX
Mean	-0.038	-0.077	-0.069	-1.569	-0.292	-0.280	-0.150	-0.604
$T$ -statistics	-5.765	-5.844	-2.609	-10.017	-9.041	-9.019	-5.404	-17.057
Median	-0.039	-0.076	-0.103	-1.383	-0.305	-0.295	-0.166	-0.626
Minimum	-0.414	-0.995	-1.225	-16.411	-1.980	-2.162	-1.820	-2.144
Maximum	0.867	1.102	2.201	11.134	1.410	1.316	1.187	1.175
Standard deviation	0.104	0.203	0.395	2.416	0.504	0.479	0.425	0.531
Skewness	1.404	0.574	1.216	-0.090	0.073	-0.046	0.068	0.394
Kurtosis	11.314	6.475	7.559	8.982	3.512	3.202	3.177	3.240
SR	—	—	—	—	0.561	0.558	0.336	1.016
SR, ref.	—	—	—	—	0.187	0.187	0.149	0.142

Notes: Summary statistics of  $(V(t, T) - K(t, T)) \times 100$ , the payoff to a long position in a variance swap with a notional amount of 100 USD and  $\log(V(t, T)/K(t, T))$ , the log excess return on a long position in a variance swap for the 5, 10, and 30 year Treasury futures and the S&P 500 equity index ( $SPX$ ). “SR” refers to the annualized Sharpe ratios of a short position in the variance swaps. “SR, ref.” refers to the annualized Sharpe ratios of a long position in the reference assets, i.e. the Treasury futures or the S&P 500 equity index.  $T$ -statistics and Sharpe ratios are computed from standard deviations estimated with the approach of Newey and West (1987) using a lag-length of 21 business days, which is the mean variance swap maturity for all the assets.

Table 2: Summary statistics of payoffs and excess returns of variance swaps

	$\alpha$	$\beta^{SPX}$	$\beta^{bond,1}$	$\beta^{bond,2}$	$\beta^{bond,3}$	$R^2$
5 year	-0.299 (-9.518)	-0.024 (-0.478)	2.588 (1.483)	0.122 (0.111)	-0.715 (-1.974)	0.093
10 year	-0.283 (-9.546)	-0.000 (-0.003)	2.517 (1.435)	-0.079 (-0.071)	-0.633 (-1.832)	0.092
30 year	-0.148 (-5.583)	-0.004 (-0.105)	3.679 (2.607)	-1.294 (-1.542)	-0.184 (-0.704)	0.094
SPX	-0.597 (-19.780)	-0.417 (-7.002)	-2.292 (-1.214)	2.724 (2.519)	-0.975 (-2.883)	0.234

Notes: Estimates of the regressions

$$r_{t,T}^{VS} = \alpha + \beta^{SPX} r_{t,T}^{SPX} + \sum_{i=1}^3 \beta^{bond,i} r_{t,T}^{bond,i} + \epsilon,$$

where  $r_{t,T}^{VS}$  denotes the log excess return on a variance swap,  $r_{t,T}^{SPX}$  denotes the log excess return on the S&P 500 index, and  $r_{t,T}^{bond,i}$ ,  $i = 1, 2, 3$  denote the log excess returns on portfolios of Treasury bonds with maturities 1–3 years, 5–7 years, and greater than 10 years, respectively. Results are displayed for variance swaps on the 5, 10, and 30 year Treasury futures and the S&P 500 equity index (*SPX*). Regressions are estimated by OLS. The  $T$ -statistics are reported in parentheses. These are computed using the Newey and West (1987) estimator with a lag-length of 21 business days, which is the mean variance swap maturity for all the assets.

Table 3: Explaining the variance risk premium with bond and equity market risk factors



	$V(t, T) = a + bK(t, T) + \epsilon$			$\log V(t, T) = a + b \log K(t, T) + \epsilon$		
	$a$	$b$	$R^2$	$a$	$b$	$R^2$
5 year	0.000 (3.971)	0.648 (-7.747)	0.363	-1.194 (-3.981)	0.858 (-2.997)	0.473
10 year	0.001 (2.733)	0.688 (-5.335)	0.410	-0.968 (-3.467)	0.877 (-2.478)	0.493
30 year	0.001 (3.031)	0.769 (-3.892)	0.422	-0.635 (-2.497)	0.900 (-1.923)	0.478
SPX	0.001 (0.334)	0.626 (-4.951)	0.449	-0.467 (-2.484)	1.041 (0.746)	0.627

Notes:  $K(t, T)$  is the variance swap rate, and  $V(t, T)$  is the realized variance. Results are displayed for the 5, 10, and 30 year Treasury futures and the S&P 500 equity index (*SPX*). Regressions are estimated by OLS. The  $T$ -statistics under the null hypotheses of  $a = 0$  and  $b = 1$  are reported in parentheses. These are computed using the Newey and West (1987) estimator with a lag-length of 21 business days, which is the mean variance swap maturity for all the assets.

Table 4: Time-variation in the variance risk premium

	$N = 1$	$N = 2$	$N = 3$
$\alpha_{0,1}$	0.0143 (0.0132)	0.0086 (0.0018)	0.0048 (0.0001)
$\alpha_{1,1}$	0.0026 (0.0024)	0.0037 (0.0008)	0.0021 (0.0001)
$\gamma_1$	0.1033 (0.0011)	0.1347 (0.0008)	0.0844 (0.0005)
$\rho_1$	-0.0200 (0.0033)	-0.1339 (0.0205)	-0.1251 (0.0086)
$\lambda_1$	-0.1933 (0.0778)	-0.1731 (0.0876)	-0.1252 (0.0791)
$\alpha_{0,2}$	—	0.0058 (0.0013)	-0.0113 (0.0003)
$\alpha_{1,2}$	—	0.0048 (0.0011)	0.0307 (0.0009)
$\gamma_2$	—	0.7406 (0.0042)	0.6611 (0.0012)
$\rho_2$	—	0.3539 (0.0492)	0.3155 (0.0044)
$\lambda_2$	—	-0.0916 (0.0653)	-0.0674 (0.0872)
$\alpha_{0,3}$	—	—	0.0013 (0.0001)
$\alpha_{1,3}$	—	—	0.0213 (0.0006)
$\gamma_3$	—	—	1.5394 (0.0051)
$\rho_3$	—	—	0.0800 (0.0200)
$\lambda_3$	—	—	-0.0194 (0.1054)
$\varphi$	0.0344 (0.0023)	0.0573 (0.0002)	0.0336 (0.0012)
$\kappa$	1.0980 (0.2467)	0.8320 (0.1444)	0.8346 (0.1344)
$\theta$	0.7153 (0.1242)	1.1842 (0.2477)	1.4516 (0.1925)
$\lambda_{N+1}$	-0.6105 (0.1283)	-0.4825 (0.1276)	-0.4687 (0.1032)
$\sigma_{rates}$	0.0050 (0.0000)	0.0010 (0.0000)	0.0003 (0.0000)
$\sigma_{swaptions}$	0.0239 (0.0000)	0.0208 (0.0000)	0.0171 (0.0000)
Log-likelihood	-77701.4	-37459.7	-9442.3
$\bar{\kappa}$	1.7045	1.2878	1.2810
$\bar{\theta}$	0.4608	0.7650	0.9458

Notes: Maximum-likelihood estimates of the model specifications with  $N = 1, 2,$  and  $3$ . The estimation period is from January 23, 1997 to April 30, 2008 (2836 daily observations). Outer-product standard errors are in parentheses.  $\sigma_{rates}$  denotes the standard deviation of errors in LIBOR and swap rates and  $\sigma_{swaptions}$  denotes the standard deviation of errors in swaption prices scaled by their Black (1976) “vegas” (which approximately corresponds to errors in swaption log-normal implied volatilities). For the specifications to be identified, I set  $\sigma = 1$ .

Table 5: Parameter estimates

		$N = 1$	$N = 2$	$N = 3$
$SR_{ZCB}$	2yr	0.1312	0.1711	0.1268
	5yr	0.1312	0.1664	0.1321
	10yr	0.1312	0.1605	0.1394
	30yr	0.1312	0.1564	0.1330
$SR_{FUT}$	2yr	0.1312	0.1710	0.1252
	5yr	0.1312	0.1661	0.1317
	10yr	0.1312	0.1603	0.1394
	30yr	0.1312	0.1563	0.1331
$SR_{VAR}$	—	-0.4117	-0.3987	-0.4341
$SR_{USV}$	—	-0.4144	-0.4220	-0.4558
$SR_{bonds}^{tan}$	—	0.1312	0.1713	0.1396
$SR_{deriv}^{tan}$	—	0.4347	0.4554	0.4767

Notes: Model implied unconditional instantaneous Sharpe ratios.  $SR_{ZCB}$  denotes the Sharpe ratios on zero-coupon bonds with maturities 2, 5, 10, and 30 years.  $SR_{FUT}$  denotes the Sharpe ratios on 1 month futures contracts on zero-coupon bonds with maturities 2, 5, 10, and 30 years.  $SR_{VAR}$  denotes the Sharpe ratios on a derivative exposed solely to variance.  $SR_{USV}$  denotes the Sharpe ratios on a derivative exposed solely to the unspanned variance factor.  $SR_{MV}^{bonds}$  denotes the Sharpe ratio of the tangency portfolio when the investment universe only consists of bonds.  $SR_{MV}^{deriv}$  denotes the Sharpe ratio of the tangency portfolio when the investment universe also includes interest rate derivatives.

Table 6: Model implied Sharpe ratios

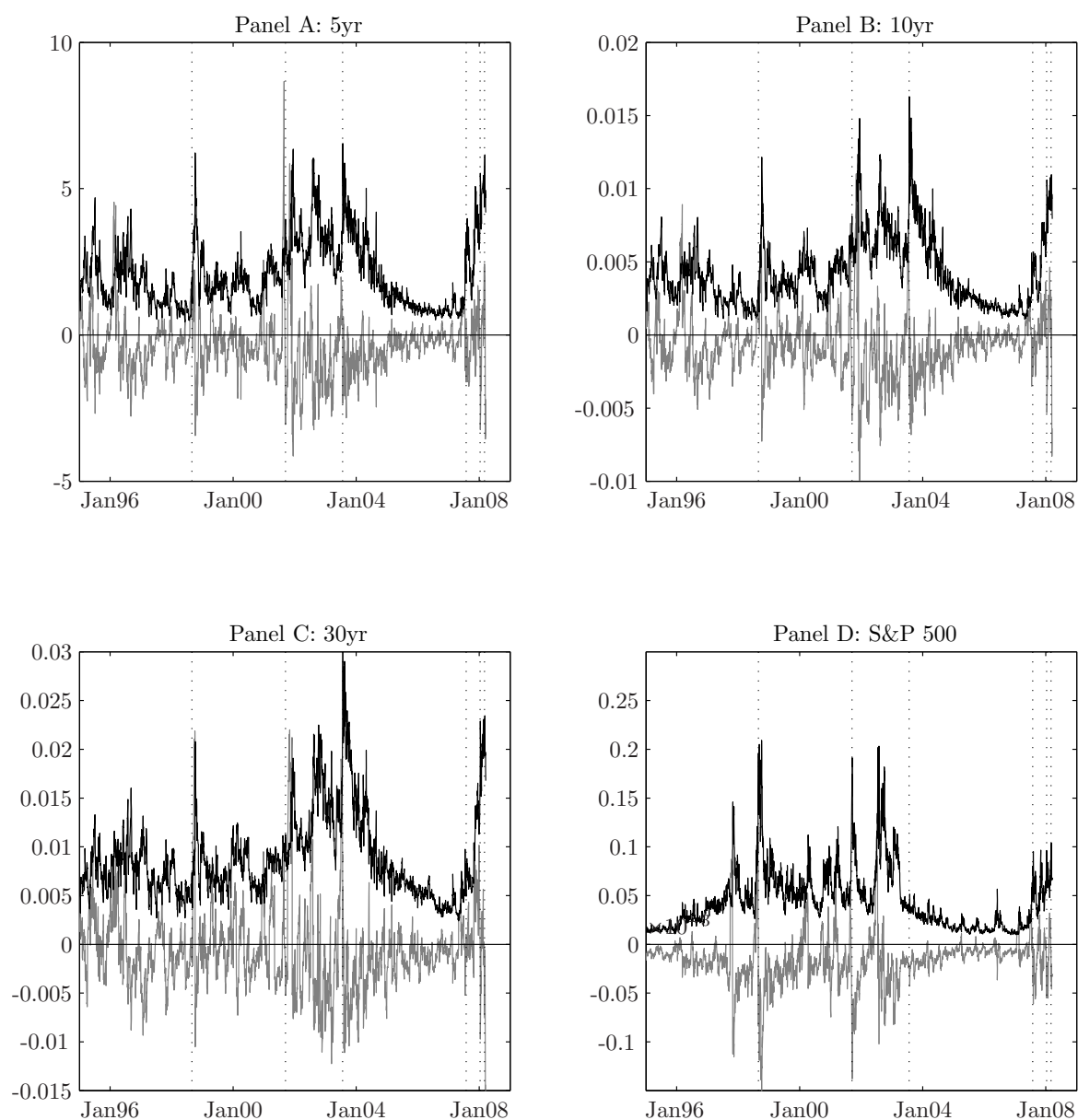


Figure 1: Time series of variance swap rates and payoffs

The black lines display the time-series of the variance swap rates,  $K(t, T)$ . The grey lines display the time-series of the payoff on long positions in variance swaps,  $V(t, T) - K(t, T)$ . The vertical dotted lines mark the onset of the LTCM crisis in early August 1998, the September 11, 2001 terrorist attacks, the sharp increase in interest rates in late July, 2003, which caused massive convexity hedging of MBS portfolios, the escalation of the credit crisis on August 9, 2007, the 75bp Fed inter-meeting rate cut on January 22, 2008, and the collapse of Bear Stearns on March 18, 2008, respectively. Each time series consists of a maximum of 3334 daily observations from January 3, 1995 to March 28, 2008.

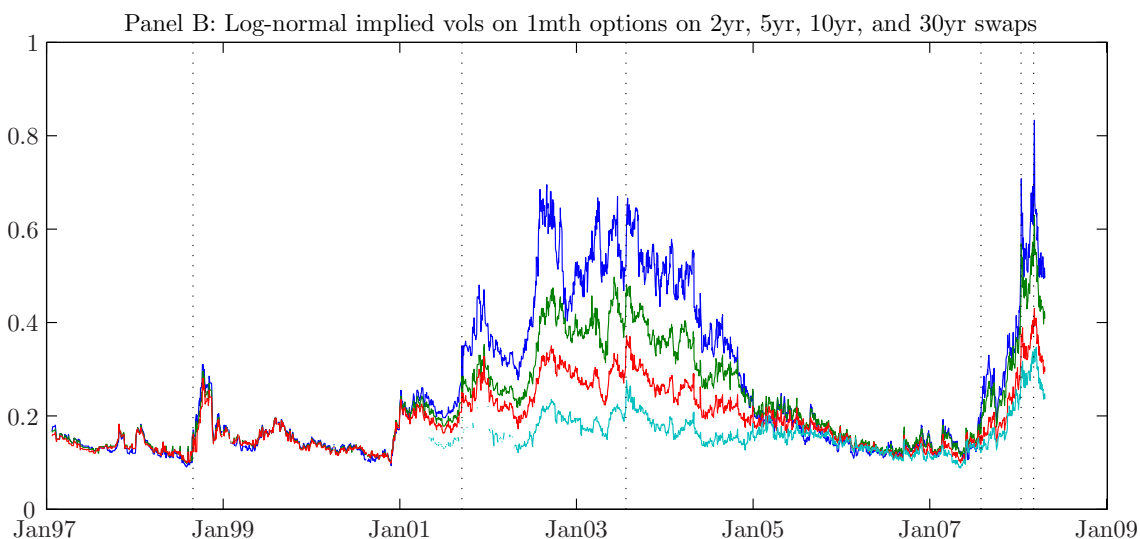
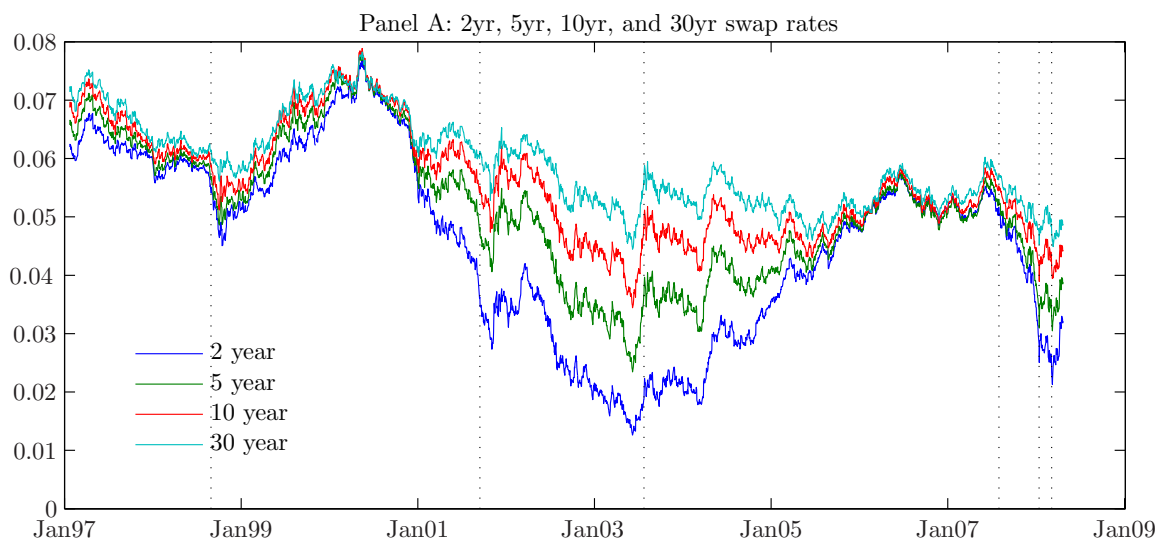


Figure 2: Time series of swap rates and ATM swaption volatilities

Panel A shows time series of 2, 5, 10, and 30 year swap rates (the estimations also includes 6 and 12 month LIBOR rates and 3, 7, and 15 year swap rates). Panel B shows time series of the corresponding 1 month ATM log-normal implied swaption volatilities. The vertical dotted lines mark the onset of the LTCM crisis in early August 1998, the September 11, 2001 terrorist attacks, the sharp increase in interest rates in late July, 2003, which caused massive convexity hedging of MBS portfolios, the escalation of the credit crisis on August 9, 2007, the 75bp Fed inter-meeting rate cut on January 22, 2008, and the collapse of Bear Stearns on March 18, 2008, respectively. Each time series consists of a maximum of 2836 daily observations from January 23, 1997 to April 30, 2008.

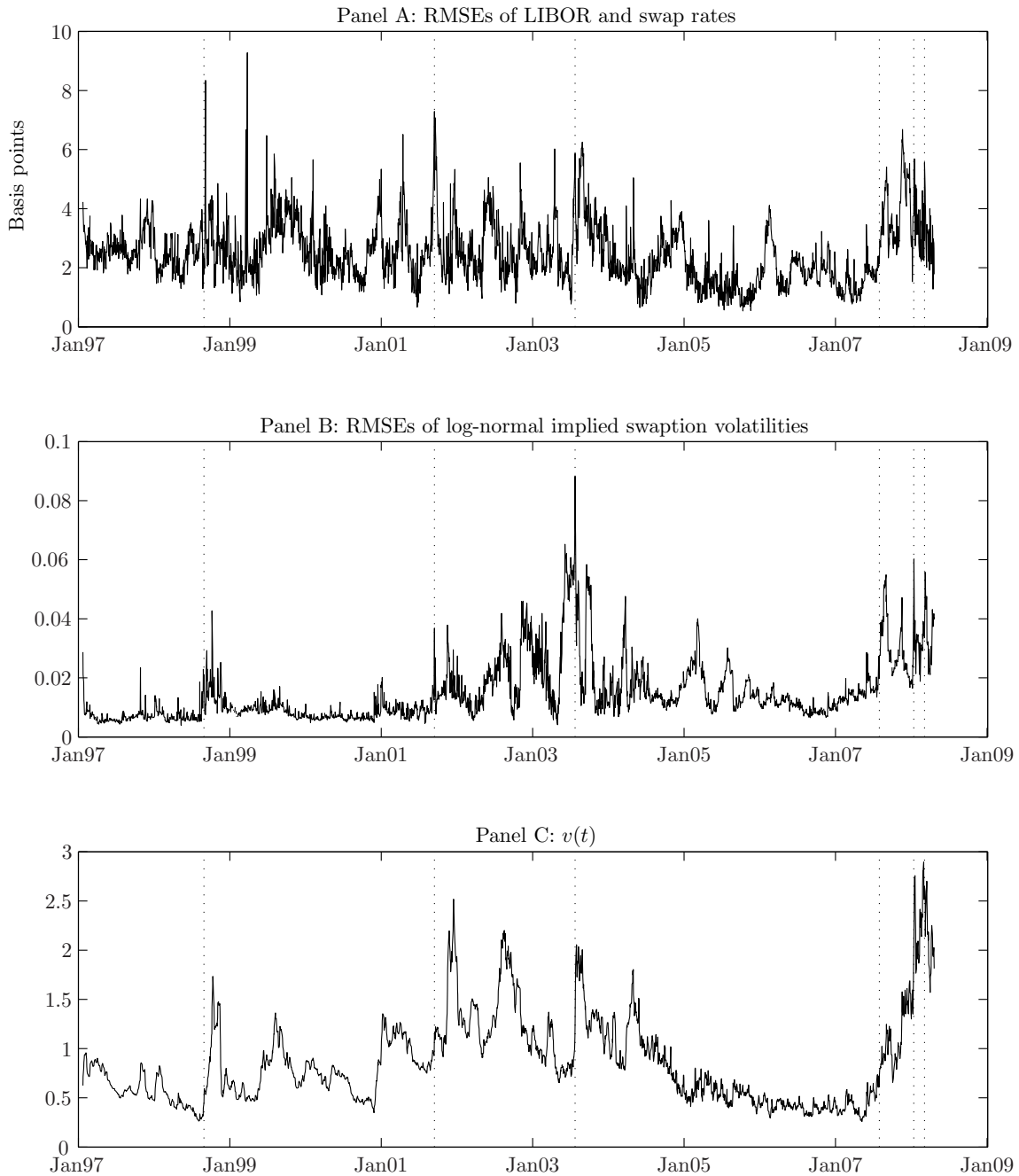


Figure 3: Time series of RMSEs and the variance state variable for the  $N = 3$  specification. Panel A shows the time series of root-mean-squared-errors (RMSEs) of the differences between fitted and actual LIBOR and swap rates. Panel B shows the time series of RMSEs of the differences between fitted and actual log-normal implied swaption volatilities. Panel C shows the time-series of the variance state variable,  $v(t)$ . The vertical dotted lines mark the onset of the LTCM crisis in early August 1998, the September 11, 2001 terrorist attacks, the sharp increase in interest rates in late July, 2003, which caused massive convexity hedging of MBS portfolios, the escalation of the credit crisis on August 9, 2007, the 75bp Fed inter-meeting rate cut on January 22, 2008, and the collapse of Bear Stearns on March 18, 2008, respectively. Each time series consists of 2836 daily observations from January 23, 1997 to April 30, 2008.

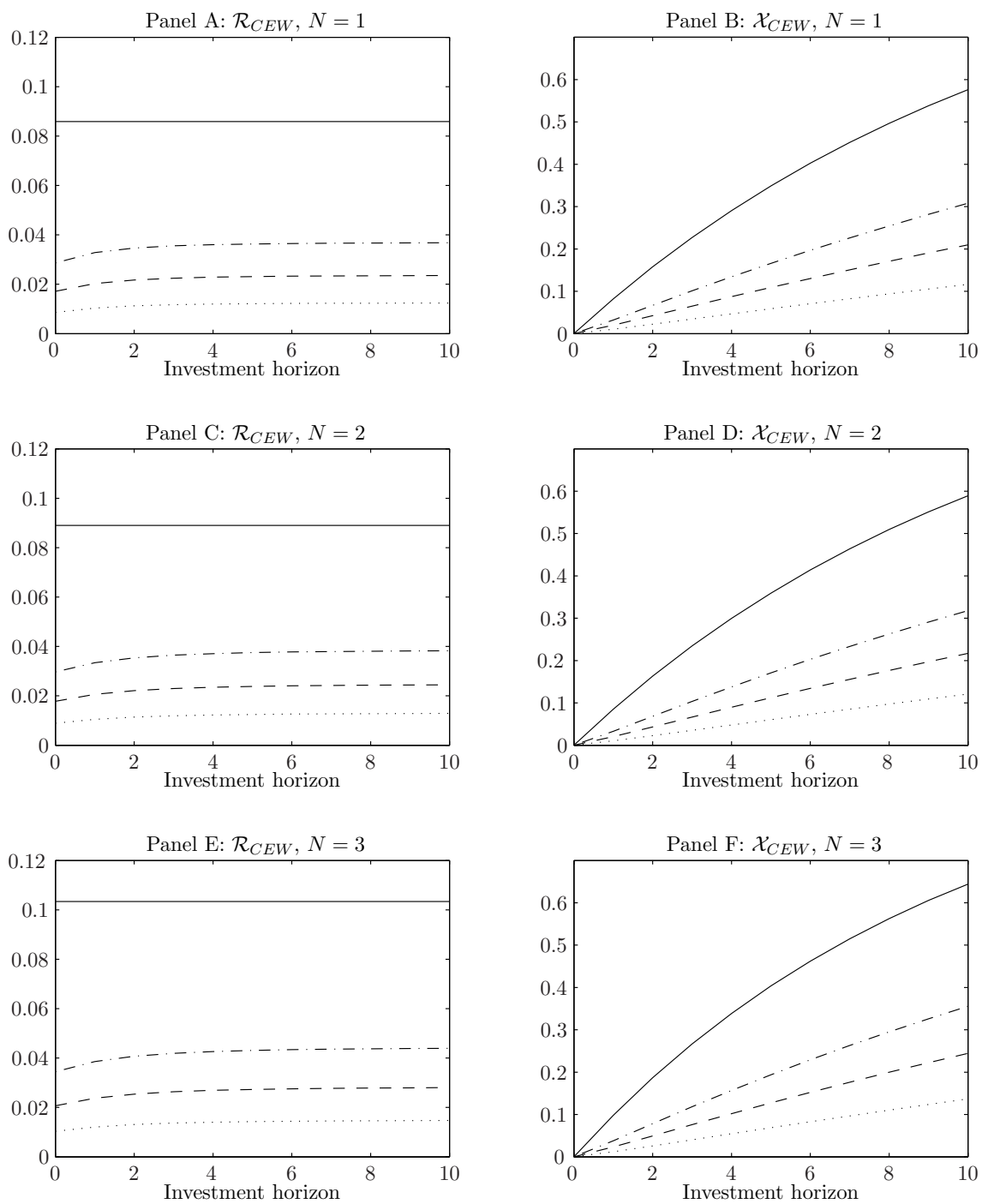


Figure 4: Utility gains from participating in the interest rate derivatives market

Panels A, C, and E show  $\mathcal{R}_{CEW}$ , the gain in CEW in terms of continuously compounded annualized returns (see (52)), for each of the three model specifications. Panels B, D, and F show  $\mathcal{X}_W$ , the fraction of wealth that a bond investor would be willing to give up to gain access to the interest rate derivatives market (see (53)), for each of the three model specifications. The solid, dashed-dotted, dashed, and dotted lines correspond to  $\eta = 1, 3, 5,$  and  $10$ , respectively.  $v(t)$  and the term structure equal their unconditional means.

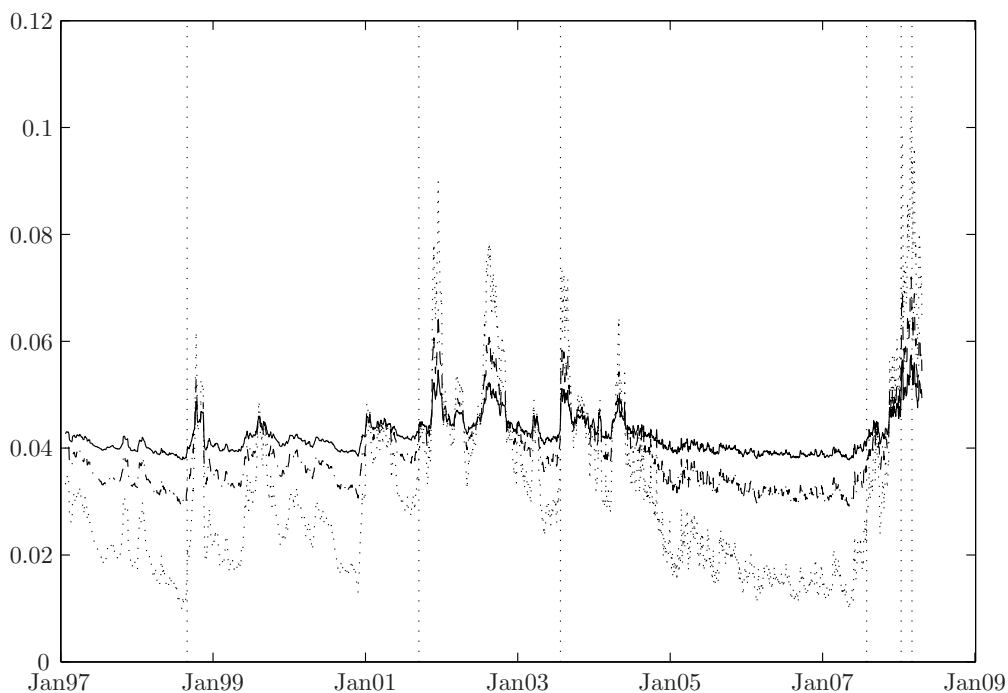


Figure 5: Time variation in utility gains from participating in the interest rate derivatives market

The figure shows the time series of  $\mathcal{R}_{CEW}$ , the gain in CEW in terms of continuously compounded annualized returns (see (52)), for the  $N = 3$  specification with  $\eta = 3$ . The dotted, dashed-dotted, and solid lines correspond to investment horizons of 1 day, 2 years, and 5 years, respectively. The vertical dotted lines mark the onset of the LTCM crisis in early August 1998, the September 11, 2001 terrorist attacks, the sharp increase in interest rates in late July, 2003, which caused massive convexity hedging of MBS portfolios, the escalation of the credit crisis on August 9, 2007, the 75bp Fed inter-meeting rate cut on January 22, 2008, and the collapse of Bear Stearns on March 18, 2008, respectively. Each time series consists of 2836 daily observations from January 23, 1997 to April 30, 2008.



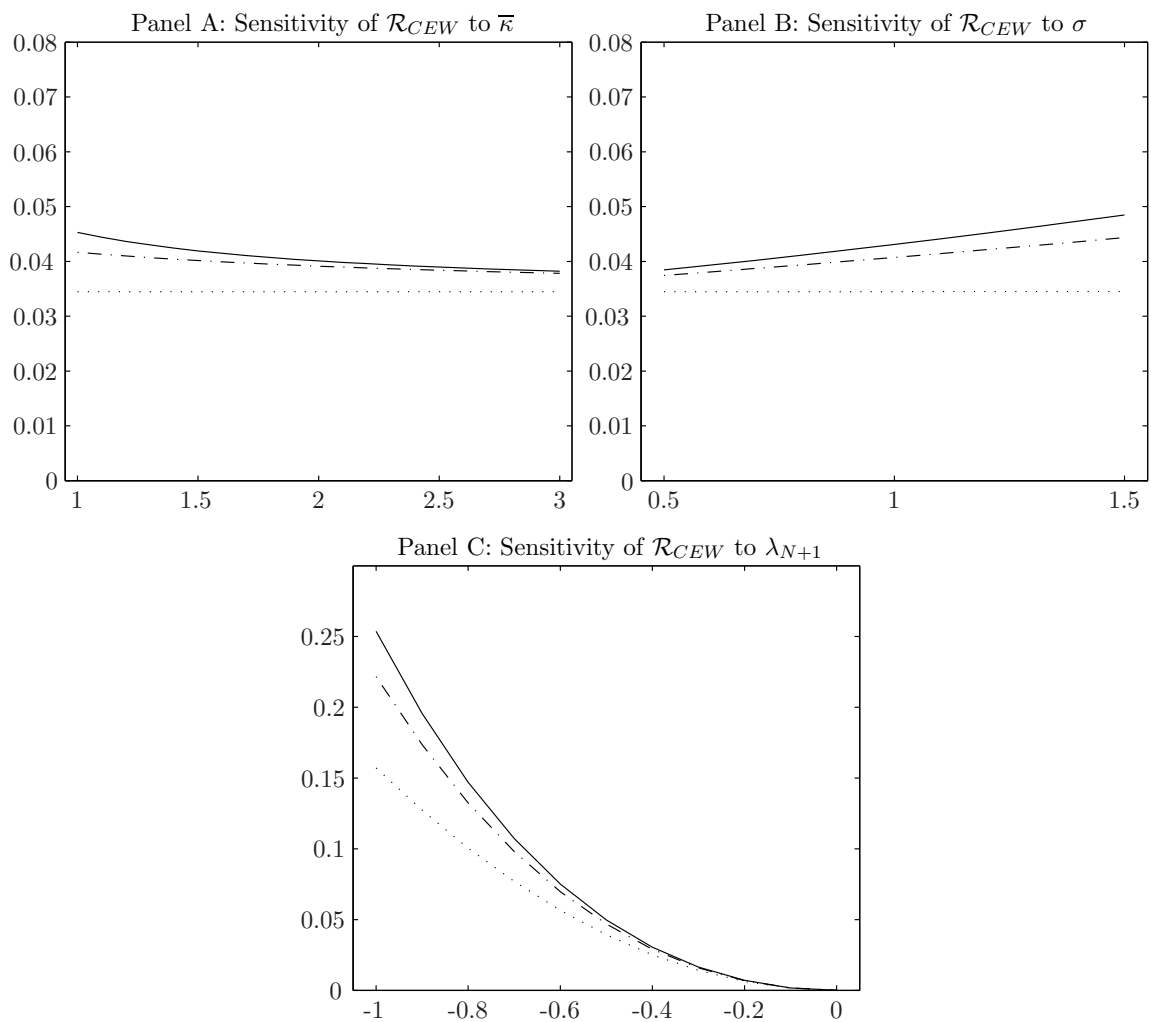


Figure 6: Sensitivity of utility gains to key parameters

The figure shows the sensitivity of  $\mathcal{R}_{CEW}$ , the gain in CEW in terms of continuously compounded annualized returns (see (52)), to  $\bar{\kappa}$  (Panel A),  $\sigma$  (Panel B), and  $\lambda_{N+1}$  (Panel C) for the  $N = 3$  specification with  $\eta = 3$ . The dotted, dashed-dotted, and solid lines correspond to investment horizons of 1 day, 2 years, and 5 years, respectively.  $v(t)$  and the term structure equal their unconditional means.

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